7/24/2025 SECOND HOURLY Practice 1 Maths 21a, O.Knill, Summer 2025

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) T F — Assume that $f(t,x)$ is a solution of the PDE $f_t = f_{xx}$, and $(0,0)$ is a critical point of f at which D is not zero, then $(0,0)$ is a saddle point of f.

2) T F — $|\nabla f(0,0)|$ is the maximum among all directional derivative $D_{\vec{v}} f(0,0)$ that can be computed at $(0,0)$.

3) T F — If $f(x,y)$ is continuous and the integral $\iint_G f(x,y)\, dA$ is zero, then $f(x,y)$ is zero somewhere in G.

4) T F — The improper integral $\int_{-\infty}^{\infty} e^{x^2}\, dx$ is equal to π.

5) T F — It is possible that a function $f(x,y)$ has a local maximum at $(0,0)$ and where $f_{yy}(0,0) = 0$.

6) T F — $(0,0)$ is a local minimum of the function $f(x,y) = x^3 + y^3$.

7) T F — If $(0,0)$ is a solution of the Lagrange equations for $f(x,y)$ under the constraint $g(x,y) = 0$, then $(0,0)$ can not be a critical point of g.

8) T F — The chain rule tells that $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'\,(f(t))$.

9) T F — If R the unit square $[0,1] \times [0,1]$ then $\iint_R x^2 + y^2\, dA = 2/3$.

10) T F — If $f(x,y) = \sin(x^2 y) x^5 y = 0$ defines y as a function $y = g(x)$ near the point $(0,0)$ then by implicit differentiation, $g'(0) = -f_x(0,0)/f_y(0,0)$.

11) T F — If $(0,0)$ is a critical point of f and $f_{xx}(0,0) = 1$ and $f_{yy}(0,0) = -1$, then $(0,0)$ is saddle point.

12) T F — If the discriminant D at a critical point is positive and $f_{yy} < 0$, then $f_{xx} < 0$.

13) T F — The result $\int_0^1 \int_0^1 f(x,y)\ dydx = \int_0^1 \int_0^1 f(x,y)\ dxdy$ is called the Toricelli theorem.

14) T F — By linear approximation, we can estimate $\sqrt{98} = 10 - 2/20 = 9.9$.

15) T F — If $(3,3)$ is a critical point of $f(x,y)$, then $(3,3)$ is also a critical point for the function $g(x,y) = \sin(f(x,y))$.

16) T F — The gradient of $f(x,y)$ at $(0,0)$ is a vector perpendicular to the level curve $f(x,y) = f(0,0)$.

17) T F — If (x_0, y_0) is a saddle point of $f(x,y)$ then (x_0, y_0) is a critical point under the constraint $g(x,y) = c = f(x_0, y_0)$.

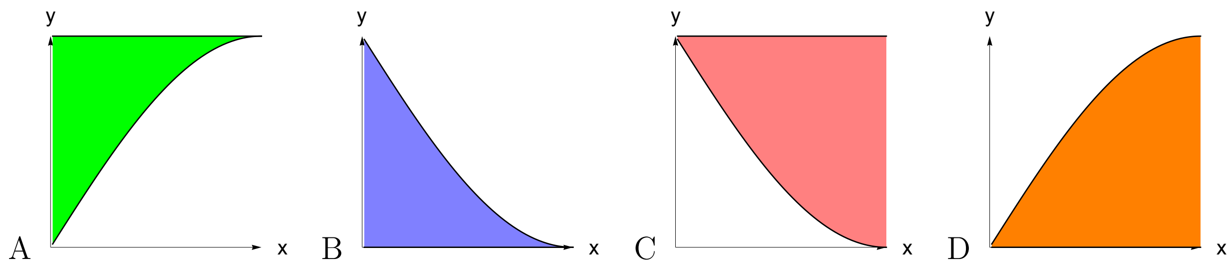
18) T F — The area of of a region R is the integral $\iint_R 1\, dA$.

19) T F — If $\vec{v}$ is a unit vector and $(0,0)$ is minimum of f with positive discriminant $D > 0$, then the second directional derivative $D_{\vec{v}} D_{\vec{v}} f(0,0) > 0$.

20) T F — There are continuous functions $f(x,y)$ for which Fubini fails.

