**7/24/2025 SECOND HOURLY Practice 4 Maths 21a, O.Knill, Summer 2025**

**"I affirm my awareness of the standards of the Harvard College Honor Code."**

| Name: |
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| |

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have exactly 90 minutes to complete your work.

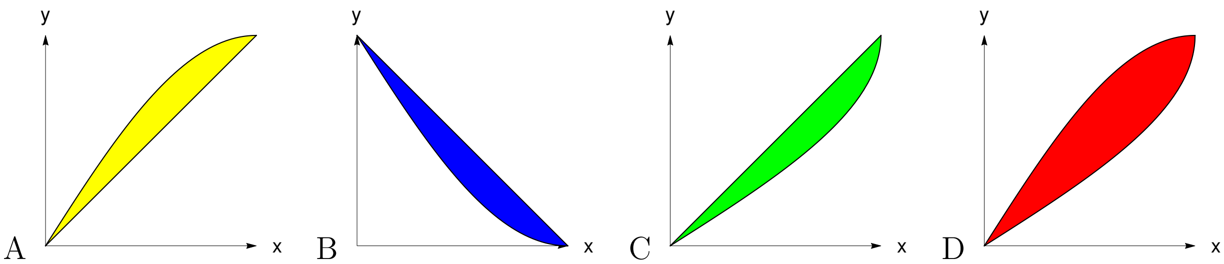
| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| Total: | | 100 |

Problem 1) (20 points) No justifications are needed.

| | | | |
|---|---|---|---|
| 1) | T | F | The gradient of the function $f(x,y) = x^2 + y^2$ at a point $(1,1)$ is perpendicular to the gradient of $g(x,y) = x^2 - y^2$ at $(1,1)$. |
| 2) | T | F | The function $f(x,y) = xy$ has only one critical point $(0,0)$. |
| 3) | T | F | If the discriminant $D$ and $f_{xx}$ are positive at a critical point $(1,1)$, then $(1,1)$ is a minimum. |
| 4) | T | F | The integral $\int_0^{\pi} \int_0^5 1 \, dr d\theta$ is the area of a half disk of radius 5. |
| 5) | T | F | If $f_{xy} = f_{xx} = f_{yy}$ at a critical point $(0,0)$ then the discriminant $D$ must be negative at $(0,0)$. |
| 6) | T | F | The function $L(x,y) = (2x)(x-1) + (4y)(y-2)$ is the linearization of $f(x,y) = x^2 + 2y^2$ at the point $(1,2)$. |
| 7) | T | F | The gradient of $f(x,y) = x^3 + y^3$ at the point $(1,1)$ is a vector perpendicular to the surface $z = x^3 + y^3$ at the point $(1,1,2)$. |
| 8) | T | F | Assume $f$ and $g$ are functions which both have the same critical point $(0,0)$ then $f$ must be a multiple of $g$. |
| 9) | T | F | Assume $\vec{r}(t)$ is a planar curve with constant speed $\lvert\vec{r}\,'(t)\rvert = 1$ and $f(x,y)$ a function, then $d/dt f(\vec{r}(t)) = D_{\vec{r}\,'(t)} f(\vec{r}(t))$. |
| 10) | T | F | If $f_{yy}(x,y) > 0$ for all points $(x,y)$, then $f$ can not have any local maximum. |
| 11) | T | F | If $f_{xx} = 0$ for all $x, y$, then at every critical point of $f$ with $D \neq 0$, we have a saddle point. |
| 12) | T | F | If $(0,0)$ is saddle point for a function $f$, then there are directions $\vec{v}, \vec{w}$ such that $D_{\vec{v}} f(0,0) > 0, D_{\vec{w}} f(0,0) < 0$. |
| 13) | T | F | $f_{tt} = f f_{xxx}$ is an example of a partial differential equation. |
| 14) | T | F | The Fubini identity assures that $\int_0^1 \int_1^3 f(x,y) \, dydx = \int_1^3 \int_0^1 f(x,y) \, dydx$. |
| 15) | T | F | Assume $(0,0)$ is not a critical point of $f$. Then the direction of steepest increase of $f$ at $(0,0)$ is $\nabla f(0,0)/\lvert\nabla f(0,0)\rvert$. |
| 16) | T | F | If $(0,0)$ is a maximum of a function $f(x,y)$ under the constraint $g(x,y) = 0$, then either $(0,0)$ is a critical point of $g$ or then a critical point of $f$. |
| 17) | T | F | The linearization of $f(x,y) = 3x + 4y + 5$ at the point $(1,1)$ is $L(x,y) = 3x + 4y + 5$. |
| 18) | T | F | If $f(x,y)$ has a maximum at $(0,0)$, then $D > 0$ and $f_{xx} < 0$. |
| 19) | T | F | The area the region $4 \leq x^2 + y^2 \leq 9, x \leq 0$ is $\int_{\pi/2}^{3\pi/2} \int_2^3 r \, dr d\theta$. |
| 20) | T | F | For any invertible function $0 \leq g(x) \leq 1$ we have $\int_0^1 \int_{g(x)}^1 f(x,y) \, dydx = \int_0^1 \int_0^{g^{-1}(y)} f(x,y) dxdy$. |

