Diss. ETH Nr. 10189

Spectral, ergodic and cohomological problems in dynamical systems

A dissertation submitted to the
SWISS FEDERAL INSTITUTE OF TECHNOLOGY ZÜRICH

for the degree of
Doctor of Mathematics

presented by
OLIVER KNILL
Dipl. Math. ETH
born Oktober 22, 1962
citizen of Appenzell/AI

accepted on the recommendation of
Prof. Dr. Oscar Lanford III, examiner
Prof. Dr. Alain-Sol Sznitman, co-examiner

1993

Spectral, ergodic and cohomological problems in dynamical systems

Spectral, ergodic and cohomological problems in dynamical systems

Abstract

- We showed for $\mathsf{K} = \mathsf{R}, \mathsf{C}$ that every $SL(2,\mathsf{K})$ cocycle over an aperiodic dynamical system can be perturbed in $L^\infty(X, SL(2,\mathsf{K}))$ on a set of arbitrary small measure, so that the perturbed cocycle has positive Lyapunov exponents. We applied these results to show that coboundaries in $L^\infty(X, \mathsf{T}^1)$ or $L^\infty(X, SU(2))$ are dense.
- We proved, that Lyapunov exponents of $SL(2,\mathsf{R})$ cocycles over an aperiodic dynamical system depend in general in a discontinuous way on the cocycle. We related the problem of positive Lyapunov exponents to a cohomology problem for measurable sets.
- We showed the integrability of infinite dimensional Hamiltonian systems obtained by making isospectral deformations of random Jacobi operators over an abstract dynamical system. Each time-1 map of these so-called random Toda flows can be expressed by a QR decomposition.
- We proved that a random Jacobi operator L over an abstract dynamical system can be factorized as $L = D^2 + E$, where E is real and below the spectrum of L and where D is again a random Jacobi operator but defined over a new dynamical system which is an integral extension. An isospectral random Toda deformation of L corresponds to an isospectral random Volterra deformation of D. The factorization leaded to super-symmetry and commuting Bäcklund transformations.
- We showed that transfer cocycles of random Jacobi operators move according to zero curvature equations, when the Jacobi operators are deformed in an isospectral way. We showed that every $SL(2,\mathsf{R})$ cocycle over an aperiodic system is cohomologuous to a transfer cocycle of a random Jacobi operator. We attached to any $SU(N)$ random discrete gauge field a random Laplacian. From the density of states of this operator, it can be decided whether the gauge field has zero curvature or not. A cohomology result of Feldmann-Moore leaded to the existence of random Harper models with arbitrary space-dependent magnetic flux.
- We gave a new short integration of the periodic Toda flows using the representation of the Toda flow as a Volterra flow. We made the remark that the Toda lattice with two particles is equivalent to the mathematical pendulum. This gives a Lax representation for the mathematical pendulum. We rewrote the first Toda flow as a conservation law for the Green function of the deformed operator. We described a functional calculus for abelian integrals obtained by looking at an abelian integral on the hyperelliptic Toda curve as a Hamiltonian of a time-dependent Toda flow.
- By renormalisation of dynamical systems and Jacobi operators, we constructed almost periodic Jacobi operators in $\mathcal{B}(l^2(\mathsf{Z}))$ having the spectrum on Julia sets J_E of the quadratic map $z \mapsto z^2 + E$ for real $E < -R$ with R large enough. The density of states of these operators is equal to the unique equilibrium measure μ_E on J_E. The set of so constructed random operators forms a Cantor set in the space of random Jacobi operators over the von Neumann-Kakutani system $T : X \to X$, a group translation on the compact topological group of dyadic integers which is a fixed point of a renormalisation map in the space of dynamical systems. The

1

Cantor set of operators is an attractor of the iterated function system built up by two renormalisation maps $\Phi_E^\pm$.

• We proved that a sufficient conditions for the existence of a Toda orbit through a higher dimensional Laplacian L is that L is not a stationary point of the first Toda flow and that it is possible to factor $L = D^2 + E$, where D is a random Laplacian over an integral extension. Random Laplacians appeared in a variational problem which has as critical points discrete random partial difference equations.

• We considered differential equations in $L^\infty(X)$ which form a thermodynamic limit of cyclic systems of ordinary differential equations. We considered also infinite dimensional dynamical systems describing the motion of infinite particles with pairwise interaction. The motion of random point vortex distributions can have a description as a motion of Jacobi operators.

• We constructed an analytic map $U : \mathbb{C}^4 \to \mathbb{C}^4$, having a one-parameter family of two-dimensional real tori S_γ invariant, on which U is the Standard map family T_γ. We provided a rough qualitative picture of the dynamics of U and gave some arguments supporting the conjecture that the metric entropy of the Standard map T_γ is bounded below by $\log(\gamma/2)$.

• We introduced a generalized Percival variational problem of embedding an abstract dynamical systems in a monotone twist maps like for example the Standard map S_γ. Using the anti-integrable limit of Aubry and Abramovici, we showed that there exists a constant $\gamma_0 > 0$ such that every ergodic abstract dynamical system (X, T, m) with metric entropy $h_m(T) \le \log(2)$ and $|\gamma| \ge \gamma_0$ can be embedded in the twist map S_γ. For such γ, the topological entropy of S_γ is at least $\log(2)$. Using a generalized Morse index, the integrated density of states of the Hessian at a critical point, we proved the existence of uncountably many different embeddings of some aperiodic dynamical systems.

• We studied several cohomologies for dynamical systems: For a group dynamical system $(\mathcal{R}, \mathcal{G})$ (the abelian group $\mathcal{R}$ is acting on the abelian group $\mathcal{G}$ by automorphisms) there is the Eilenberg-McLane cohomology. For a group dynamical system $(\mathbb{Z}, \mathcal{G})$ we define a sequence of Halmos homology and cohomology groups. For an algebra dynamical system $(\mathbb{Z}^d, \mathcal{M}, \mathrm{tr})$ or for an group dynamical system $(\mathbb{Z}^d, \mathcal{G})$, there is a discrete version of de Rham's cohomology.

• We studied the hyperbolic properties of bounded $SL(2, \mathbb{R})$ cocycles over a dynamical system. We investigated the relation between the rotation number of Ruelle for measurable matrix cocycles and the hyperbolic behavior of the cocycle. We showed that a cocycle is uniformly hyperbolic if and only if the rotation number is locally constant along a special deformation of the given cocycle. We proved that the spectrum of a cocycle acting on $L^2(X, \mathbb{C}^2)$ is the same as the Sacker-Sell spectrum.

Kurzfassung

- Wir zeigten, dass für $\mathsf{K} = \mathsf{R}, \mathsf{C}$ jeder $SL(2,\mathsf{K})$ Kozyclus über einem aperiodischen dynamischen System im Raum $L^\infty(X, SL(2,\mathsf{K}))$ auf einer Menge von beliebig kleinem Mass gestört werden kann, so dass der gestörte Kocyclus einen positive Lyapunovexponenten hat. We wendeten dieses Resultat an, um zu zeigen, dass Koränder dicht in $L^\infty(X, \mathsf{T}^1)$ oder $L^\infty(X, SU(2))$ liegen.

- Wir bewiesen, dass Lyapunovexponenten von $SL(2,\mathsf{R})$ Kozyclen über einem aperiodischen dynamischen System im Allgemeinen unstetig vom Kozyclus abhängen. Wir finden eine Beziehung zwischen dem Problem, positive Lyapunovexponenten zu zeigen und dem Kohomolgieproblem für messbare Mengen.

- Wir zeigten die Integrabilität von unendlich dimensionalen Hamilton'schen Systemen, die durch isospektrale Deformation von zufälligen Jacobioperatoren über einem dynamischen System erhalten werden. Die Zeit-1-Abbildung von diesen sogenannten zufälligen Todaflüssen kann durch eine QR-Zerlegung ausgedrückt werden.

- Wir bewiesen, dass ein zufälliger Jacobioperator L faktorisiert werden kann als $L = D^2 + E$, wo E reell und unterhalb des Spektrums von L liegt und D wieder ein zufälliger Jacobioperator über einer Integralerweiterung des alten Systems ist. Einer isospektralen zufälligen Todadeformation von L entspricht eine isospektrale zufällige Volterradeformation von D. Die Faktorisierung führte zu Supersymmetrie und kommutierenden Bäcklundtransformationen.

- Wir zeigten, dass Transferkozyklen unter isospektraler Deformation von zufälligen Jacobioperatoren sich gemäss Nullkrümmungsgleichungen bewegen. Wir zeigten, dass jeder $SL(2,\mathsf{R})$-Kozyklus über einem aperiodischen dynamischen system kohomolog zu einem Transferkozyklus eines zufälligen Jacobiopertators ist. Wir adjungierten zu jedem diskreten $SU(N)$ Eichfeld einen zufälligen Laplaceoperator dessen Zustandsdichte entscheidet ob das Eichfeld Krümmung Null hat oder nicht. Ein Kohomologieresultat von Feldmann und Moore führte zur Existenz von zufälligen Harpermodellen mit beliebigem ortsabhängigem magnetischem Fluss.