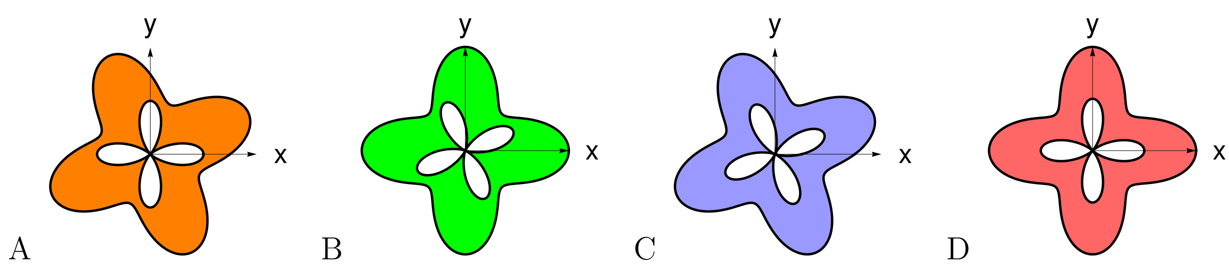
Problem 2) (10 points) No justifications are needed in this problem.

a) (4 points) Please match the regions with the area formulas. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^1 \int_{2\arcsin(y)/\pi}^1 1\ dxdy$
	$\int_0^1 \int_0^{1-\sin(\pi x/2)} 1\ dydx$
	$\int_0^1 \int_0^{2\arcsin(y)/\pi} 1\ dxdy$
	$\int_0^1 \int_{1-\sin(\pi x/2)}^1 1\ dydx$

b) (4 points) Please match **polar regions** with area integrals. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^{2\pi} \int_{1+\sin(4t)}^{3+\cos(4t)} r dr d\theta$
	$\int_0^{2\pi} \int_{1+\cos(4t)}^{3+\sin(4t)} r dr d\theta$
	$\int_0^{2\pi} \int_{1+\sin(4t)}^{3+\sin(4t)} r dr d\theta$
	$\int_0^{2\pi} \int_{1+\cos(4t)}^{3+\cos(4t)} r dr d\theta$

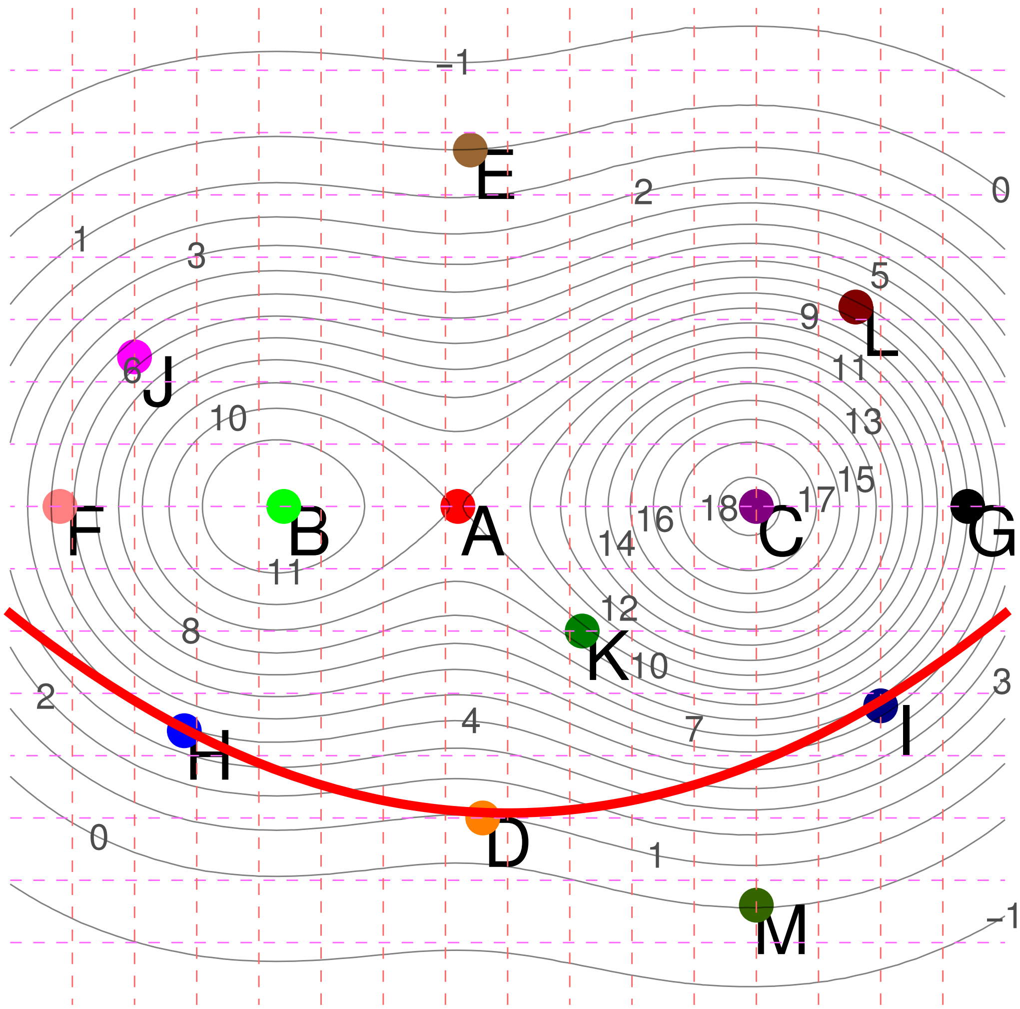
c) (2 points) Recall the differential equations for the unknown function of two variables.