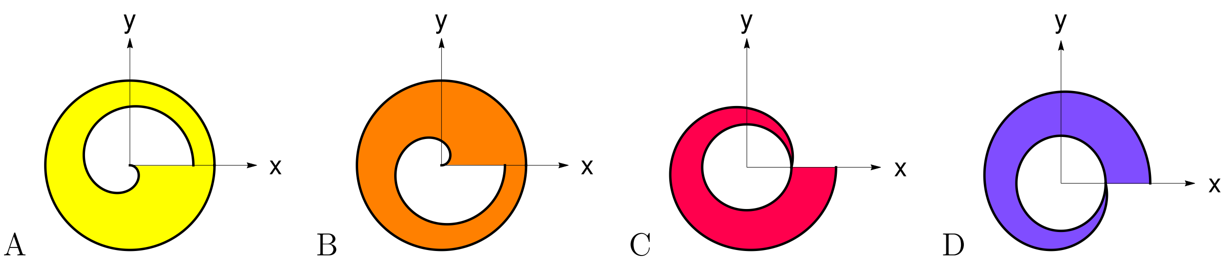
Problem 2) (10 points) No justifications are needed in this problem.

a) (4 points) Match the regions with their area formulas. A-D are used exactly once.




| Enter A-D | Area integral |
|---|---|
| | $\int_0^1 \int_{2\arcsin(x)/\pi}^{x} 1\ dydx$ |
| | $\int_0^1 \int_{1-\sin(\pi x/2)}^{1-x} 1\ dydx$ |
| | $\int_0^1 \int_{2\arcsin(x)/\pi}^{\sin(\pi x/2)} 1\ dydx$ |
| | $\int_0^1 \int_{x}^{\sin(\pi x/2)} 1\ dydx$ |

b) (4 points) Match the regions with their area integrals. A-D are used exactly once.




| Enter A-D | Area integral |
|---|---|
| | $\int_0^{2\pi} \int_2^{2+2\sin(t/4)} r\ drdt$ |
| | $\int_0^{2\pi} \int_2^{2+2\cos(t/4)} r\ drdt$ |
| | $\int_0^{2\pi} \int_{3\sin(t/4)}^{4} r\ drdt$ |
| | $\int_0^{2\pi} \int_{3\cos(t/4)}^{4} r\ drdt$ |

c) (2 points) We want you to write down the formulas for two partial differential equations for the unknown function $f(x,t)$.

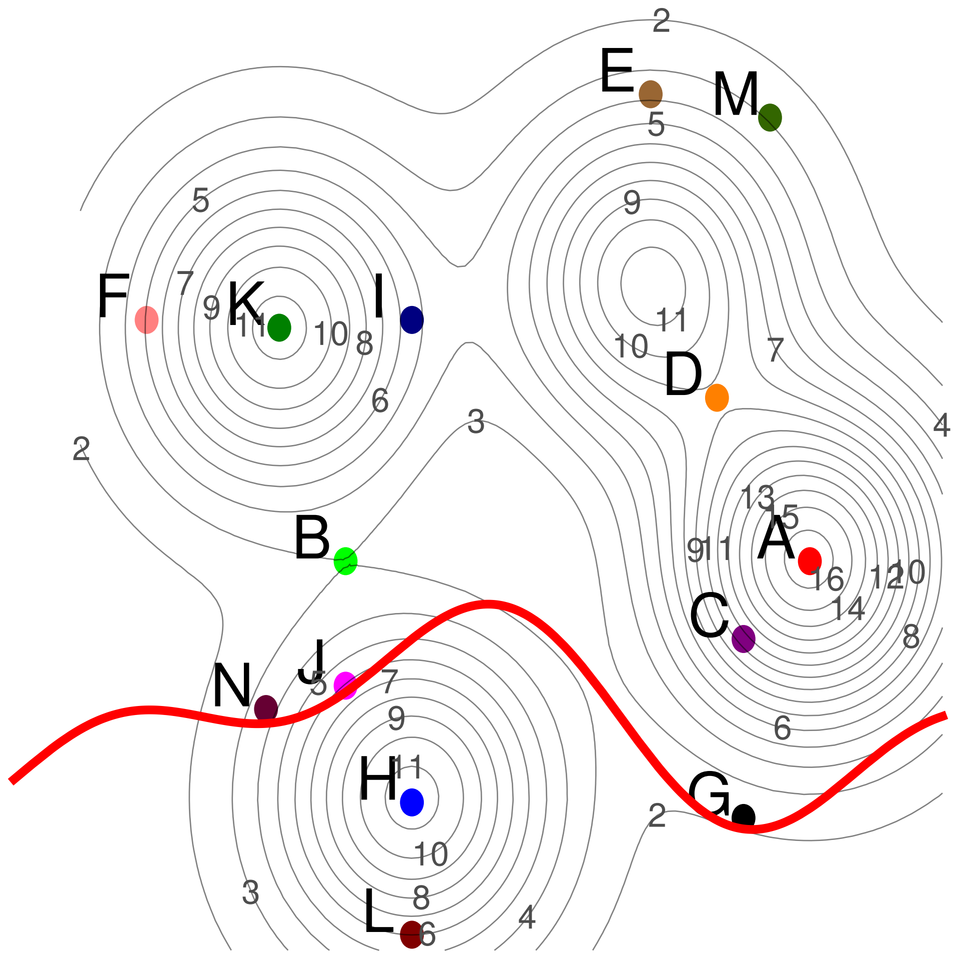
Wave equation: 