- Wir gaben eine neue kurze Integration des periodischen Toda Systems. Wir machten eine Bemerkung, die zu einer Lax-darstellung des mathematischen Pendels führte. Wir transformierten den ersten Todafluss als Erhaltungssatz für die Greenfunktion des deformierten Operators. Wir beschrieben ein Funktionalkalkül für abelsche Integrale indem ein abel'sches Integrale auf einer hyperelliptischen Todakurve als Hamiltonfunktion für ein zeitabhängiges Todasystem betrachtet wurde.

- Mittels Renormalisierung von dynamischen Systemen und Jacobioperatoren konstruierten wir fastperiodische Jacobioperatoren in $\mathcal{B}(l^2(\mathsf{Z}))$, die das Spektrum auf Juliamengen J_E der quadratischen Abbildung $z \mapsto z^2 + E$ haben. Die Zustandsdichte von diesen Operatoren ist gleich dem eindeutigen Gleichgewichtsmass μ_E auf J_E. Die Menge der so konstruierten zufälligen Operatoren bilden eine Cantormenge im Raum der zufälligen Jacobioperatoren über dem von Neumann-Kakutani-system $T : X \to X$, einer Gruppentranslation auf der kompakten topologischen Gruppe der dyadischen ganzen Zahlen die ein Fixpunkt einer Renormalisierungsabbildung

im Raum der dynamischen Systeme ist. Die Cantormenge von Operatoren ist ein Attraktor eines iterierten Funktionensystems, das durch zwei Renormalisierungsabbildungen $\Phi_E^{\pm}$ gebildet wird.

• Wir zeigten, dass eine hinreichende Bedingung für die Existenz von einem Todafluss durch einen höher dimensionalen diskreten Laplaceoperator L ist, dass L nicht ein stationärer Punkt vom ersten Todafluss ist und dass es möglich ist, eine Faktorisierung $L = D^2 + E$ zu machen, wo D ein zufälliger Laplaceoperator über einer Integralerweiterung ist. Zufällige Laplaceoperatoren gibt es in einem Variationsproblem, dessen kritische Punkte durch partielle Differenzengleichungen beschrieben werden.

• Wir betrachteten Differentialgleichungen in $L^{\infty}(X)$, die einen thermodynamischen Limes von zyklischen Systemen von gewöhnlichen Differentialgleichungen oder die Bewegung von unendlich vielen Teilchen mit Paarwechselwirkung beschreiben. Die Bewegung von zufälligen Punktwirbelkonfigurationen hat manchmal eine Beschreibung als Bewegung von einem Jacobioperator.

• Wir konstruierten eine analytische Abbildung $U : \mathbb{C}^4 \to \mathbb{C}^4$, die eine einparametrige Familie von zwei-dimensionalen reellen Tori S_γ invariant hat, auf denen U die Standardabbildung T_γ ist. Wir machten eine grobe qualitative Beschreibung von U und gaben ein paar Argumente, die die Vermutung unterstützen, dass die metrische Entropie der Standardabbildung von unten durch $\log(\gamma/2)$ abschätzbar ist.

• Wir führten ein verallgemeinertes Percival'sches Variationsproblem zur Einbettung von abstrakten dynamischen Systemen in einer monotone Twistabbildung S_γ ein. Unter Benützung des anti-integrablen Limes von Aubry und Abramovici zeigten wir, dass für grosse γ jedes ergodische abstrakte dynamische System (X, T, m) mit metrischer Entropie $h_m(T) \leq \log(2)$ in die Standardabbildung S_γ eingebettet werden kann. Für grosse γ ist die topologische Entropie von S_γ mindestens $\log(2)$. Unter Benützung eines verallgemeinerten Morseindex bewiesen wir die Existenz von überabzählbar vielen verschiedenen Einbettungen von dynamischen Systemen.

• Wir studierten verschiedene Kohomologieen für dynamische Systeme: Für ein dynamisches System $(\mathcal{R}, \mathcal{G})$, wo eine abelsche Gruppe $\mathcal{R}$ auf einer abelschen Gruppe $\mathcal{G}$ operiert, gibt es die Eilenberg-McLane-Kohomologie. Im Falle $\mathcal{R} = \mathbb{Z}$ definierten wir eine Folge von Halmos Kohomologie- und Halmos Homologie gruppen. Für ein dynamisches System, $(\mathbb{Z}^d, \mathcal{M}, \mathrm{tr})$, wo $\mathbb{Z}^d$ auf der Algebra $\mathcal{M}$ mittels Spur erhaltenden Algebraautomorphismen operiert, definierten wir eine diskrete Version von de Rham's Kohomologie.

• Wir studierten das hyperbolische Verhalten von beschränkten, messbaren $SL(2, \mathbb{R})$-Kozyklen über einem dynamischen System. Wir untersuchten die Relation zwischen der Rotationszahl von Ruelle für messbare Matrixkozyklen und dem hyperbolischen Verhalten der Kozyklen. Wir zeigten, dass ein Kozyklus uniform hyperbolisch ist, genau dann wenn die Rotationszahl konstant ist in einer Umgebung des Kozyklus. Wir betrachteten auch verschiedene Spektren von Matrixkozyklen und bewiesen, dass das Spektrum des Kozyklus als Operator auf $L^2(X, \mathbb{C}^2)$ gleich dem Sacker-Sell Spektrum ist.

4

Contents

Prolog

Contents

1 Guide for the Reader

Each chapter is self-contained. This implies for example that *some definitions occur several times and in different versions* according to the generality needed at the corresponding places.

Because of the inhomogeneity of the themes, *the notation is not uniform in different chapters* throughout the theses.

Several chapters are already published and we plan to publish some parts separately hoping also to obtain in future more answers to some questions not yet solved. The already published chapters are partially updated to newer developments and sometimes, parts were added to the published version. Especially some illustrations and examples have been added.

Each chapter contains its own *bibliography*. There is also a collected bibliography at the end.

We added quite freely *questions* following each chapter. These questions help to keep organized the loose ends.

The current state of the chapters is the following:

- **Chapter "Three problems"**:
This chapter introduces into three circles of problems which are treated in the theses.

- **Chapter "Density results for positive Lyapunov exponents"**:
A large part of this chapter is published in [Kni 2]. We added a proof of a similar result for $SL(2, \mathbb{C})$ cocycles and updated some references.

- **Chapter "Discontinuity and positivity of Lyapunov exponent"**:
This is essentially the version in [Kni 1]. We added two appendices concerning cohomology and a theorem of Baire.

- **Chapter "Isospectral deformations of random Jacobi operators"**:
published in [Kni 3]. We added appendices in which the Thouless formula and integration of aperiodic Toda lattices is shown. These results in the appendices are known but not available in collected form. We added a short appendix about the definition of the continuous analogue of random Toda flows, the *random KdV* system.

- **Chapter "Factorization of random Jacobi operators and Bäcklund transformations"**:

published in [Kni 4]. We clarified one aspect in the published proof of the fact that Bäcklund transformations are isospectral. We also added a short appendix about Bäcklund transformations for KdV flows.

- **Chapter "Cohomology of $SL(2, \mathbb{R})$ cocycles and zero curvature equations for random Toda flows":**
This chapter can be viewed as part III of the two previous chapters. It is quite inhomogeneous and leads to different topics like zero curvature equations, lattice gauge fields, cohomology of cocycles and Harpers Laplacian.

- **Chapter "Some additional results for random Toda flows":**
This is a continuation and unpublished work of research done for the previous chapters on random Jacobi operators. Also this chapter is not homogeneous and it has some loose ends.

- **Chapter "Renormalisation of Jacobi matrices: Limit periodic operators having the spectrum on Julia sets":**
A more compact version of this chapter will soon be submitted. This chapter is a continuation of the study of one dimensional Jacobi operators. Iteration of the factorization $L = D^2 + E$ constructed in [Kni 4] leads to "renormalisation" in the fiber bundle of Jacobi operators over the complete metric space of dynamical systems. Projecting this renormalisation to the spectrum of the operators leads to the quadratic map $z \mapsto z^2 + E$. Relations of random Jacobi operators with complex dynamical systems appear like for example that the density of states of the operators in the limit of renormalisation is the unique equilibrium measure on the Julia set or that the determinant $\det(L - E)$ is the Böttcher function for the quadratic map. We hope that results from the theory of iteration of rational maps will shed light on what happens at points where the renormalisation set up breaks down.

- **Chapter "Isospectral deformations of discrete Laplacians":**
We study higher dimensional Laplacians and isospectral deformations of such Laplacians. We show that the existence of a Dirac operator is related to isospectral deformations of Laplacians. We consider also random partial differential equations over a $\mathbb{Z}^d$ dynamical system. We use the anti-integrable limit of Aubry to prove the existence of such equations. An version of this chapter is planned to be presented in July 1993 at Leuven.

