ELEMENTS OF FINITE GEOMETRY

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Unit 8: Sphere formula

If G is any finite abstract simplicial complex, then $\sum_{x \in G} \omega(x)\chi(S(x)) = 0$. So, if all unit spheres have the same non-zero Euler characteristic, like for odd dimensional manifolds, the Euler characteristic of the complex is zero.

8.1. Let G be a finite abstract simplicial complex. The **unit sphere** $S(x)$ of $x \in G$ is the boundary of the **star** $U(x) = \{y \in G, x \subset y\}$, the smallest open set that contains x. As this is $\overline{U}(x) \setminus U(x)$, the unit sphere is a simplicial complex. The topological definition links it to a natural topology $\mathcal{O}$ on G, even so it is never Hausdorff, if the dimension of G is positive.

8.2. For a vertex v in a graph, the unit sphere $S(v)$ is the graph generated by the neighbors of v. For a simplex x in a complex G, the unit sphere is $S(x) = \delta U(x)$. There is a connection. If $\Gamma(G) = (V, E)$ denotes the graph with vertices $V = G$ and $E = \{(x, y), x \subset y \text{ or } y \subset x\}$, we can naturally associate $S(x)$ with $\Gamma(S^+(x)) \oplus \Gamma(S^-(x))$. We write $S(x) = S^+(x) \oplus S^-(x)$ even so we join an open with a closed set. This is a form of hyperbolicity because for a hyperbolic fixed point of a smooth map T in $\mathbb{R}^n$, we have stable and unstable manifolds $W^-(x)$ and $W^+(x)$ which when intersected with a small sphere $S(x) = S_r(x)$ produce two spheres $S^-(x)$ and $S^+(x)$ which have the property that their join is $S(x)$. In G, the hyperbolic structure comes from the **dimension functional** dim on G. Here is the **hyperbolicity lemma**:

Theorem: $S(x) = S^+(x) \oplus S^-(x)$ for all $x \in G$.

Proof. We show the graph join $\Gamma(S(x)) = \Gamma(S^+(x)) \oplus \Gamma(S^-(x))$. A simplex $y \in G$ is a vertex in $\Gamma(S(x))$ if either y is a subset of some $z \in S(x))$ meaning y is in $S^-(x)$ or if y contains some $z \in S(x)$ meaning y is in $S^+(x)$. So, the vertex sets of the two graphs agree. If two points y, z in $\Gamma(S(x))$ are connected, we either have $y \subset z$ or $z \subset y$. There are four cases: either both are in $S^-(x)$ which means they are connected in $S^-(x)$ and so connected in the join. The same hold if both are in $S^+(x)$. Any pair $y \in S^-(x)$ and $z \in S^+(x)$ satisfies $y \subset x \subset z$ so that they are connected in $S(x)$. $\square$

Theorem: $\sum_{x \in G} \omega(x)\chi(S(x)) = 0$.

Proof. (i) For any x we have the local valuation formula $\chi(B(x)) = \chi(U(x))+\chi(S(x))$. Proof: For any subsets $A, B \subset G$, whether open or closed or neither, we have the valuation formula $\chi(A) + \chi(B) = \chi(A \cup B) - \chi(A \cap B)$ because each of the basic valuations $f_k(G)$ counting the number of k-dimensional simplices satisfies the formula and χ is a linear combination of such basic valuations. (ii) $\chi(B(x)) = 1$ for all x. Proof: Use induction with respect to the number of elements in $B(x)$. If $B(x)$ has one element, it has $\chi(B(x)) = 1$. We can reduce the size of $B(x)$ by taking a way an element $y \in B(x)$ different from x. The complex $B'(x) = B(x) \setminus U(y)$ is now a unit ball $B'(x)$ in a smaller $G \setminus U(y)$. The induction assumption assures that $\chi(B'(x)) = 1$. The local valuation formula gives that $\chi(B(x))) = \chi(B'(x)$.
(iii) the energy formula $\sum_{x \in G} \omega(x)\chi(U(x)) = \chi(G)$ was proven in the unimodularity unit. It told that the super trace of g was Euler characteristic. Here is an other proof: Replace the ± 1-valued function $\omega(x)$ by a more general function $h(x)$ and define $\chi_h(A) = \sum_{x \in A} h(x)$ and $\psi_h(x) = \omega(x)\chi_h(U(x))$. Both maps $h \to \chi_h(A)$ and $h \to \psi_h(A)$ are linear in h for fixed A so that we only need to consider the case where $h(x_0) = 1$ and $h(x) = 0$ for $x \neq x_0$. The right hand side is then $\chi_h(G) = 1$. The left hand side is $\sum_{x \in G} \omega(x)\chi_h(U(x)) = \sum_{x \in G, x \subset x_0} \omega(x)$, which is the Euler characteristic of the simplicial complex $\overline{\{x_0\}} = \{x \subset x_0\}$ which is always 1. (iv) $\omega(G) = \sum_{x \in G} \omega(x)\chi(B(x)) = \sum_{x \in G} \omega(x) = \chi(G)$. □

8.3. The value $g(x, x) = \omega(x)^2\chi(U(x) \cap U(x) = \chi(U(x))$ is a diagonal entry in the **Green function**. It is the **self-energy** of $x \in G$. Now $\chi(S(x)) = 1 - \chi(U(x)$ gives $\sum_{x \in G} \omega(x)\chi(S(x)) = \sum_{x \in G} \omega(x)(1 - \chi(U(x)) = \chi(G) - \chi(G) = 0$, which is the sphere formula.

8.4. What is $\sum_{x \in G} \chi(S(x))$? This is the trace of $L - g$. We called this the **Hydrogen functional** because of formal similarity with the quantum mechanical Hamiltonian of the Hydrogen atom, where the potential part involves the kernel of the inverse of the kinetic part - the $1/r$ potential being determined by the Laplacian Δ in $\mathbb{R}^3$ for example. We called $L - L^{-1} = L - g$ the **Hydrogen operator**.

Theorem: $\sum_{x \in G} \chi(S(x)) = \mathrm{tr}(L - L^{-1})$.

8.5. Here is an application of the sphere formula. The Euler-Gem and sphere formula together immediately give the zero Euler characteristic result:

Theorem: All odd-dimensional manifolds satisfy $\chi(G) = 0$.

Proof. Euler-Gem formula assures that odd-dimensional manifolds have unit spheres with $\chi(S(x)) = 2$. The sphere formula implies $0 = \sum_x \omega(x)\chi(S(x)) = \sum_x \omega(x)2 = 2\chi(G)$. □

8.6. Example: for $G = K_3 = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, we have