Black-Scholes equation: 

Problem 3) (10 points) No justifications are needed in this problem.

(10 points) We see the contours of an unknown smooth function, $f(x,y)$. The thick red curve is a constraint $g(x,y)=0$. Use each of the labels A-N only once. Read carefully: in the last two questions, we want you to enter two letters You will therefore use 12 of the 14 letters $A-N$ and none of them twice.

| | Enter A-N here |
|---|---|
| The point among A-N with maximal $\lvert\nabla f\rvert$. | |
| A point, where $f_x < 0, f_y = 0$. | |
| A point, where $f_x > 0, f_y = 0$. | |
| A point, where $f_y > 0, f_x = 0$. | |
| A point, where $f_y < 0, f_x = 0$. | |
| A global maximum of $f(x, y)$. | |
| A local maximum of $f(x, y)$ on $\{g(x, y) = 0\}$. | |
| A local minimum of $f(x, y)$ on $\{g(x, y) = 0\}$. | |
| Two points that are saddle points | and |
| Two points that are local but not global max | and |



| Problem 4) (10 points) |
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a) (8 points) Classify the critical points of the function

$$f(x,y) = x^{10} - 5x^2 + y^2 + 2y$$

using the second derivative test.

| Point | $D$ | $f_{xx}$ | nature |
|---|---|---|---|
| | | | |

b) (2 points) Is there a global maximum or minimum of $f(x,y)$? (No explanation is necessary for this part b).)

| | Yes | No |
|---|---|---|
| There is a global max for $f$ | | |
| There is a global min for $f$ | | |

| Problem 5) (10 points) |
|---|

Use the Lagrange method to solve the problem to extremize

$$f(x,y) = 5 + x^3 + y^3$$

under the constraint $g(x,y) = 9x + 4y = 35$. There is only one solution with positive $x$ and with positive $y$. Find this solution.

| Problem 6) (10 points) |
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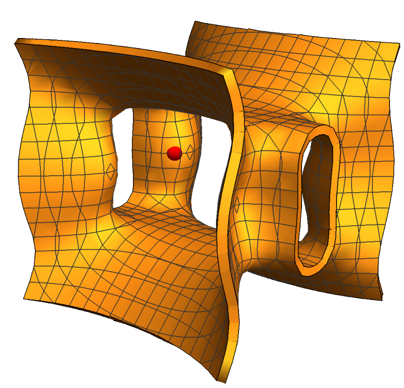

a) 5 points) Find the equation $ax+by+cz = d$ of the tangent plane to the surface

$$f(x,y,z) = x^4 - y^4 + z^2 + x^2y^2 - x^2z^2 + y^2 - z^2 = -3$$

at the point $(1,0,2)$.

b) (5 points) Estimate $f(1.01, 0.02, 1.97)$ using linearization.

Problem 7) (10 points)

Find the **surface area** of the surface

$$\vec{r}(u,v) = [2u, 4 - u^2 - v^2, 2v]$$

with parameters satisfying $u^2 + v^2 \leq 25$.

Problem 8) (10 points)

a) (5 points) You know $D_{\vec{v}} f(1,2) = \sqrt{2}$ for $\vec{v} = [1,1]/\sqrt{2}$. You also know $D_{\vec{w}} f(1,2) = 1$ for $\vec{w} = [3,4]/5$. Find the directional derivative $D_{\vec{u}} f(1,2)$ for $\vec{u} = [1,-1]/\sqrt{2}$.
b) (5 points) Write down the equation $ax + by = d$ of the **tangent line** to the level curve $f(x,y) = f(1,2)$ at the point $(1,2)$, where $f(x,y)$ refers to the same function than in a).

Problem 9) (10 points)

a) (5 points) Evaluate the following double integral

$$\iint_G \frac{1}{(x^2+y^2)^3} \, dxdy \ ,$$

where $G$ is region given by

$$\{4 \leq x^2 + y^2 \leq 9, y < 0\} \ .$$

b) (5 points) As usual, $\log = \ln$ is the natural log. Compute the following integral:

$$\int_1^e \int_0^{\sqrt{\log(x)}} \frac{12x}{e^2 - e^{(2y^2)}} \, dydx \ .$$