- **Chapter "Infinite particle systems":**
We study the idea to use ergodic theory in order to obtain the thermodynamic limit of infinite particle systems in one real or complex dimension. This generalization applies for ordinary differential equations describing particles. One idea is to get the particle coordinates $q_n = q(T^n x)$ along the orbit of a point x. The thermodynamic

13

limit is then a well defined ordinary differential equation in the Banach space of the coordinate field q. An other idea is to look at a vortex configuration in the complex plane as the spectrum of an operator and to describe the vortex flow as a flow of operators.

- **Chapter "Embedding abstract dynamical systems in monotone twist maps"**:
This chapter [Kni 6] was submitted in October 1992 to Inventiones and in December 1992 to Ergodic Theory and Dyn. Syst. In both cases the paper was not accepted. We take the opportunity to comment on the results. The main point of the chapter is that it allows to give an *quantitative explicit bound* on the topological entropy for monotone twist maps. This is a new application of the anti-integrable limit of Aubry and is (according to our opinion) an interesting result. It has not been obtained by other methods and other approaches (like finding homoclinic points) for positive topological entropy are more complicated. The use of the *generalized Morse index* in the paper is also new. The multiplicity result of uncountably many different embeddings has not been obtained by other methods.

- **Chapter "An analytic map containing the standard map family"**:
This chapter was submitted in October 1992 to Nonlinearity [Kni 5]. We formulate a quantitative conjecture about the metric entropy of the standard map and show that the whole standard map family can be embedded in one complex analytic map of C^4. We begin a qualitative study of the map.

- **Chapter "Cohomology of dynamical systems"**:
In this chapter we discuss some cohomological constructions for abstract dynamical systems with the aim to build algebraic invariants of the systems. We think that interesting research in this direction is still possible and necessary. The chapter illustrates that the definition of the cohomologies is quite easy but that the explicit computation of the cohomology groups seems to be very difficult even in the simplest cases.

- **Chapter "Nonuniform and uniform hyperbolic cocycles"**:
In this chapter, we discuss the relation of uniform and nonuniform hyperbolicity for cocycles. We study also the relation between Lyapunov exponents and rotation numbers and the relation between Lyapunov exponents and spectra of cocycles treated as operators.
This chapter was written in an early stage of the theses and does not contain so much new material. It can be viewed as an appendix to the first two chapters on matrix cocycles and one can find for example detailed proofs of a theorem of Ruelle and a theorem of Wojtkowsky in the special case of measurable $SL(2, \mathbb{R})$ cocycles.

14

2 The basic objects

The basic (sometimes hidden) mathematical structure appearing in all the chapters is a **non-commutative von Neumann algebra** $\mathcal{X}$ obtained as a crossed product of the von Neumann algebra $\mathcal{M} = L^\infty(X, M(N, \mathbb{C}))$ with the $\mathbb{Z}^d$- dynamical system

$$(X, T_1, T_2, \ldots, T_d, m),$$

where $T_1, \ldots, T_d$ are commuting automorphisms of the Lebesgue probability space (X, m).

> **The basic question is to find spectral, ergodic and cohomological invariants of elements or subsets in $\mathcal{X}$ and relate them to invariants of the dynamical system.**

• **Spectra** are obtained by choosing a representation of $\mathcal{M}$ in an algebra of operators and taking as the spectrum of an element A in $\mathcal{M}$ either the set of complex values z such that $z - A$ is not invertible or to define a spectrum by taking the set of points z, where $z \cdot A$ is not invertible. Choosing special cocycles (like for example the Jacobean of a map, or the Koopman operator associated to an abstract dynamical system) leads to invariants of the dynamical system.

• **Ergodic invariants** are numbers obtained by ergodic averaging. Examples are the Lyapunov exponent, the rotation number or the total curvature of a cocycle or field. Choosing special cocycles leads to invariants of the dynamical system like for example the entropy or the index of an embedded system.

• **(Co)homology groups** or **Moduli spaces** of conjugacy classes of a subset in $\mathcal{X}$ give algebraic informations about a dynamical system. Examples are cohomology groups $\mathcal{H}^1(T, G)$ defined by the quotient of all cocycles with values in the group G modulo the space of coboundaries.

The elements in $\mathcal{X}$ we are going to study are:

• $d = 1, N = 2, A\tau$ is called a *matrix cocycle* over the abstract dynamical system (X, T, m). Such cocycles appear as *transfer cocycles* of one dimensional discrete Schrödinger operators, as *Jacobeans* of diffeomorphisms on compact two dimensional manifolds and which are also called *weighted composition operators* or *weighted translation operators*.

15

- $\sum_{i=1}^{d} A_i \tau_i$ are called *one-forms*, or *connections* or *non-abelian gauge fields*.

- $d = 1, N = 1, L = a\tau + (a\tau)^* + b$ is a discrete version of a one dimensional Schrödinger operator. Such operators serve as models in solid state physics. They appear also as *Hessians* of variational problems in monotone twist maps. For general N, they are called *Jacobi operators on a strip*.

- $L = \sum a_i \tau_i + (a_i \tau_i)^* + b$ is a discrete version of a *Laplace operator*. We call it *random Laplace operator*. We allow also $N > 1$.

3   The role of Lyapunov exponents

The title of the theses could also be "Some topics about Lyapunov exponents" because Lyapunov exponents will appear in almost all chapters, sometimes with other names. They are connected with

- the "entropy" of a dynamical system,

- the "determinant" of a random Jacobi operators,

- "integrals" of random Toda flows,

- the "Hamiltonian" used for the interpolation of Bäcklund transformations,

- the "energy" of a random Coulomb gas in two dimensions,

- the potential theoretical "Green function" of some Julia sets,

- the logarithm of a "spectral radius" of a weighted composition operator,

- some "gauge invariants" of random gauge fields.

4   The intuitive idea

We will formulate in the first chapter three problems which should give a first comment on the key words "ergodic invariants, spectral invariants, cohomological invariants" in the title.

A *guiding idea* for the whole theme is the following *analogy* between differential topology on manifolds and the ergodic theoretical concepts treated here.

16

If we think of the probability space (X, m) as a *manifold*, the group action defines a *geometry* on this space. The orbit of a point x is a lattice is playing the role of a *chart* at the point and the set of all orbits serves as an *atlas*.

$L^\infty(X, M(N, \mathbb{C}))$ or a multiplicative subgroup $L^\infty(X, G)$ corresponds to a *fiber bundle*. The crossed-product $\mathcal{X}$ is an operator algebra playing the role of *differential operators* and contains objects like Laplacians or connections.

The *classification of differentiable manifolds* is analogous to the *classification of group actions*. *Spectral problems* of differential operators, *numerical* or *cohomological invariants* of manifolds are analogous to spectral problems of random operators, ergodic and cohomological invariants of dynamical systems.

References

[Kni 2]   O.Knill. *Positive Lyapunov exponents for a dense set of bounded measurable $SL(2, R)-$ cocycles.*
Ergod. Th. Dynam. Sys., 12, 319-331, 1992

[Kni 1]   O.Knill. *The upper Lyapunov exponents of $SL(2, R)$ cocycles: Discontinuity and the problem of positivity.*
Lect. Notes Math., 1486, Springer 1991

[Kni 3]   O.Knill. *Isospectral Deformations of Random Jacobi Operators.*
Commun. Math. Phys., 151:403-426, 1993

[Kni 4]   O.Knill. *Factorisation of Random Jacobi Operators and Bäcklund transformations.*
Commun. Math. Phys., 151:589-605, 1993

[Kni 5]   O.Knill. *An analytic map containing the standard map family.*
Submitted to Nonlinearity

[Kni 6]   O.Knill. *Embedding abstract dynamical systems in monotone twist maps.*
To be submitted

Three Problems

Contents

Abstract

We give three examples of problems which should illustrate some topics of this theses.

The first problem is the problem of proving positive metric entropy in a given conservative dynamical system.

The second problem is the problem of determining the spectrum of random Schrödinger operators.

The third problem is to determine a cohomology group for abstract dynamical systems.

All three problems are related to the problem of calculating Lyapunov exponents of matrix cocycles over an abstract dynamical system.

1   An ergodic problem: positive metric entropy

"Is there positive metric entropy ("chaos") in a given conservative dynamical system?"

One of many definitions of "chaos" in a Hamiltonian system is the property of having positive metric entropy. Hamiltonian systems with this property show sensitive dependence on initial conditions on a set of positive measure. Moreover, one can find quantities accessible to measurements which are actually independent random variables. The system could be used as a true *random number generator* !
(We refer to the Les Houches Lecture of Lanford [Lan 83], the Bernard Lecture [Rue 90] or the Lezioni Lincee [Rue 87] of Ruelle for an introduction into some of the topics.)

From the physical point of view, there is evidence that chaos is the rule and zero metric entropy is the exception. The entropy has been measured in many systems and found to be positive. It seems, however, that *mathematical proofs* for chaos are difficult. It could even be that most measurements show numerical artifacts and that positive metric entropy is the exception.

There are milder requirements for a system to belong to the fashion class of "chaotic systems". One of these is *positive topological entropy* which is easier to *prove*. For topological dynamical systems, there is a definition of chaos due to Devaney [Dev 89] which requires that the homeomorphism of the metric space is (i) transitive and (ii) that periodic orbits are dense. (A third requirement of Devaney, sensitive dependence on initial conditions, turned out to follow from the requirements (i),(ii) if the

system is not itself periodic [Ban 92].)