Transport equation for $f(t,x)$:

Wave equation for $f(t,x)$:

Problem 3) (10 points) No justifications are needed in this problem.

Answer the following 5 questions for the **contour map** of a function $f(x, y)$. There is a constraint curve drawn which passes through the points H, D, I. This curve is referred to as $g(x, y) = 0$. Each problem is worth two points. In each entry, you need to enter at least one point, but there could be more than one points. So, for example, if both L and G would be an answer somewhere, you would enter $\boxed{\text{LG}}$ in that box.



	Enter from $\{A - M\}$
Local maxima or saddle points of f	
Maxima of f on the constraint $g(x, y) = 0$.	
Non critical points of f, where $f_x = 0$	
Non critical points of f , where $f_y = 0$	
Non critical points of f, where $f_y = 0, f_{xx} < 0$	

Problem 4) (10 points)

A cylindrical water tower of radius x and height y has **volume** $\pi x^2 y$ but **construction costs** grow like $\pi x^2 + \pi y$. Engineers decide to look for parameters x, y for which the volume minus the costs are extreme. They therefore look for critical points of

$$f(x,y) = x^2 y - x^2 - y \ .$$

a) (8 points) Find all the critical points of f and classify them. Include also points if x or y should be negative even so they represent radius and height.

b) (2 points) Does f have a global maximum or minimum when x, y ranges over the entire plane?

The Burlington tower near the Mary Cummings park: Photo shot by Oliver on 7/21/2024.

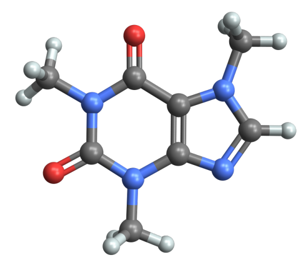
Problem 5) (10 points)

In the first practice exam you have maximized the **entropy** $S(x,y) = -x\log(x) - y\log(y)$ under the constraint $g(x,y) = x + y = 1$, where $\log(x) = \ln(x)$ is the natural log. If the system has **energy** $H(x,y) = 2x+4y$, we want to extremize the **free energy** $f = S - H$ under the same constraint. So, here is the task: maximize

$$f(x,y) = -x\log(x) - y\log(y) - 2x - 4y$$

under the constraint

$$g(x,y) = x + y = 1 \ .$$



The caffeine molecule. Free energy is important in chemistry. Apropos caffeine: Paul Erdoes once said: "A mathematician is a device for turning coffee into theorems".

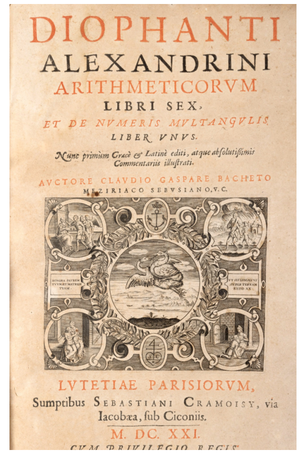
Problem 6) (10 points)

a) (5 points) Find the **tangent plane** to the Diophantine equation

$$f(x,y,z) = x^3 + 2y^4 - 3z^5 = 7 \ ,$$

at a solution point $(2,1,1)$.

b) (5 points) Estimate $f(2.01, 1.001, 0.99)$ using linear approximation.



Diophantine equations look for integer solutions to polynomial equations.

Problem 7) (10 points)

Compute the surface area of the surface which has the parametrization

$$\vec{r}(u,v) = \begin{bmatrix} 5+u+v^2 \\ 1+2v^2 \\ 3+u \end{bmatrix}$$

and where the parameter domain is $0 \leq u \leq 2$ and $0 \leq v \leq 1$.

AI drew this picture to the theme "Karma".

There would be a short cut to get the surface area. We want you go the standard way and give details. No credit except karma is given for the short cut.

Problem 8) (10 points)

Evaluate the double integral

$$\int_0^1 \int_{-y^{1/4}}^{y^{1/4}} e^{x-x^5/5} \, dx \, dy \, .$$

We did not want to draw anything helpful here

in order not to give away too much information.

May this Karma AI incarnation inspire!

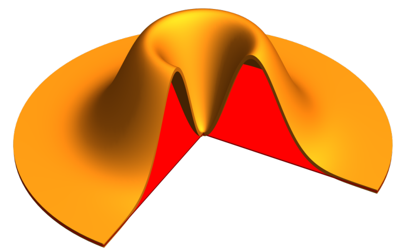
Problem 9) (10 points)

Evaluate the double integral

$$\iint_G (x^2 + y^2) e^{-(x^2+y^2)^2} \, dxdy \,,$$

where G is region given as the intersection of

$$\{x \geq 0 \text{ or } y \geq 0\} \text{ with } \{0 \leq x^2 + y^2 \leq 4\}.$$



The picture is not only here for Karma but could help to understand the region G.