7/24/2025 SECOND HOURLY Practice 1   Maths 21a, O.Knill, Summer 2025

"I affirm my awareness of the standards of the Harvard College Honor Code."

Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) No justifications are needed.

1) **T** F

Assume that $f(t, x)$ is a solution of the PDE $f_t = f_{xx}$, and $(0, 0)$ is a critical point of f at which D is not zero, then $(0, 0)$ is a saddle point of f.

Solution:

Since $f_t = 0$ we have $f_{xx} = 0$ and so $D = -f_{xy}^2 < 0$.

2) **T** F

$|\nabla f(0, 0)|$ is the maximum among all directional derivative $D_{\vec{v}} f(0, 0)$ that can be computed at $(0, 0)$.

Solution:

Angles dance upwards. The maximal slope indeed is the length of the gradient.

3) **T** F

If $f(x, y)$ is continuous and the integral $\iint_G f(x, y)\, dA$ is zero, then $f(x, y)$ is zero somewhere in G.

Solution:

Otherwise it would be everywhere positive (and give a positive integral) or everywhere negative (and give a negative integral)

4) T **F**

The improper integral $\int_{-\infty}^{\infty} e^{x^2}\, dx$ is equal to π.

Solution:

As in the movie "gifted", the sign is off

5) **T** F

It is possible that a function $f(x, y)$ has a local maximum at $(0, 0)$ and where $f_{yy}(0, 0) = 0$.

Solution:

Take $f(x, y) = -x^4 - y^4$ for example.

6) T **F** $(0,0)$ is a local minimum of the function $f(x,y) = x^3 + y^3$.

Solution:

The function can become negative and positive arbitrarily close.

7) T **F** If $(0,0)$ is a solution of the Lagrange equations for $f(x,y)$ under the constraint $g(x,y) = 0$, then $(0,0)$ can not be a critical point of g.

Solution:

It is possible that $(0,0)$ is also a critical point of g. It just implies $\lambda = 0$.

8) T **F** The chain rule tells that $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'\,(f(t))$.

Solution:

There is no $f(t)$ at the very end.

9) **T** F If R the unit square $[0,1] \times [0,1]$ then $\iint_R x^2 + y^2 \; dA = 2/3$.

Solution:

Just compute.

10) **T** F If $f(x,y) = \sin(x^2 y) x^5 y = 0$ defines y as a function $y = g(x)$ near the point $(0,0)$ then by implicit differentiation, $g'(0) = -f_x(0,0)/f_y(0,0)$.

Solution:

By the formula $y' = -f_x/f_y$.

11) **T** F If $(0,0)$ is a critical point of f and $f_{xx}(0,0) = 1$ and $f_{yy}(0,0) = -1$, then $(0,0)$ is saddle point.

Solution:

Indeed, then $D < 0$.

12) | T (marked) | F |

If the discriminant D at a critical point is positive and $f_{yy} < 0$, then $f_{xx} < 0$.

Solution:

This is the second derivative test

13) | T | F (marked) |

The result $\int_0^1 \int_0^1 f(x,y)\ dydx = \int_0^1 \int_0^1 f(x,y)\ dxdy$ is called the Toricelli theorem.

Solution:

It is the Fubini theorem.

14) | T (marked) | F |

By linear approximation, we can estimate $\sqrt{98} = 10 - 2/20 = 9.9$.

Solution:

Yes, this is how we do linear estimation.

15) | T (marked) | F |

If $(3,3)$ is a critical point of $f(x,y)$, then $(3,3)$ is also a critical point for the function $g(x,y) = \sin(f(x,y))$.

Solution:

$\nabla \sin(f(x,y)) = \cos(f(x,y))[f_x, f_y]$.

16) | T (marked) | F |

The gradient of $f(x,y)$ at $(0,0)$ is a vector perpendicular to the level curve $f(x,y) = f(0,0)$.

Solution:

Yes, this follows from the gradient theorem.

17) T **F** If (x_0, y_0) is a saddle point of $f(x,y)$ then (x_0, y_0) is a critical point under the constraint $g(x,y) = c = f(x_0, y_0)$.

Solution:

Yes, $\nabla f(x,y) = 0$ automatically also gives a solution to the Lagrange equations.

18) **T** F The area of of a region R is the integral $\iint_R 1 \, dA$.

Solution:

The first set contains the second set.

19) **T** F If $\vec{v}$ is a unit vector and $(0,0)$ is minimum of f with positive discriminant $D > 0$, then the second directional derivative $D_{\vec{v}} D_{\vec{v}} f(0,0) > 0$.