We want to insist on positive metric entropy as the *more relevant quantity* for Hamiltonian systems. It is based on the belief, that physically significant quantities for Hamiltonian systems must be accessible on sets of positive measure. Also from a mathematical point of view it is not satisfactory to use a quantity like topological entropy from the category of topological dynamical systems when having a natural invariant measure, the Lebesgue measure. (Of course, invariant sets like periodic orbits or *horse-shoes*, which have in general zero measure, can also have consequences for sets of positive measure. For example, *invariant curves* of zero measure can form barriers in a two dimensional phase space and prevent ergodicity.)

The answers to the following *general* problems in classical mechanics are not known. (We assume implicitly that the Hamiltonian systems considered have a compact energy surface and that the entropy is understood with respect to the flow on such a surface.)

• How does one decide if a given finite dimensional Hamiltonian system has positive metric entropy or not?

• How big is the class of Hamiltonian systems having positive metric entropy? Does positive metric entropy occur generically?

• Does every non-integrable Hamiltonian system have positive metric entropy?

• Does there exist Hamiltonian systems which have positive topological entropy but zero metric entropy?

• Does there exist a Hamiltonian system in which true coexistence occurs? ([Str 89]) True coexistence means that a part of phase space is the union of two flow-invariant dense subsets both having positive Lebesgue measure such that the metric entropy of the flow is zero on the first subset and positive on the other subset.

The main obstacle in solving the above problems is that proving positive metric entropy is a hard problem. Because a formula of Pesin shows that in many cases the entropy is the sum of the integrated positive Lyapunov exponents, the problem of positive metric entropy is often equivalent to prove that there are positive Lyapunov exponents on a set of positive measure.

There are several examples, where positive metric entropy is *conjectured but not proven* (compare [Str 89]). From special interest is the example of our solar system, where positive metric entropy was also measured [Las 90]. (See also the review article [Wis 87] for chaotic motion in the solar system.) We will give now a list

of examples which include discrete Hamiltonian systems (which are mappings) and classical Hamiltonian systems (which are flows). We consider also *conservative dynamical systems*, flows or maps which leave the Lebesgue measure invariant but which need not to be Hamiltonian.

- For the Chiricov mapping *Chiricov map* or Standard mapping on the torus $\mathsf{T} = \mathsf{R}^2/(2\pi\mathsf{Z})$

$$T_\gamma : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y + \gamma\sin(x) \\ y + \gamma\sin(x) \end{pmatrix}$$

the entropy is measured to be greater or equal then $\log(|\gamma/2|)$. Some orbits of the map are shown with the following Mathematica program which produced a little "film" which shows the situation for larger and larger "coupling constant" γ.

```
T[{x_Real,y_Real},g_Real]:=N[Mod[{x+y+g*Sin[x],y+g*Sin[x]},2 Pi]];
Orbit[p0_,n_Integer,g_Real]:=Module[{},S[p_]:=T[p,g];NestList[S,p0,n-1]];
OrbitSet[k_Integer,n_Integer,g_]:=Module[{s={}},
  Do[s=Union[s,Orbit[{0.0,N[2Pi*(0.741629+(i-1)/k)]},n,g]],{i,k}];s];
Pict[k_,m_,g_]:=ListPlot[OrbitSet[k,m,g],PlotRange->{{0,N[2Pi]},{0,N[2Pi]}},
  DisplayFunction->Identity,Frame->True,Axes->False,FrameTicks->None,
  FrameLabel->{FontForm[g,{"Helvetica",12}],"","",""}];
Film[k_,m_,l_]:=Table[Pict[k,m,0.0+N[(j-1)/4]],{j,1}];
Display["!psfix -land -stretch > standard.ps",
  Show[GraphicsArray[Partition[Film[7,500,12],3]],
  DisplayFunction->$DisplayFunction,Frame->False,
  PlotLabel->FontForm["The Standard Map family",{"Helvetica",12}]]]
```

This program has generated the following picture:

The Standard Map family
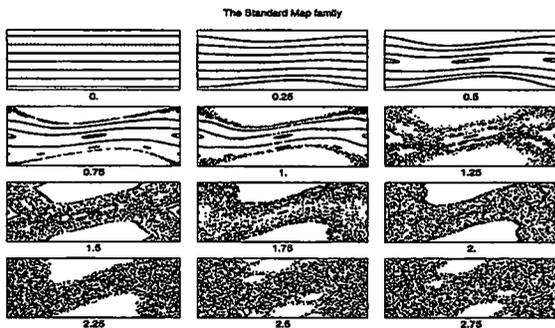
If one considers the Jacobian dT_γ of the map T_γ as a *cocycle* over the dynamical system (T, T_0, m) and calculates the Lyapunov exponent of this cocycle, one gets indeed with a method developed by M.Herman) the lower bound $\log(|\gamma/2|)$. But it is an unsolved problem whether there exists a value γ such that the metric entropy is positive for the standard map S_γ. (This entropy problem has been mentioned already in [Spe 86]). For the topological entropy more is known: there is a result of Angenent [Ang 92] which states that a $C^{1+\epsilon}$ twist diffeomorphism of the annulus has either positive topological entropy (and therefore a transversal homoclinic point) or else has invariant circles for any rotation number in the twist interval. In [Ang 90] is shown that for $\gamma > 1$, there must be positive topological entropy. Fontich has also shown [Fon 90], that for $\gamma > 0$ the standard map has a heteroclinic point and therefore a horse-shoe embedded and one has therefore positive topological entropy. In the chapter "Embedding of abstract dynamical systems in monotone twist maps", we will show that for γ large enough the topological entropy of the Standard map is at least $\log(2)$.

- Among monotone twist mappings, *billiards* have been investigated extensively. There is no known example of a strictly convex *smooth* billiards with positive metric entropy. (On the other hand one has also the unsolved question, which tables produce integrable billiard maps. Guillmin conjectures that every billiards which is not integrable must be an ellipse ([Gui 87]). An other open question is whether every billiards with zero metric entropy is an ellipse.) There are known classes of examples of not smooth billiards having positive metric entropy. The most famous example is Bunimovitch's *stadium billiards*

(see [Woj 86] or [Don 91] and references therein for newer results). A candidate for a smooth billiards with positive metric entropy is the *Robnik billiards*. It is defined by the curve

$$\gamma : t \;\mapsto\; \left(\begin{array}{c} \cos(t) + \gamma\cos(2t) \\ \sin(t) + \gamma\sin(2t) \end{array} \right) \; ,$$

where $\gamma \in (0, 1/4)$. One measures positive metric entropy with values like for example $\gamma = 0.2$. Other candidates are C^1 billiards, where the curve consists of 4 arcs (see [Hay 87]) or the *Benettin-Strelcyn Billiard* [Ben 78],[Hen 83].

- A far as we know, no smooth *dual billiards map* with positive metric entropy has been constructed. The dual billiards or *exterior billiards* is a mapping defined outside a convex curve in the plane. It preserves Lebesgue measure. Application of KAM methods [Dua 82] (III-12) show that for smooth dual billiards curves, there exist invariant curves of the dynamics. Therefore, the dynamics is defined on regions of finite Lebesgue measure and the question of positive metric entropy is well defined. Tabatchnikov [Tab 93] has recently shown that if two dual billiards maps are commuting then the tables are conformal ellipses. Dual billiards at polygons is already very interesting and there are many open questions (see [Viv 87], [Gut 92] in this case and general references).

The following Mathematica program allows the numerical calculation of a dual billiards with arbitrary real analytic table. The program calculates and plots a picture of 20 orbits each consisting of 1000 points.

```
a={1.0,0.0,0.0,0.0,0.0,0.0,0.01,0.001};
r[t_]:=Sum[a[[n]] Cos[(n-1)*t],{n,Length[a]}];
rdot[t_]:=Sum[-a[[n]]*(n-1)*Sin[(n-1)*t],{n,2,Length[a]}];
BilliardMap[{x_,y_}]:=Module[{psi=Arg[x+I*y],eta},
  eta=t /. FindRoot[ (r[t]*Cos[t]-x)*(rdot[t]*Sin[t]+r[t]*Cos[t])==
               -(r[t]*Sin[t]-y)*(-rdot[t]*Cos[t]+r[t]*Sin[t]),{t,psi+Pi/2}];
  {-x+2*r[eta]*Cos[eta],-y+2*r[eta]*Sin[eta]}];
T=ParametricPlot[{r[t]*Cos[t],r[t]*Sin[t]},{t,0,2 Pi},DisplayFunction->Identity];
Orbit[p_,n_]:=NestList[BilliardMap,p,n];
OrbitSet[n_,m_]:=Module[{s={}},Do[s=Join[s,Orbit[{1.0+i*0.05,0.0},n]],{i,m}];s];
OrbitSetPict[n_,m_]:=ListPlot[OrbitSet[n,m],DisplayFunction->Identity];
JoinedPict[n_,m_]:=Show[{T,OrbitSetPict[n,m]},
```

```
DisplayFunction->$DisplayFunction,PlotRange->All,Frame->True,Axes->False];
Display["!psfix -land -stretch > billiard.ps",
  Show[JoinedPict[1000,20],DisplayFunction->$DisplayFunction,Frame->True,
  PlotLabel->FontForm["Exteriour billiards",{"Helvetica",12}]]]]
```

