A SUMMER OF GEOMETRY

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Appendix 6: Calculus

6.1. A vector field F on a simplicial complex G is a map that assigns to $x \in G$ an element $v \in V$ with $v \in x$. It defines so a map $x \to x \setminus v$ from G_k to G_{k-1}. A map $i_X f(x) = f(x \setminus v)\text{sign}(x, x \setminus v)$ satisfies $i_X^2 = 0$ is called the **inner derivative** of X. Any vector field F defines an inner derivative $i_X f(x) = f(x \setminus F(x))\text{sign}(x, x \setminus v)$.

6.2. A simplicial complex G defines an exterior derivative d on $l^2(G)$. It is defined as $df(y) = \sum_{x \subset y, \dim(x)=\dim(y)-1} f(x)\text{sign}(y, x)$ Together with an inner derivative $i_X : l^2(G_i) \to l^2(G_{i-1})$ we have a **Lie derivative** $L_X = i_X d + d i_X$ that acts on k-forms. The space of k-forms is left invariant under the Lie derivative.

6.3. For example, if $G = C_n$ and $df(a, b) = f(b) - f(a)$ and $F((a, b)) = b$ and $i_X f(a) = f(a, a+1)$. On scalar functions, the map $L_X f(a) = i_X df(a) = df(a, a+1) = f(a+1) - f(a)$ is the usual discrete derivative in Business calculus. On 1-forms and $e \in G_1$ $L_X f(e) = d i_X f(e) = d(f(b) - f(a)) = f(e+1) - f(e)$. The generator L_X of translation is the product of two nilpotent operators d and i_X. They satisfy $d^2 = 0, i_X^2 = 0$ but become potent in the form of $D_X = d + i_X$, which is a **directional Dirac operator** which defines a flow that is in general invertible in that there is an opposite field $-X$ which reverses the flow. The Dirac operator $D = d + d^*$ on the other hand produces a wave dynamics.

6.4. Unlike d, i_X which are nilpotent, L_X is not. It now gives us quantum evolutions like $\exp(i(d + d^*)t)$ or $\exp((d + i_X)t)$ which for $t = 2n$ involves powers $(d + i_X)^n$. This involves terms which are of the form $d i_X d i_X + ... + d i_X$ or $i_X d i_X d + ... + i_X d$. In one dimensions, this produces translation. The reason is the Gregory Taylor expansion $f(x + n) = \exp((d + i_X)2n) = \exp(L_X n)$.

6.5. The notation is powerful however in general, not only in one dimensions. The **Dirac type** operator $D_X = d + i_X$ in general generates a translation in a simplicial complex G. As in the continuum, a vector field generates a flow on G.

6.6. School calculus comes in two flavors. There is a Bosonic calculus which is orientation oblivious and there is a Fermionic calculus in which orientation matters. Terms like "volume" is a Bosonic notion. Terms like "work" is a Fermionic notion. Terms like Gauss-Curvature or Euler characteristic are Bosonic. Terms like geodesic curvature are Fermionic.

6.7. When computing lengths, areas and volumes, one works with a Bosonic calculus. Surface area $\int_U |r_u \times r_v| \; dudv$ for example does not depend on the orientation of the parametrization $r : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Gauss Curvature is Bosonic. It does not matter how we orient the surface. Torsion, the highest curvature element for a 1-manifold embedded in a host manifold is Fermionic. It comes with a sign. Multivariable calculus notions like flux $\int_U F(r(u,v)) \cdot r_u \times r_v \; dudv$ depend on orientation. It integrates a 2-form F over a 2-manifold $S = r(U)$. The result depends on the orientation of S.

6.8. Introduction courses in single variable mix these things. It is not communicated to early learners of course, but teachers designing calculus courses better are aware of this. Terms like Δx or dx or ΔA and dA can mean different things. Terms like $dxdy$ or $dx \wedge dy$ can mean different things as $dxdy = dydx$ but $dx \wedge dy = -dy \wedge dx$. When working with quantized calculus it becomes important however. Integrating a 2-form over a 2-manifold is a finite sum that is orientation sensitive in the sense that the actual computation depends on the given basis that has been chosen. The actual result does not depend on the orientation. It is like with determinants. The determinant of a matrix does not depend on the basis chosen, but the actual computation of the determinant very much does. In a basis, where the matrix is in Jordan form for example makes the computation simpler.

6.9. To dwell a bit more on this: in the context of the fundamental theorem of calculus $\int_a^b f'(t) \; dt = f(b) - f(a)$ has a different sign than $\int_b^a f'(t) \; dt$. When computing the area under the curve of a positive function f on $[a, b]$ one interprets $\int_a^b f(t) \; dt$ as positive. The Riemann sum is not signed. Bosonic calculus is more like integral geometry which in the discrete means looking at valuations like $f_k(G)$, the k dimensional volume. Fermionic calculus deals with an orientation dependent exterior derivative d. Most properties, like Betti vectors do not depend on the orientation but orientation stats to matter when we look at questions like orientability.

6.10. Connection calculus meaning calculus related to the connection matrix L is Bosonic. Incidence calculus, meaning calculus related to the Dirac operator D is Fermionic. Both lead to Laplacians L^2 and $H = D^2$ but the former is positive definite. When looking at the wave equation $u_{tt} = -L^2 u$ or $u_{tt} = -D^2 u$ we have completely different behavior stability wise. When adding a nonlinear term for example then the nonlinear Bosonic wave equation like $u_{tt} = -L^2 u + \epsilon \sin(u)$ has a stable vaccuum: for small enough ϵ there is no other solution of this wave equation besides $u = 0$. This is remarkable. In the Hodge case $u_{tt} = -D^2 u + \epsilon \sin(u)$. The stability is already non-trivial in the 0-dimensional case, where if time is discretized too we have the standard map and KAM phenomena matter. The connection calculus has stability because of the standard implicit function theorem and linear algebra. The incidence calculus needs heavy strong implicit function theorem machinery for establishing stability and in higher dimensions the stability is only of Nekhoroshev type, in the sense that the time scales for instability are so large due to slow Arnold diffusion that they do not matter. While the kernel of the Hodge Laplacian H is a source for technical difficulties when dealing with Green functions, the kernels are important topologically. Harmonic forms play an important role in physics.

6.11. All such conundrums in classical calculus (which mostly manifest pedagogically when presenting the material to students who have not seen calculus before) become much more transparent in the discrete, where we just have a finite set of sets G. There are orientation oblivious notions like Euler characteristic or curvatures. And then there are orientation sensitive notions like integrating a k-form over a k-manifold. Given a Moebius strip G for example, which is non-orientable and a 2-form f, there is no ambiguity at all with what we mean with $\int_G f$. The 2-simplices is just a finite set G_2 of triangles given with an orientation and the function f is a function on this set. The integral is $\sum_{x \in G_2} f(x)$.

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