Solution:

Indeed this is just the second directional derivative in the $\vec{v}$ direction and so concave up.

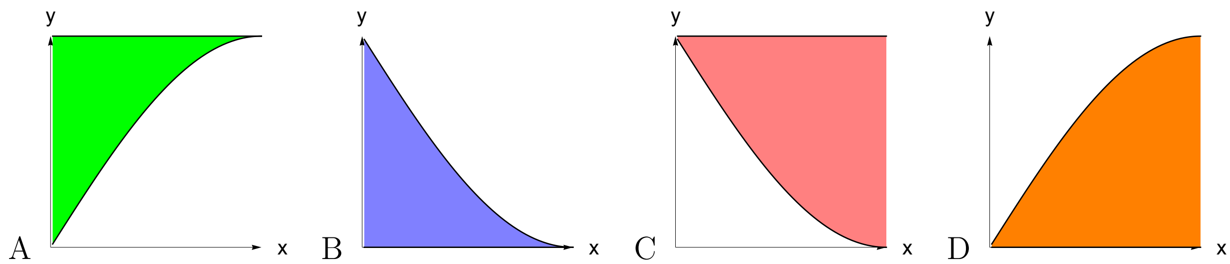
20) T **F** There are continuous functions $f(x,y)$ for which Fubini fails.

Solution:

No, Fubini works for all continuous functions.

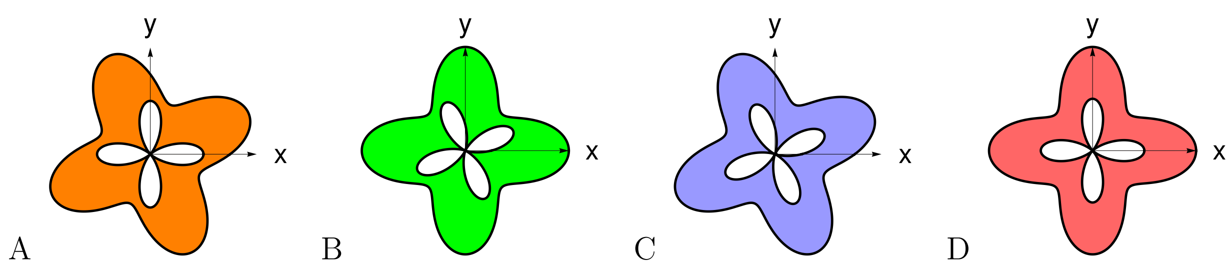
Problem 2) (10 points) No justifications are needed in this problem.

a) (4 points) Please match the regions with the area formulas. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^1 \int_{2\arcsin(y)/\pi}^1 1\ dxdy$
	$\int_0^1 \int_0^{1-\sin(\pi x/2)} 1\ dydx$
	$\int_0^1 \int_0^{2\arcsin(y)/\pi} 1\ dxdy$
	$\int_0^1 \int_{1-\sin(\pi x/2)}^1 1\ dydx$

b) (4 points) Please match **polar regions** with area integrals. A-D are used exactly once.



Enter A-D	Area integral
	$\int_0^{2\pi} \int_{1+\sin(4t)}^{3+\cos(4t)} r dr d\theta$
	$\int_0^{2\pi} \int_{1+\cos(4t)}^{3+\sin(4t)} r dr d\theta$
	$\int_0^{2\pi} \int_{1+\sin(4t)}^{3+\sin(4t)} r dr d\theta$
	$\int_0^{2\pi} \int_{1+\cos(4t)}^{3+\cos(4t)} r dr d\theta$

c) (2 points) Recall the differential equations for the unknown function of two variables.

Transport equation for $f(t,x)$:

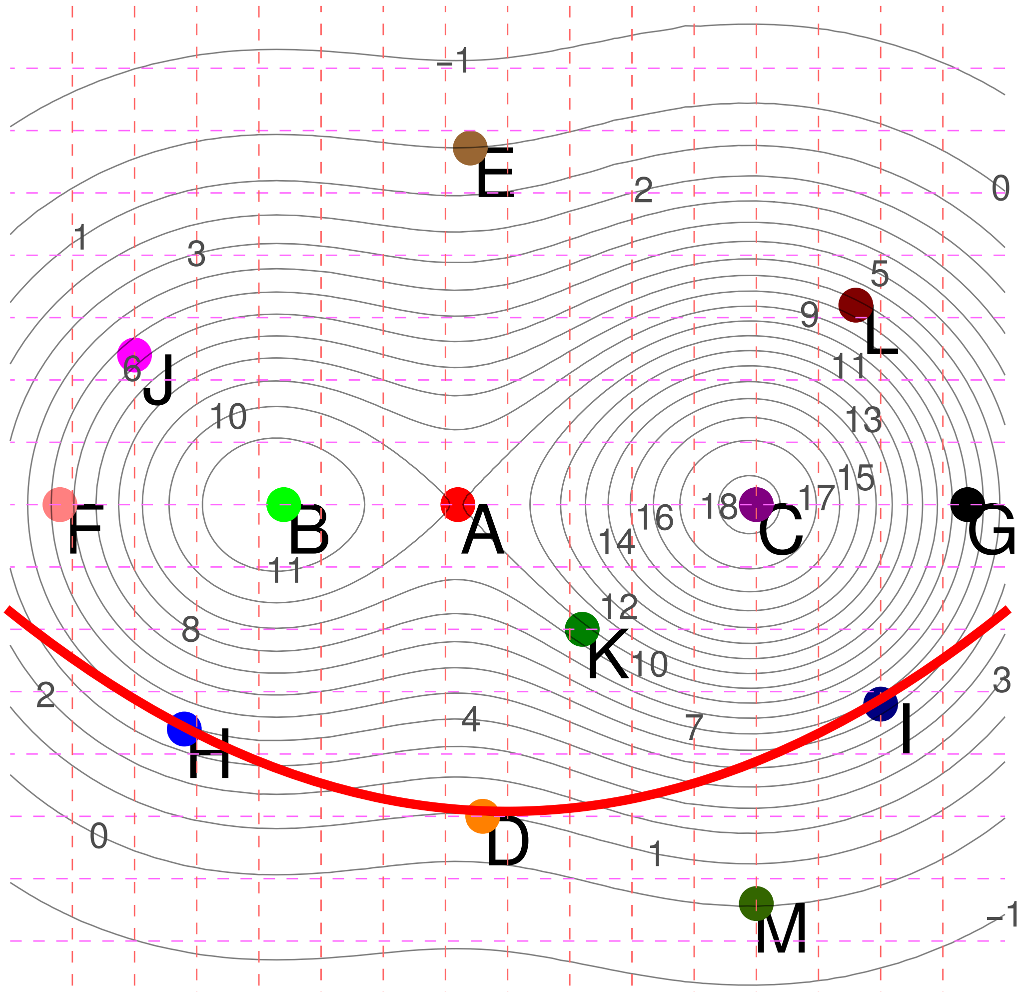
Wave equation for $f(t,x)$:

Solution:

a) DBAC
b) BACD
c) $f_t = f_x$ and $f_{tt} = f_{xx}$.

Problem 3) (10 points) No justifications are needed in this problem.

Answer the following 5 questions for the **contour map** of a function $f(x, y)$. There is a constraint curve drawn which passes through the points H, D, I. This curve is referred to as $g(x, y) = 0$. Each problem is worth two points. In each entry, you need to enter at least one point, but there could be more than one points. So, for example, if both L and G would be an answer somewhere, you would enter LG in that box.



	Enter from $\{A - M\}$
Local maxima or saddle points of f	
Maxima of f on the constraint $g(x, y) = 0$.	
Non critical points of f, where $f_x = 0$	
Non critical points of f , where $f_y = 0$	
Non critical points of f, where $f_y = 0, f_{xx} < 0$	

Solution:

ABC, IH, MED, FG, NONE

Problem 4) (10 points)

A cylindrical water tower of radius x and height y has **volume** $\pi x^2 y$ but **construction costs** grow like $\pi x^2 + \pi y$. Engineers decide to look for parameters x, y for which the volume minus the costs are extreme. They therefore look for critical points of

$$f(x,y) = x^2y - x^2 - y \; .$$

a) (8 points) Find all the critical points of f and classify them. Include also points if x or y should be negative even so they represent radius and height.

b) (2 points) Does f have a global maximum or minimum when x, y ranges over the entire plane?

The Burlington tower near the Mary Cummings park: Photo shot by Oliver on 7/21/2024.