and here is the picture

Exteriour billiards
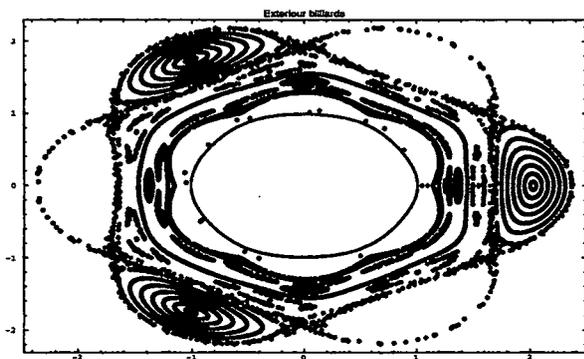
- There are also interesting monotone twist maps in the plane R^2 of the form

$$T = T_f : R^2 \to R^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f(x) - y \\ x \end{pmatrix} ,$$

where f is a continuous function $f : R \mapsto R$. Examples are the *Henon map* $f(x) = ax^2 + b$, the *Lozi map* $f(x) = 1 + a|x|$ or the map given by $f(x) = \sqrt{a^2 + x^2}$. The entropy question makes sense when these maps are restricted to a measurable invariant subset of R^2 which has finite nonzero Lebesgue measure. For $f(x) = \sqrt{a^2 + x^2}$, Aharonov and Elias [Aha 90] have shown the existence of invariant curves far away (even for a more general case when T is the composition of two such maps with possibly different a_i). It seems however (a conjecture of H.Cohen communicated by Y.Colin-de Verdier

to J.Moser April 10 1993) that the map with $f(x) = \sqrt{a^2 + x^2}$ is integrable! (For $a = 0$ there is an additional integral $F(x, y) = y + |y - |x|| + |x - |y - |x||| + |y - |x - |y||| + |x - |y| + |y - |x - |y||||$). For the Lozi map, it is known that it has positive metric entropy for example for $a = 1$ [Dev 84]. Nothing about the metric entropy seems to be known for the conservative Henon map.

- A promising strategy to find true coexistence is to start from a system with positive metric entropy and to move into a region with zero entropy. One expects then to pass a region of true coexistence. Coexistence of stable (existence of an elliptic fixed point) and unstable (positive metric entropy) behavior has been constructed by Przytycki [Prz 82] by constructing a real analytic arc starting at an Anosov system. It is however not known if this example satisfies true coexistence. An other candidate of a perturbation of an Anosov map is

$$T_f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x + y \\ x + y + \gamma \sin(2x + y) \end{pmatrix} .$$

It is not known if one can deform this Anosov map T_0 to a system which has zero metric entropy. It would also be interesting to know what happens at the boundary, where the system looses uniform hyperbolicity.

- Among Hamiltonian systems, there are many candidates with two degrees of freedom which are believed to have positive metric entropy:

- The Henon Heils system : [Hen 64]

$$H(p, q) = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1^2 q_2 - \frac{1}{3}q_2^2$$

on the energy surface $H = E$ with $0 < E < 1/6$ is a famous example of a Hamiltonian system with two degrees of freedom.

- The Störmer problem (see [Bra 81]) :

$$H(p, q) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(\frac{1}{q_1} - \frac{q_1}{(q_1^2 + q_2^2)^{3/2}})^2$$

for energies $0 < E < 1/32$ is an example of great physical interest because it describes the motion of electrons and protons in the *van Allen radiation belts*. It would be desirable to know more about this system. There are mathematical proofs that particles can be trapped for ever [Bra 81], but one still does not know why the actual existing van Allen belts around the earth are so stable. Numerical experiments indicate that the system shows chaotic behavior. Braun has proved that a related discrete model problem, a so called

linked twist map has homoclinic points [Bra 81a]).

- The Contopoulos,Barbanis system [Con 89]

$$H(p,q) = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1^2 q_2 - q_1 q_2^2 .$$

or the Caranicolas-Vozikas system [Car 87]

$$H(p,q) = \frac{1}{2}(p_1^2 + p_2^2) + q_1^4 + q_2^4 + 2\gamma q_1^2 q_2^2$$

for $E = 1$, $\gamma = 6$, or the Sahos-Bountis system

$$H(p,q) = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 q_2^2 + \gamma(q_1^2 + q_2^2)).$$

are interesting because of the simplicity of their Hamiltonians.

- Unequal mass Toda system:

$$H(p,q) = \frac{1}{2}\left(\frac{p_1^2}{m_1} + \frac{p_2^2}{m_2}\right) + e^{q_1 - q_2} + e^{q_2 - q_1}$$

for $m_1 \neq m_2$. For $m_1 = m_2$ the system is integrable and is related to the physical pendulum.

- The planar isosceles 3 body problem (see for example [Dev 80])

$$H(p,q) = \frac{p_1^2}{m_1} + \frac{(2m_1 + m_3)p_2^2}{4m_1 m_3} - \frac{m_1^2}{q_1} - \frac{4m_1 m_3}{(q_1^2 + 4q_2^2)^{1/2}} .$$

Here $m_1 = m_2, m_3$ are the masses of the body. q_1 is the distance from q_1 to q_2 which are lying symmetric with respect to the $y-$ axes. q_2 is the distance from the center of mass of q_1 and q_2 to q_3.

- Interesting Hamiltonian systems arise from *geodesic flows*. The question is then which smooth Riemannian metrics on a compact manifold have positive metric entropy for the geodesic flow. Donney [Don 88a],[Don 88b] proved that on every compact orientable surface, there exists a C^∞ Riemannian metric, for which the geodesic flow is ergodic and has positive metric entropy.

- There are interesting *cyclic systems of differential equations* which are candidates for systems having positive metric entropy.

- The Orszag-McLaughlin flow is given by the differential equation

$$\dot{x}_i = a \cdot x_{i+1} x_{i+2} + b \cdot x_{i-1} x_{i-2} + c \cdot x_{i+1} x_{i-1} \ ,$$

where the index i is taken modulo n with $n \geq 3$. If one takes $a + b + c = 0$ and $abc \neq 0$, this system preserves the Lebesgue measure and leaves invariant the spheres

$$S_r = \{ \sum_{i=1}^{n} x_i^2 = r \} \ .$$

The measurements indicate that the dynamics restricted to such spheres is chaotic.

- The Arnold-Beltrami-Childress flow or *ABC-flow* (see [Zha 93] for more information) is a flow on T^3 defined by

$$
\begin{aligned}
\dot{x} &= A\sin(z) + C\cos(y) \\
\dot{y} &= B\sin(x) + A\cos(z) \\
\dot{z} &= C\sin(y) + B\cos(x) \ ,
\end{aligned}
$$

where A, B, C are nonzero constants. More generally with given real numbers $a_i \in \mathsf{R}$, a system of differential equations is given by

$$\dot{x}_i = a_{i-1}\sin(x_{i-1}) + a_{i+1}\cos(x_{i+1}) \ ,$$

where $x_i \in \mathsf{R}/\mathsf{Z}$ and i is taken modulo n. This system preserves Lebesgue measure on the torus T^n.

- The *heavy top* and its higher dimensional generalizations. Given $M \in sl(n,\mathsf{R})$, $B \in so(n,\mathsf{R})$ and $J = J^* \in GL(n,\mathsf{R})$. For $L = JB + BJ$ the differential equation

$$\dot{L} = [B, L] + M$$

generalizes the motion of the heavy top in any dimension. For $M = 0$ one obtains the motion of the free n dimensional top which is an integrable system and can be viewed as a geodesic flow on the Lie group $SO(n,\mathsf{R})$ (see [Arn 89]). For $M \neq 0$, one expects to have positive metric entropy in general.

- The Hamiltonian for n vortices $z_i = q_i + ip_i$ in the complex plane C is

$$H(z) = \frac{1}{2\pi i} \sum_{i<j} \log |z_i - z_j| \ .$$

For $n \leq 3$ the system is integrable. For $n \geq 4$ one expects that the dynamics

$$\dot{q} = \frac{\partial}{\partial p} H, \ \dot{p} = -\frac{\partial}{\partial q} H$$

27

has positive metric entropy. Point vortices can also be defined on any two dimensional manifold. Interesting is the situation on the cylinder which is also called the *periodic Karman vortex street*. The Hamiltonian is in this case

$$H(z) = \sum_{i<j} \log(\sin|z_i - z_j|) \ .$$

We summarize: for conservative dynamical systems the problem of positive metric entropy is often reduced to the decision whether the highest Lyapunov exponents for symplectic cocycles is positive or not (the flow case can be reduced by a time-one map or a Poincaré map to the case when time is discrete). Calculating the highest Lyapunov exponent of a general measurable symplectic cocycle over an abstract dynamical system is an example of an *ergodic problem*.

