ELEMENTS OF FINITE GEOMETRY

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Unit 9: Level sets

If G is a discrete q-manifold and $g : V \to K_k = \{0, \dots, k\}$ is a function, then $M_g = \{x \in G | g(x) = K_k$ is a $(q-k)$-manifold if not empty.

9.1. A q**-manifold** is a finite abstract simplicial complex G such that every unit sphere $S(x)$ is a $(q-1)$-sphere. A q**-sphere** is a q-manifold G such that $G \setminus U(x)$ and $S(x)$ are both contractible. A complex G is **contractible** if there exists x such that both $G \setminus U(x)$ and $S(x)$ are both contractible. The initial object **void** 0 is the (-1) sphere and the terminal object 1 is contractible.

9.2. The vertex set $V = \bigcup_{x \in G} x$ in G can be identified with the 0-dimensional objects G_0 in G. A function $g : V \to K_k = \{0, \dots, k\}$ defines a **level set** of G denoted by $M_g = \{x \in G, g(x) = K_k\}$, where $g((x_0, x_1, \dots, x_)) = \{g(x_0), \dots g(x_k)\}$ In the case $k = 1$ for example, this is the set of x on which $g(x) = \{0, 1\}$ takes both values, meaning that $\{v \in x, g(v) = 0\}$ and $\{v \in x, g(v) = 1\}$ are both non-empty. The set M_g is a priori only open. This means $x \in M_g$, and $y \in G$ contains x then $y \in M_g$. But M_g defines a simplicial complex: take the set M_g as the vertices of a graph Γ and connect two vertices (a, b) if $a \subset b$ or $b \subset a$. The Whitney complex of this graph is then a simplicial complex and if we say M_g is a manifold, we mean that its simplicial complex is.

9.3.

Theorem: Any $g : V(G) \to K_k$ on a q-manifold G has a level set M_g that is a $q-k$ manifold if it is not empty.

9.4. The proof uses the hyperbolic structure of any simplicial complex. In the case of manifolds, we can even deal with a Morse complex because every $x \in G$ is a critical point of $x \to \dim(x)$. The Morse index of x is $m(x) = \dim(S^-(x)) = k$, where $S^-(x) = \{y \in G, y \subset x, y \neq x\}$. Complementary is the $S^+(x) = \{y \in G, x \subset y, y \neq x\}$. G is a manifold if and only if every $S^+(x)$ is a sphere. Again, $S+(x)$ is only an open set at first and we refer to the Whitney complex of its graph. Hyperbolicity is:

Theorem: If G is a q-manifold and $\dim(x) = k$, then $S(x)$ is the join of the $(k-1)$-sphere $S^-(x)$ and $(q-k-1)$ sphere $S^+(x)$.

Proof. The graph of $S(x)$ contains both simplices $S^-(x)$ strictly contained in x and simplices $S^+(x)$ that strictly contain x. Every $y \in S^-(x)$ is contained in any $z \in S^+(x)$. $\square$

9.5. In the special case, when G is the boundary complex $S^-(x)$ of a k-simplex x, level sets are always spheres. For the 2-simplex $x = K_3$ for example, the boundary sphere is $S^-(x) = \{\{1\},\{2\},\{3\},\{1,2\},\{2,3\},\{1,3\}\}$. For any function $g : \{1,2,3\} \to \{0,1\}$ the level set is always a 0-sphere.

9.6. A special case is if G be a complete n-dimensional complex. Every surjective $g : V(G) \to K_k$ produces a level set M_g that is a $(n-k)$-sphere. The sets $V_j = \{g = j\} \subset V$ partition V into $k+1$ sets of size $n_j > 0$ such that the sum is $n+1$. The level set M_g is the join of all these $(n_j - 1)$-spheres which is a $(n-k)$ - sphere because $(\sum_j n_j) - 1 = n + 1 - k - 1 = n - k$.

9.7. Now the proof of the main theorem:

Proof. We use induction with respect to dimension q. The unit sphere $S_G(x)$ in G is the join of $S^+(x)$ and $S^-(x)$. The unit sphere $S_M(x)$ in G is the join of $S^+_M(x)$ and $S^-_M(x)$. We have $S^+_M(x) = S^+(x)$ because if y is in $S(x)$ and g takes all values on x, then g takes all values on y so that y is also in $S^+_M(x)$. As seen above, the level surface in the $k-1$ sphere $S^-(x)$ is $k-2$-sphere. So $S_M(x)$ is the join of a $(k-2)$ sphere with a $q-k-1$ sphere which is a $q-k-1$ sphere $\square$

9.8. The set-up can be generalized if K_k is replaced with a $k+1$ partition complex $P = K_{n_0,n_1,\dots n_k}$. If a function $g : V(G) \to V(P)$ is given, define $M_g = \{x \in G, g(x)$ contains at least one facet of $P\}$. In that case again M_g is a $q-k$ manifold if it is not empty. Every facet f of P now produces a patch $M_{g,f} = \{x \in g, f \subset g(x)\}$ which is a $q-k$ manifold with boundary. The union of all these patches is the manifold M_g without boundary.

Theorem: Any $g : V(G) \to P$ from a q-manifold G to a partition complex P of dimension k has a level set M_g that is a $q-k$ manifold or $\emptyset$.

9.9. In the continuum, level sets in q-manifolds are not manifolds in general, like for $g(x,y) = \sin(x)\sin(y) = 0$ on the 2-manifold $M = \mathbb{T}^2$. The Morse Sard theorem assures that for almost all c in the range of $g : M \to \mathbb{R}^k$, the level set $\{g = c\}$ is a $q-k$ manifold. In the discrete, no regularity at all is needed.

9.10. Examples:

- If g is 1 on some vertex v and 0 else, then $M_g = S(v)$ is the unit sphere.
- If G is a hexagonal lattice and $g : V \to \{0,1\}$ is a function, then M_g consists of all triangles and edges on which g takes two values. If an edge e is in M_g then the two attached triangles are there. If t is a triangle in M_g, then there are exactly two edges contained in M_g where g takes two values. Since every point in M_g has exactly two neighbors, the level set must be a union of circular graphs.