A SUMMER OF GEOMETRY

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Appendix 5: Homotopy

5.1. If G is a finite abstract simplicial complex, a **contraction step** is defined as a removal of a star $U(x)$ with contractible boundary $S(x) = \delta U(x)$ from G. Formally, this means $G \to G \backslash U(x)$ under the assumption that $S(x) = \overline{U(x)} \backslash U(x)$ is contractible. The inverse operation is called an **expansion step**. Two complexes are called **homotopic** if there exists a finite number of homotopy expansion or contraction steps, deforming one to an other.

5.2. If G can be reduced to 1 by a finite number of contraction steps, the complex G is called **contractible**. This is very different from being homotopic to 1. The **dunce hat** is an example of a complex that is homotopic to 1 but not contractible.

5.3. If G is contractible and H is arbitrary, then $G \oplus H$ is contractible. For example $G \oplus 1$, the cone extension is always contractible. Every ball $B(x) = \overline{U(x)}$ is contractible. Contractible simplicial complexes form a **semi-group** within the monoid of all simplicial complexes. It is not a sub-monoid because the zero element is not there.

5.4. No q-manifold is contractible unless it is $G = 1$, the zero dimensional one point complex. No sphere is contractible: the empty set which is the (-1)-sphere is not contractible by definition. Recursively, no q-sphere for $q > 0$ is contractible. The semi-group of contractible complexes is disjoint from the sphere monoid.

5.5. Contractibility can be computed in a time complexity that grows maximally quadratically in the number of elements of G: just go through all the $x \in G$ and check in each case whether $S(x)$ and $G \setminus U(x)$ are both contractible. Both simplicial complexes $S(x)$ and $G \setminus U(x)$ are smaller.

5.6. On the other hand, there is no fixed Turing machine that takes as an input a simplicial complex G and tells whether G is homotopic to 1 or not. The distinction between contractibility and **homotopic to 1** could not be bigger. This is a reason not to use words like **collapsibility** and **contractibility** to make the distinction since both terms are in colloquial language associated with a shrinking process. But homotopic to 1 can mean that we have to take a complex and blow it up again and again until it eventually can be shrunk to a point.

5.7.

- Every tree is contractible.
- Every wheel graph is contractible.
- Every unit ball $B(x)$ in a simplicial complex is contractible.

- Two circular complexes C_n and C_m are homotopic, but not within one dimensional complexes.
- The only manifold that is contractible is the 0-manifold 1.
- The dunce hat can be implemented as a graph with 17 vertices.

5.8. One can see an extension step as a disjoint union of $K = G \setminus U(y)$ with $U = U(y)$. We have then $\chi(U(y)) = 0$ and K, U form a complementary closed-open pair in G.

5.9.

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ToGraph[G_]:=Graph[n=Length[G];
  Select[Flatten[Table[G[[k]]<->G[[l]],{k,n},{l,k+1,n}],1],(SubsetQ[#[[2]],#[[1]]])&]];

G=Sort[{{1,2,3,4},{1,2,3},{1,2,4},{1,2},{1},{2}}];
G=Generate[{{a,b,c},{a,b,d}}]
ToGraph[G]


DunceHat=UndirectedGraph[Graph[
{1->2,1->4,1->5,1->6,1->7,1->8,1->9,1->10,1->11,1->12,
1->13,2->3,2->5,2->7,2->13,3->4,3->5,3->7,3->8,3->10,3->11,
3->13,3->14,3->15,3->16,4->8,4->10,4->11,5->6,5->16,5->17,
6->7,6->17,7->14,7->17,8->9,8->14,8->17,9->10,9->17,10->15,
10->17,11->12,11->16,11->17,12->13,12->17,13->15,13->17,14->17,
15->17,16->17}]];
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