2   A spectral problem: random Jacobi operators

Does a given 1-dimensional Schrödinger operator of an electron have absolutely continuous spectrum ("good conductivity")?

In classical quantum mechanics, the evolution of a system is governed by a Hamiltonian L which is a self adjoint operator on a Hilbert space. The evolution of a wave function u is given by the Schrödinger equation $i\dot{u} = Lu$. An important problem is to calculate and analyze the spectrum of the Hamiltonian because many physical properties of the system are determined by the spectrum. Because the spectrum is accessible by measurements, one would also like to solve the inverse problem, namely to find out the Hamiltonian of the system from the spectrum.

These mathematical problems are not easy even for the motion of a particle in one dimension in a given external field. In such a *one body approximation* one neglects the interaction of the electrons. One often considers the so called *tight binding approximation*, where the continuum is replaced by a lattice. This is technically more simple and the model is commonly used for describing the electronic properties of disordered media.

An important qualitative spectral problem is the question whether the spectrum is absolutely continuous or not. For one-dimensional Laplacians or Laplacians on the strip, this question can be reduced to a question about positive Lyapunov exponents for symplectic cocycles.

A main problem is: Given a random Schrödinger or Jacobi operator L.

- What is the spectrum of L?

- What type of spectrum (point, singular, or absolutely continuous) does occur?

- What spectra do occur generically? Is there generically no absolutely continuous spectrum?

There is an inverse spectral problem which is however closely related to the above spectral problem. It can be formulated as:

- Can one describe the set of all Jacobi operators which are unitarily equivalent or the set of operators which have the same density of states. Can one reconstruct the isospectral set back from the spectrum. How large is the set of operators which one can reach by isospectral deformations?

Similarly to the positive entropy question, which has the same mathematical problem in the background, there are several examples, where absence of absolutely continuous spectrum is conjectured but not proved. The general belief is that for high disorder, there is no absolutely continuous spectrum any more. A more precise formulation of this conjecture could be:

Given any aperiodic dynamical system (X, T, m) such that T^n is ergodic for each $n \in \mathsf{Z}$. Given a one- parameter family of discrete random Schrödinger operators

$$(L_g u)_n = u_{n+1} + u_{n-1} + g \cdot V_n u_n$$

with non-constant potential $V \in L^\infty(X, \mathsf{R})$ and with $V_n = V(T^n x)$. Does there exist $g \in \mathsf{R}$, such that L_g has no absolutely continuous spectrum?

We list now some classes of operators.

- The *Anderson model*

$$(Lu)_n = u_{n+1} + u_{n-1} + V_n u_n \ ,$$

where V_n are non-constant independent identically distributed random variables. Fürstenberg's theorem assures positive Lyapunov exponents for all $E \in \mathsf{R}$ and there is no absolutely continuous spectrum.

- *Almost periodic operators.*
  Let T be an ergodic translation of a compact topological group X and a, b two continuous real-valued functions $X \mapsto \mathsf{R}$. Given $x \in X$, define $a_n = a(T^n x), b_n = b(T^n x)$. The operator

$$(Lu)_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n$$

is called *almost periodic*.

Example. If the group X is the circle and T is an irrational rotation $x \mapsto x + \alpha$ and $V(x) = \cos(x)$, one obtains the *almost Mathieu operator*

$$(L(x)u)_n = u_{n+1} + u_{n-1} + \gamma \cos(x) \ .$$

This is a famous model for an electron in a quasi-crystal. About the spectrum $\sigma = \sigma_{pp} \cup \sigma_{sc} \cup \sigma_{ac}$ is known that for $\lambda < 4/(2 + \pi/\sqrt{5})$ the absolutely continuous spectrum σ_{ac} is not empty. For almost all α, the measure of the absolutely continuous spectrum is $4 - \gamma$. This was proved recently by Last [Las 93] following previous work of Avron, van Mouche and Simon [Avr 90]. The general conjecture of Aubry and André that this should hold for any irrational α is still not proved. For large γ, one knows that in the case of Diophantine α, there is point spectrum [Fro 90] and that for Liouville numbers α and any $\gamma > 2$, there is purely singular continuous spectrum (see [Cyc 87]). For any irrational α, there is no absolutely continuous spectrum for $\gamma > 2$. In general, the question is open, whether for irrational α and all γ, the spectrum is a Cantor set (Martini problem). In the *critical Hofstadter case* $\lambda = 2$, which is also called *Harper's model*, Last has recently shown [Las 93] that the spectrum is in this case a zero measure Cantor set for almost all α. Plotting the family of spectra parameterized by α gives the *Hofstadter butterfly*. A review with other developments can be found in [Bel 92].

A numerical illustration. The periodic functions a, b on T^1 are fixed and the spectra of the operators $L = a\tau_\alpha + (a\tau_\alpha)^* + b$ over dynamical system $(\mathsf{T}^1, x \mapsto x + \alpha, dx)$ are calculated with the following program

```
a[alpha_,n_Integer]:=Table[N[2+0.01*Sin[k*alpha 2 Pi]],{k,n}];
b[alpha_,n_Integer]:=Table[N[ 2*Cos[k*alpha 2 Pi]],{k,n}];
m[i_Integer,n_Integer]:=Mod[i-1,n]+1;
d[k_Integer,l_Integer,n_Integer]:=IdentityMatrix[n][[m[k,n],m[l,n]]];
JacobiMatrix[a_List,b_List]:=Module[{n=Length[a]},
 Table[d[k,i,n]*b[[k]]+d[k,i+1,n]*a[[i]]+d[k,i-1,n]*a[[m[i-1,n]]],{k,n},{i,n}]];
Spec[a_List,b_List]:=Sort[Eigenvalues[JacobiMatrix[a,b]]];
anti[a_List]:=Block[{n=Length[a]},Table[a[[k]]-2*d[k,n,n]*a[[k]],{k,n}]];
AntiSpec[a_List,b_List]:=Sort[Eigenvalues[JacobiMatrix[anti[a],b]]];
DoubSpec[a_List,b_List]:=Sort[Join[Spec[a,b],AntiSpec[a,b]]];
SpecInterv[a_List,b_List]:=Partition[DoubSpec[a,b],2];
Hofstadter[m_Integer,n_Integer]:=Block[{p={},S,alpha=0.0},
Do[S=SpecInterv[a[alpha,n],b[alpha,n]];
   Do[p=Append[p,Line[{{S[[i]][[1]],alpha},{S[[i]][[2]],alpha}}]],
   {i,Length[S]}];alpha=alpha+1/m,{m+1}]; Show[Graphics[p],
   PlotLabel->FontForm["Spectra of Jacobi operators",{"Helvetica",12}] ] ];
Display["!psfix -land -stretch > spectrum.ps",Hofstadter[200,23]];
```

which produces the following picture which shows the spectrum of such an operator changes when the parameter α (which is increasing along the $x-$ axis) of the dynamical system is varied.

Spectra of Jacobi operators
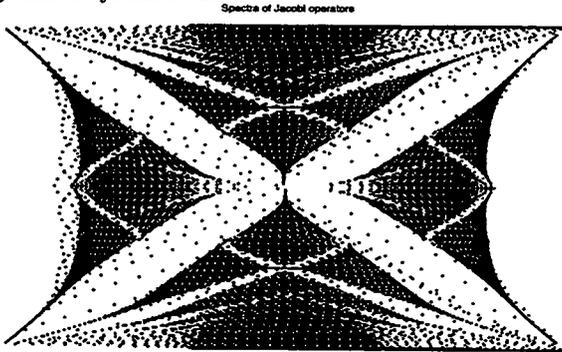
• *Operators generated by substitutions.*

A substitution dynamical system (we follow [Hof 92]) is defined by a map S from a finite alphabet A into the set of finite words A^* built by A. This substitution generates a fixed point $\tilde{x} \in A^{\mathbb{N}}$ of the map S if there exists a symbol $a \in A$ such that Sa begins with a. Take any word $\overline{x} \in A^{\mathbb{Z}}$ which coincides with $\tilde{x}$ on the positive integers and form the set X of all limit points (in the product topology) of $T^n \overline{x}$, where T is the shift. This gives a dynamical system (X, T) which is uniquely ergodic and minimal. The potential V for the Jacobi operator is $V(x) = x_0$. The spectrum is expected to be singular continuous in general and having zero Lebesgue measure. This has been proved for the *Thue-Morse* systems defined by $S(a) = ab$, $S(b) = ba$ or *Fibonacci sequences* defined by $S(a) = ab$, $S(b) = a$ and other examples. Kotani [Kot 89] dealt in a more general context with potentials over an ergodic dynamical system which take values only in a finite target space. He proved that in such a case, there is no absolutely continuous spectrum. See [Hof 92] or [Ghe 92] for references and recent results.

