ELEMENTS OF FINITE GEOMETRY

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Unit 7: Brouwer Lefschetz

The sum of the indices of a simplicial map T on a simplicial complex G is equal to the Lefschetz number $\chi_T(G)$, the super trace of the Koopman operator of T restricted to the harmonic forms of G.

7.1. A map $T : G \to G$ on a finite abstract simplicial complex G is a **simplicial map** if it maps the set V of 0-dimensional simplices into itself and if it is order-preserving, meaning that $x \subset y$ implies $T(x) \subset T(y)$. The map T does not have to be invertible. A constant map T that maps G to $x \in G$ for example is a simplicial map only if x is 0-dimensional.

7.2. If T is a simplicial map, its **attractor** $\bigcap_{k \geq 0} T^k(G)$ is a simplicial complex on which T is invertible. As we are interested in fixed points of T and fixed points must be in the attractor, we can assume T is invertible.

7.3. An invertible **simplicial** map T induces a **Koopman operator** U on the Hilbert space Λ of differential forms. It is $Uf(x) = f(Tx)\text{sign}(T|x)$. If T is invertible, it is a unitary operator. The linear map U commutes with the exterior derivative d and so commutes with the Hodge Laplacian. It therefore preserves the linear space of k-forms Λ_k and also preserves the harmonic k-forms.

7.4. Let U_k be the restriction of U to k-forms. Its super trace $\text{str}(U|\ker(H)) = \sum_{k=0}^{q}(-1)^k \text{tr}(U_k|\ker(H_k))$ of U is the **Lefschetz number** of T. We call it $\chi_T(G)$ because if T is the identity, then $\chi_T(G)$ is the Euler characteristic $\chi(G)$.

7.5. Let $F = \{x \in G, T(x) = x\}$ denote the **fixed point set**. T induces a permutation of the vertices of each fixed point x for which $\text{sign}(T|x)$ is the sign of. The number $i_T(x) = \omega(x)\text{sign}(T|x)$ is called the **index** of x.

7.6. The Lefschetz fixed point theorem is

Theorem: $\chi_T(G) = \sum_{x \in F} i_T(x)$.

Proof. The operator U commutes with the heat flow e^{-tL}. Define the operator $U_t = Ue^{-tL}$. The McKean-Singer symmetry implies that $\text{str}(U_t)$ is independent of t. For $t = 0$, it is the super trace of U which is $\sum_{x \in F} i_T(x)$ by the theorem below. Since $\lim_{t\to\infty} e^{-tL} = P$ is the projection on the harmonic forms, we have $\text{str}(Ue^{-tL}) \to \chi_T(G)$, proving the theorem. $\square$

Theorem: $\text{str}(U) = \sum_{x \in F} i_T(x)$

Proof. $\sum_{x \in G} \omega(x) U(x,x) = \sum_{x \in F} \omega(x) U(x,x) = \sum_{x \in F} \omega(x) \text{sign}(T|x) = \sum_{x \in F} i_T(x)$. $\square$

7.7. Lets see what happens if G is the complex of a complete graph K_n and T is a permutation of $V = \{1, \dots, n\}$. In this case $\text{str}(U) = 1$ which is $\chi_T(G)$. The map U encodes the fixed points in the diagonal If the permutation belonging to T is $\pi = (c_1)(c_2)\dots(c_m)$, where c_i are the cycles, then every non-empty collection of cycles produces a fixed simplex of T.

7.8. The Lefschetz theorem for contractible spaces is **Brouwer's fixed point theorem**:

Theorem: A map on a contractible G has at least 1 fixed point.

Proof. If G is contractible, the cohomology only consists of constant 0-forms. The Lefschetz number is 1. $\square$

7.9. Let $\text{Aut}(G)$ denote the **automorphism group** of G, the set of symmetries. Given a subgroup A of $\text{Aut}(G)$, the **average Lefschetz number** is $\chi(G/A) = \frac{1}{|A|} \sum_{T \in A} \chi_T(G)$. If G is trivial it is $\chi(G)$.

Theorem: If $H = G/A$ is a complex, $\frac{1}{|A|} \sum_{T \in A} \chi_T(G) = \chi(G/A)$.

Proof. Burnside's theorem for A acting on G gives $f_k(G/A) = \frac{1}{|A|} \sum_{T \in A} |F_k(T)|$ with $F_k(T) = \{x \in G_k, T(x) = x\}$ so that $\chi(G/A) = \sum_{k=0}^{q} (-1)^k f_k(G/A)$ is

$$\frac{1}{|A|} \sum_{T \in A} (-1)^k |F_k(T)| = \frac{1}{|A|} \sum_{T \in A} \sum_{x \in F} i_T(x) = \frac{1}{|A|} \sum_{T \in A} \chi_T(G) \; .$$

$\square$

7.10. The curvatures of G/A are $\kappa(x) = \frac{1}{|A|} \sum_{T \in A} i_T(x)$. They add up to $\chi(G/A)$. We can see G as **branched cover** over $H = G/A$. If A is the cyclic group generated by T, then G is a $k:1$ cover over $H = G/A$. The fixed points of T are branch points. If $T, T^2, \dots, T^{k-1}$ have no fixed points then $\chi_{T^l}(G) = 0$ and $\chi_{Id}(G) = \chi(G)$ so that $\chi(H) = \chi(G)/|A|$, a special **Riemann-Hurwitz** result.

7.11. Examples: 1) For $G = [0,1,2,3]$ and $T(x) = T(3-x)$, there is single fixed point $(1,2)$ of index 1. 2) If G a figure 8 complex that carries an involution with 3 fixed points of index 1. The trace on harmonic 0 forms is 1. The trace on harmonic 1 forms is -2. 3) Invertible orientation preserving maps on a $2k$-sphere, or invertible orientation reversing maps on a $(2k+1)$-sphere have at least 2 fixed points.