ELEMENTS OF FINITE GEOMETRY

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Unit 6: Unimodularity

A simplicial complex G defines a connection matrix L that is unimodular. The inverse matrix g of L is explicitly given in the form of the Green-Star formula $g(x,y) = \omega(x)\omega(y)\chi(U(x) \cap U(y))$. The total potential $\sum_{x,y} g(x,y)$ is $\chi(G)$.

6.1. If G is a simplicial complex, the **stars** $U(x) = \{y \in G, x \subset y\}$ are open sets, while the **cores** $C(x)\{y \in G, y \subset x\}$ are closed. For $A \subset G$, the **Fermi characteristic** $\phi(G) = \prod_{x \in G} \omega(x)$ of is a multiplicative sibling of the additive Euler characteristic $\chi(A) = \sum_{x \in A} \omega(x)$, where $\omega(x) = (-1)^{\dim(x)}$. Define the **connection matrix** $L(x,y) = \chi(C(x) \cap C(y))$. The entries are 1 if $x \cap y \neq \emptyset$ and 0 else. Define also $\psi(G) = \det(L(G))$. The **unimodularity theorem** is:

Theorem: $\det(L) = \phi(G) \in \{-1, 1\}$.

Proof. The simplicial complex G can be built up recursively from a smaller complex H by adjoining a new k-simplex x to an already existing $(k-1)$-sphere A in H that is not yet the boundary of a ball in H. This enlargement $H \to G = H \cup_A \{x\}$ changes the Fermi characteristic $\phi(G)$ to $\phi(G) = \phi(G)\omega(x)$. A path interpretation of the determinant verifies that this also changes the determinant $\psi(G) = \det(L)$ by a factor $\omega(x)$. It is the content of the multiplicative Poincaré-Hopf analog below. $\square$

6.2. Poincaré-Hopf is the additive statement $\chi(G \cup_A \{x\}) = \chi(G) + (1 - \chi(A))$. It used the inclusion-exclusion formula and $\chi(B(\{x\})) = 1$. Multiplicatively, we have:

Theorem: $\psi(G \cup_A \{x\}) = \psi(G)(1 - \chi(A))$.

Proof. $Y : A \to (\psi(G \cup_A x) - \psi(G))$ is a valuation that satisfies $\psi(G \cup_A x) = 0$ if A is a complete complex so that $Y(A) = -\psi(G)$ if A is a complete complex. The scaled valuation $-Y(A)/\psi(G)$ takes the value 1 for complete sub-complexes. This characterizes Euler characteristic: we have $-Y(A)/\psi(G) = \chi(A)$ meaning $Y(A) = -\chi(A)\psi(G)$ But $(\psi(G \cup_A x) - \psi(G)) = -\chi(A)\psi(G)$ is equivalent to $\psi(G \cup_A \{x\}) = \psi(G)(1 - \chi(A))$. $\square$

6.3. Let $g(x,y) = \omega(x)\omega(y)\chi(U(x) \cap U(y))$ be the **Green function**. It is like L a $n \times n$ matrix. It invokes the **stars** $U(x), U(y)$ of x and y. The next result is the **Green-star formula**:

Theorem: $L^{-1} = g$.

Proof. Write the matrix entries of L and g are linear expressions in $\omega(x)$:

$$L(u,v) = \sum_{x \in G, x \subset u \cap v} \omega(x) ,$$

$$g(v,w) = \sum_{y \in G, v \cup w \subset y} \omega(v)\omega(w)\omega(y) .$$

Now multiply the two matrices $\sum_v L(u,v)g(v,w)$. This is a sum over x, y, v. Distinguish two cases. If $u = w$, we divide the sum up into two parts, one part where $x = y$ which implies $u = w, x, y = v$ and giving 1 and a part where $x \neq y$, where the sum over v is zero. The diagonal entries are 1.
In the second case $u \neq w$ the situation is similar, but $x = y$ is no more possible. We end up with 0. ☐

6.4. If $g(x,y)$ is interpreted as the **potential energy** between the simplices x and y, the number $E(G) = \sum_{x,y \in G} g(x,y)$ is the **total energy**. The **energy theorem** is:

Theorem: $\sum_{x,y \in G} g(x,y) = \chi(G)$

Proof. We have $g(x,x) = \chi(U(x)) = 1 - \chi(S(x))$. Call it $i(x)$. By the Cramer formula, $g(x,x)$ is $\psi(G \setminus x)/\psi(G)$ by the multiplicative Poincaré-Hopf theorem this is $1 - \chi(S(x))$. Now think of G is the vertex set of a graph in which two vertices are connected if one is contained in the other. Take the functional $\dim(x)$. Hyperbolicity gives $S(x) = S_f^-(x) \oplus S_f^+(x)$ so that $i_f(x) i_{-f}(x) = i(x)$. Since $i_f(x) = \omega(x)$ and $i_{-f}(x) = 1 - \chi(U(x))$ $k(x) = \omega(x)(1 - \chi(S(x)))$ is also a curvature adding up to $\chi(G)$. This implies $\sum_x \omega(x)\chi(S(x)) = 0$ which can be read as a McKean-Singer $\mathrm{str}(L^k) = \chi(G)$ for $k = -1, 0, 1$. Introduce the potential $V(x) = \sum_y g(x,y)$. It satisfies $V(x) = \omega(x) g(x,x) = k(x)$. Gauss-Bonnet $\sum_x k(x) = \chi(G)$ now proves the theorem. ☐

6.5. Examples:
1) $\det(L(K_1)) = 1$ but $\det(L(K_k)) = -1$ for $k > 1$.
2) The diamond complex $G = \{ (1),(2), (3),(4), (1,2),(1,3), (2,3),(2,4), (3,4), (1,2,3),(2,3,4)\}$ has f-vector $f = -(4,5,2)$, Euler characteristic $\chi(G) = 4 - 5 + 2 = 1$ and Betti vector $b = (1,0,0)$. Its Fermi characteristic is $(+1)^4(-1)^5(+1)^2 = -1$.

$$g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & -1 & -1 & -1 & 0 & 1 \end{bmatrix} .$$