• *Operators which are second variations of twist mappings*

If $l(x, x')$ is a generating function of a twist map and $T : X \mapsto X$ is a dynamical system embedded in the twist map by a measurable function

$$q : X \mapsto \mathsf{T}$$

31

which satisfies

$$l_1(q(x), q(Tx)) + l_2(q(T^{-1}x), q(x)) = 0 .$$

The operator

$$(L(x)u)_n = a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n ,$$

with

$$\begin{aligned} a(x) &= l_{12}(q(x), q(Tx)) , \\ b(x) &= l_{11}(q(x), q(Tx)) + l_{22}(q(T^{-1}x), q(x)) \end{aligned}$$

is the second variation of a variational problem. It would be interesting to know what are the spectra of these Hessians.

We summarize: for one-dimensional discrete Schrödinger operators (on the strip), the problem of existence of absolutely continuous spectrum is reduced to the decision whether the highest Lyapunov exponent for a one parameter family of symplectic cocycles is positive or not. Calculating the spectrum of random operators or cocycles over an abstract dynamical system is an example of a *spectral problem*.

3 A cohomological problem: cohomology of measurable sets

" What is the cohomology group defined as the group of measurable sets of a probability space modulo the sets of the form $Z(T)\Delta Z$, where T is an automorphism of the probability space? "

Given an aperiodic ergodic automorphism T of the Lebesgue space $(X, \mathcal{A}, m)$. The measurable sets $Y \in \mathcal{A}$ are called *cocycles* and form an abelian group with multiplication Δ. This group has the subgroup

$$\mathcal{C} = \{Y \Delta T(Y) \mid Y \in \mathcal{A}\}$$

of *coboundaries*. How big is the *cohomology group*

$$\mathcal{H}(T, \mathbb{Z}_2) = \mathcal{A}/\mathcal{C} ?$$

We call the problem the *cohomology problem for measurable sets*. An other problem is to decide whether a given measurable set is a coboundary or not. We call this problem the *coboundary identification problem for measurable sets*. We will see in the chapter "Discontinuity and positivity of Lyapunov exponents" that solving this problem is easier than calculating Lyapunov exponents.

For finite periodic dynamical systems, the problem can be solved: assume X is a finite set and $\mathcal{A}$ is the set of subsets of X. An ergodic measure preserving map

$T : X \mapsto X$ is just a cyclic permutation of X. The group $\mathcal{A}$ consists of all subsets of X and has $2^{|X|}$ elements. It is quite easy to see that $\mathcal{C}$ consists exactly of the sets with even cardinality. In this case $\mathcal{H}(T, \mathbb{Z}_2) = \mathcal{A}/\mathcal{C}$ is isomorphic to $\mathbb{Z}_2$.

There are a lot of other related unsolved questions. For example, there is a *conjecture of Kirillov*: assume two automorphism of a fixed probability space (X, m) have the same measurable sets as coboundaries. Are they conjugate?

The cohomology problem for measurable sets can be generalized as a question in group theory: let $\mathcal{G}$ be an arbitrary abelian group and T a group automorphism. What is the group $\mathcal{H}(\mathcal{G}, T) = \mathcal{G}/\mathcal{C}$, where

$$\mathcal{C} = \{g(T)g^{-1} \mid g \in \mathcal{G}\} \; ?$$

Let (X, T, m) be a $\mathbb{Z}^d$ action. A generalization of the cohomology problem for sets is to find the *moduli space* of all zero curvature $\mathbb{Z}_2$ fields. A gauge field $Y = (Y_1, Y_2, \ldots, Y_d)$ is given by d measurable sets Y_i. The space $\mathcal{E}$ of all fields is the d-fold direct product of the group of all measurable sets. The *curvature* of such a gauge field Y is

$$F_{ij}(Y) = Y_i \Delta Y_j(T_i) \Delta Y_i(T_j) \Delta Y_j \; .$$

A field Y has *zero curvature* if $F_{ij}(Y) = \emptyset$ for all i, j. There is a subgroup of $\mathcal{E}$ called the *group of gradients*

$$\mathcal{C} = \{Y \in \mathcal{E} \mid \exists Z, \; \forall i = 1, \ldots d, Y_i = Z \Delta Z(T_i)\} \; .$$

On $\mathcal{E}$ are defined the *Gauge transformations*

$$Y_i \to Y_i \Delta Z(T_i) \Delta Z \; .$$

The gradients are just the fields which can be gauged to the identity.
The question is to find the moduli space of all zero curvature fields. In other words, we want to find the group

$$\mathcal{H} = \mathcal{E}/\mathcal{C} \; .$$

We have used only the additive group structure of the measurable sets. Taking the *ring structure also* gives also interesting problems which lead more away from the subject. The problem is however of the same type. As an example we consider a random version of a *nonlinear cellular automata* recently studied by Bobenko, Bordemann, Gunn and Pinkall [Bbgp 92]. It is an "integrable" system and the evolution can be interpreted as a collision process of soliton like particles. The cellular automata can be described as a $\mathbb{Z}_2$-valued field X_{nm} on a two-dimensional lattice $\mathbb{Z}^2$ satisfying the rule

$$X_{n,m} + X_{n+1,m+1} = X_{n,m+1}X_{n+1,m} + X_{n,m+1} + X_{n+1,m} \quad (\text{mod } 2) \; . \qquad (1)$$

33

We translate this automata now into a random setting. Given a $\mathbf{Z}^2$ dynamical system (X, T_1, T_2, m). Assume we have given a measurable set $Y \subset X$ satisfying

$$Y \Delta Y(T_1 T_2) = Y(T_1) \cap Y(T_2) \Delta Y(T_1) \Delta Y(T_2) . \qquad (2)$$

Define the random variable $X(x) = 1_Y$. For almost all $x \in X$ we can write $X_{n,m} = X(T_1^n T_2^m)$ and get from Equation 2 the rule 1. Equation 2 is equivalent to

$$Y \Delta Y(T_1 T_2) = Y(T_1) \cup Y(T_2) .$$

The problem is to find nontrivial sets satisfying this equation. Is there for any cellular automata rule

$$Y(T_1, T_2) = F(Y, Y(T_1), Y(T_2)) ,$$

where F is one of the 2^8 possible functions a measurable set which solves it?

An abstract generalization of the cellular automata is obtained as follows. Let $(\mathcal{R}, +, \cdot)$ be a commutative ring over the field $\mathbf{Z}_2$. Given two automorphism T_1, T_2 of this ring. Use the notation $T_1(r) = r(T_1)$ for $r \in \mathcal{R}$. The question is if there exists a non-zero ring element $r \in \mathcal{R}$ satisfying

$$r + r(T_1 T_2) = r(T_1) \cdot r(T_2) + r(T_1) + r(T_2) .$$

In this case, every ring element $T_1^n T_2^m(r)$ describes the cellular automata BBGP. We summarize: the cohomology identification problem would be solved if we could calculate the Lyapunov exponents of symplectic cocycles. Cohomology problems over an abstract dynamical system with structure group is $\mathbf{Z}_2$ lead to interesting questions. The general problem of calculating the cohomology groups $H^1(T_1, G)$ is an example of a *cohomology problem*.

References

[Aha 90] D.Aharonov, U.Elias. *Invariant curves around a parabolic fixed point at infinity.*
Ergod. Th. Dynam. Sys., 10:209-229, 1990

[Ang 90] S.Angenent. Monotone recurrence relations their Birkhoff orbits and topological entropy.
Erg. Th. Dyn., Sys., 10:15-41, 1990

[Ang 92] S. Angenent S. *A remark on the topological entropy and invariant circles of an area preserving twist map.*
In "Twist mappings and their Applications", IMA Volumes in Mathematics, Vol. 44., Eds. R.Mc Gehee, K.Meyer, 1992

[Arn 89] V.I. Arnold. *Mathematical methods of classical mechanics.*
Springer, New York (etc.), 1989

[Avr 90] J.Avron, P. van Mouche, B.Simon. *On the measure of the spectrum for the almost Mathieu model.*
Commun. Math. Phys., 132:103-118, 1990

[Ban 92] J.Banks, J.Brooks,G.Cairns,G.Davis,P.Stacey. *On Devaney's definition of chaos.*
Americ. Math. Month.,99:332-334, 1992

[Bel 92] J.Bellisard. *Le papillon de Hofstadter.*
Astérisque, 206 Séminaire Bourbaki 44ème année.