Solution:

There are two critical points. Both are saddle points.

x	y	D	f_{xx}	Type	Value
-1	1	$-4\pi^2$	x	saddle	$-\pi$
1	1	$-4\pi^2$	x	saddle	$-\pi$

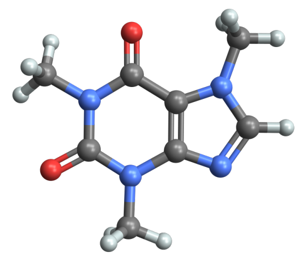
Problem 5) (10 points)

In the first practice exam you have maximized the **entropy** $S(x,y) = -x\log(x) - y\log(y)$ under the constraint $g(x,y) = x+y = 1$, where $\log(x) = \ln(x)$ is the natural log. If the system has **energy** $H(x,y) = 2x+4y$, we want to extremize the **free energy** $f = S-H$ under the same constraint. So, here is the task: maximize

$$f(x,y) = -x\log(x) - y\log(y) - 2x - 4y$$

under the constraint

$$g(x,y) = x+y = 1 \ .$$



The caffeine molecule. Free energy is important in chemistry. Apropos caffeine: Paul Erdoes once said: "A mathematician is a device for turning coffee into theorems".

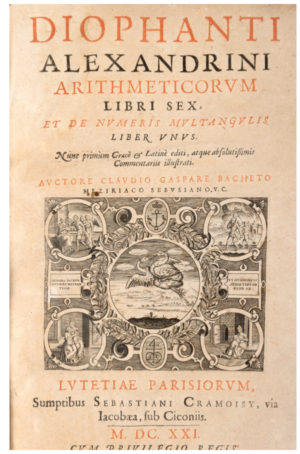
Solution:

The Lagrange equations are

$$\begin{aligned} -1 - \log(x) - 2 &= \lambda \\ -1 - \log(y) - 4 &= \lambda \\ x + y &= 1 \end{aligned}$$

The solution is $(x,y) = (e^2/(1+e^2), 1/(1+e^2))$. It is called a Gibbs distribution.

Problem 6) (10 points)



Diophantine equations look for integer solutions to polynomial equations.

a) (5 points) Find the **tangent plane** to the Diophantine equation

$$f(x,y,z) = x^3 + 2y^4 - 3z^5 = 7 \ ,$$

at a solution point $(2,1,1)$.

b) (5 points) Estimate $f(2.01, 1.001, 0.99)$ using linear approximation.

Solution:

a) $12x + 8y - 15z = 17$. b) $7 + 12 * 0.01 + 8 \cdot 0.001 - 15 \cdot (-0.01)$.

Problem 7) (10 points)

Compute the surface area of the surface which has the parametrization

$$\vec{r}(u,v) = \begin{bmatrix} 5+u+v^2 \\ 1+2v^2 \\ 3+u \end{bmatrix}$$

and where the parameter domain is $0 \leq u \leq 2$ and $0 \leq v \leq 1$.

AI drew this picture to the theme "Karma".

There would be a short cut to get the surface area. We want you go the standard way and give details. No credit except karma is given for the short cut.

Solution:

The $|r_u \times r_v|^2 = 36v^2$ gives $|r_u \times r_v| = 6v$. The surface area is 6.

Problem 8) (10 points)

Evaluate the double integral

$$\int_0^1 \int_{-y^{1/4}}^{y^{1/4}} e^{x-x^5/5} \, dx \, dy \, .$$

We did not want to draw anything helpful here

in order not to give away too much information.

May this Karma AI incarnation inspire!

Solution:

Change the order of integration. The bottom to top region has x^4 at the bottom and 1 at the top. The integral becomes $\int_{-1}^{1} \int_{x^4}^{1} e^{x-x^5/5} \; dydx$ which can be solved. The result is $2\sinh(4/5) = e^{4/5} - e^{-4/5}$.
Interestingly Mathematica fails to compute the integral directly!

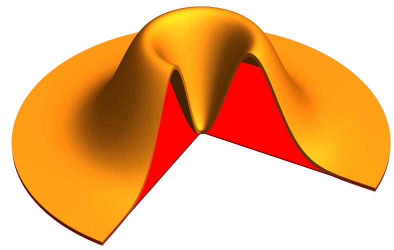
Problem 9) (10 points)

Evaluate the double integral

$$\iint_G (x^2+y^2)e^{-(x^2+y^2)^2}\,dxdy\ ,$$

where G is region given as the intersection of

$$\{x \geq 0 \text{ or } y \geq 0\} \text{ with } \{0 \leq x^2+y^2 \leq 4\}.$$



The picture is not only here for Karma but could help to understand the region G.

Solution:

Use polar coordinates. The final result is $(3\pi/8)(1-e^{-16})$.