[Ben 78] G.Benettin, J.Strelcyn. *Numerical experiments on the free motion of a point mass moving in a plane convex region: Stochastic transition and entropy.*
Phys. Review A, 17:, 1978

[Bbgp 92] A.Bobenko,M.Bordemann,C.Gunn,U.Pinkall. *On two integrable cellular automata.*
SFB 288 Preprint No.25, Technische Universität Berlin

[Bra 81] M.Braun. *Mathematical remarks on the van Allen radiation belt: A survey of old and new results.*
SIAM Review, 23:61-93, 1981

[Bra 81a] M.Braun. *Invariant curves, homoclinic points and ergodicity in area-preserving mappings.*
SIAM J.Math. Anal, 12:630-638, 1981

[Car 87] N.Caranicolas, Ch.Vozikis. Chaos in a quartic dynamical model.
Celestial Mechanics40:35-49, 1987

[Con 89] G.Contopoulos, B.Barbanis. Lyapunov characteristic numbers and the structure of phase space.
Astron. Astrophys., 222:329-343, 1989

[Cyc 87] H.L.Cycon,R.G.Froese,W.Kirsch,B.Simon. *Schroedinger Operators.*
Texts and Monographs in Physics, Springer, 1987

[Dev 80] R.Devaney. *Triple Collision in the Planar Isosceles Three Body Problem.*
Invent. Math., 60:249-267, 1980

[Dev 84] R.Devaney. *A piecewise linear model for the zones of instability of an area preserving map.*
Physica D10:387-393, 1984

[Dev 89] R.Devaney. *An introduction to chaotic dynamical systems.*
Redwood City, California (etc.) : Addison-Wesley, 1989

[Don 88a] V.Donnay. *Geodesic flow on the two-sphere, Part I: Positive measure entropy.*
Ergod. Th. Dynam. Sys., 8:531-553, 1988

[Don 88b] V.Donnay. *Geodesic flow on the two-sphere, Part II: Ergodicity.*
Lect. Notes. Math., 1342: 112-153, 1988

[Don 91] V.Donnay. *Using Integrability to produce Chaos: Billiards with positive Entropy.*
Commun. Math. Phys., 141:225-257, 1991

[Dua 82] R.Douady. *Application du théorme des tores invariants.*
These Université Paris VII, 1982

[Fon 90] E.Fontich. *Transversal Homoclinic Points of a Class of Conservative Diffeomorphisms.*
J.Diff.Equ., 87:1-27 (1990)

[Fro 90] J.Fröhlich, T.Spencer, P.Wittwer. *Localization for a Class of One Dimensional Quasi-Periodic Schrödinger Operators.*
Commun. Math. Phys.,132:5-25, 1990

[Gal 81] L.Galgani, A.Giorgilli,J.M.Strelcyn. *Chaotic Motions and Transition to stochasticity in the classical problem of the heavy rigid body with a fixed point.*
Il nuovo cimento, 61 B, No 1 , 1981

[Ghe 92] J-M. Ghez. *Schrödinger Operators generated by substitutions*
Report No. 3, Institut für Angewandte Analysis und Stochastik,   Berlin, 1992

[Gui 87] V.Guillemin. *Some inverse spectral results on Riemannian manifolds.*
AMS Lecture, Mimeographed copy of a preprint.

[Gut 92] E.Gutkin, N.Simanyi. *Dual polygonal billiards and necklace dynamics.*
Commun. Math. Phys., 143:431-449, 1992

[Hay 87] A.Hayli,T.Dumont, J.M.Ollangier, J.M. Strelcyn. *Quelques résultats nouveaux sur les billard de Robinik.*
J.Phys. A : Math. Gen. :2337-3249, 1987

[Hen 64] M.Henon, C.Heiles. *The applicability of the third integral of motion some numerical experiments.*
Astronomical Journal, 69:73-79, 1964

[Hen 83] M.Henon, J.Wisdom. *The Benettin-Strelcyn oval billiard revisited.* Physica D, 8:157-169, 1983

[Hof 92] A.Hof. *Some remarks on discrete aperiodic Schrödinger operators.* Preprint ETH Zürich, December 1992.

[Kot 89] S.Kotani. *Jacobi matrices with random potentials taking finitely many values.* Rev. in Mathematical Phys., 1:129-133, 1989

[Lan 83] O.Lanford III. *Introduction to the mathematical theory of dynamical systems.* Les Houches Session XXXVI, Editors G.Iooss, R.Helleman, R.Stora, North-Holland Publishing Company, Amsterdam 1983

[Las 90] J.Laskar. *The chaotic motion of the solar system: a numerical estimate of the size of the chaotic zones.* Icarus, 88:266-291, 1990

[Las 93] Y.Last. *A relation between a.c. spectrum of ergodic Jacobi matrices and the spectra of periodic approximants.* Commun. Math. Phys., 151:183-192, 1993

[Las 93] Y.Last. Zero measure spectrum for the almost Mathieu operator. Technion preprint March, 1993

[Prz 82] F.Przytycki. *Examples of conservative diffeomorphisms of the two-dimensional torus with coexistence of elliptic and stochastic behaviour.* Ergod. Th. Dynam.Sys, 2:439-463, 1982

[Rob 83] M.Robnik. *Classical dynamics of a family of billiards with analytic boundaries.* J.Phys.A : Math. Gen., 16:3971-3986, 1983

[Rue 90] D.Ruelle. *Deterministic chaos: the science and the fiction.* Proc. R.Soc. Lond. A, 427:241-248, 1990

[Rue 87] D.Ruelle. *Chaotic evolution and strange attractors.* Cambridge university press, Cambridge 1997

[Spe 86] T.Spencer. *Some rigorous results for random and quasiperiodic potentials.* Phys. A, 140:70-77, 1986

[Str 89] J.-M.Strelcyn. *The Coexistence Problem for conservative dynamical systems:A review.* Colloquium Mathematicum, LXII:331-345, 1991

[Tab 93] S.Tabachnikov. *Commuting dual billiard maps.*
Preprint FIM Febr. 1993

[Viv 87] F.Vivaldi, A.Shaidenko. *Global Stability of a class of discontinuous dual billiards.*
Commun. Math. Phys., 110:625-640, 1987

[Wis 87] J.Wishdom. *Chaotic behaviour in the solar system.*
in "Dynamical chaos", Proceedings of a Royal Society discussion Meeting,
Eds. M.Berry, I.Percival, N.Weiss, Princeton, 1987

[Woj 86] M.Wojtkowski. *Principles for the Design of Billiards with Nonvanishing Lyapunov Exponents.*
Commun. Math. Phys., 105:391-414, 1986.

[Zha 93] X-H.Zhao, K-H.Kwek,J-B.Li, K-L.Huang *Chaotic and resonant streamlines in the ABC flow.*
SIAM J. Appl. Math, 53:71-77, 1993

Density results for positive Lyapunov exponents

Contents

Abstract

Let T be an aperiodic automorphism of a standard probability space (X, m) and K be either C or R. We prove that the set

$$\{A \in L^\infty(X, SL(2, \mathsf{K})) \mid \lim_{n \to \infty} n^{-1} \log(\|A(T^{m-1}x) \cdots A(Tx)A(x)\|) > 0, \quad a.e.\}$$

is dense in $L^\infty(X, SL(2, \mathsf{K}))$.

We apply this to show that the set of coboundaries are dense in $L^\infty(X, SO(2, \mathsf{R}))$ or that coboundaries are dense in $L^\infty(X, SU(2))$.

1 Introduction

Lyapunov exponents are useful in different domains of mathematics: in smooth ergodic theory, one is interested in the ergodic behavior of smooth maps. The Lyapunov exponents can be used to determine, if a power of a given smooth map is equivalent to a Bernoulli automorphism on a set of positive measure . Moreover, with Pesin's formula the metric entropy can be expressed as a function of the Lyapunov exponents [Pes 77][Kat 86]. In the theory of discrete one-dimensional Schrödinger operators, Lyapunov exponents can decide if the spectrum of such operators is not absolutely continuous [Cyc 87].

Thus, there is a strong enough motivation to calculate these numbers. It seems to be a matter of fact that calculating or even estimating the Lyapunov exponents is a difficult problem. Attempts in this direction have been made by several people: Wojtkowsky estimated the Lyapunov exponents under the assumption that an invariant cone bundle in the phase space exists [Woj 86]. The subharmonicity property of the Lyapunov exponents was used by Herman [Her 83] in order to estimate them for certain special mappings. In the case of random matrices, a criterion of Furstenberg, [Fur 63] later generalized by Ledrappier [Led 84] applies. Chulaevsky has analysed the skew product action and claimed that the Lyapunov exponents are positive for certain cocycles arising in the theory of Schrödinger operators [Chu 89]. This research is a part of the problem to prove localisation for quasi-periodic Schrödinger operators [Fro 90], [Chu 91]. Young investigated certain random perturbations of cocycles [You 86] (see also [Led 91] for higher dimensional generalisations). She proved that with this random noise the Lyapunov exponents depend continuously on the cocycle. Furthermore, they converge to the Lyapunov exponents of the unperturbed cocycle as the noise is reduced to zero. While analysing holomorphic parametrized cocycles Herman [Her 83] found a surprising result, which will be crucial for our work. (See section 2.4). Ruelle [Rue 79a] investigated the case in which a continuous cone bundle in the phase space is mapped into its interior. He found that in this case the Lyapunov exponents depend real analytically on the cocycle. The method of Herman was refined in [Sor 91],[Gol 92],[Gol 92a] to much more general situations. A new method was designed by L.-S. Young in [You 92] to construct open sets of nonuniform hyperbolic cocycles in a C^1 topology of cocycles.

Numerical experiments suggest that in the case of $SL(2,\mathbb{R})$-cocycles, positive Lyapunov exponents are quite frequent. Such cocycles arise in the case of smooth area preserving mappings on a two dimensional manifold or if one investigates one-dimensional discrete ergodic Schrödinger operators. Even for very special systems like the standard mapping of Chirikov on the two-dimensional torus, no estimates for the numerically measured Lyapunov exponents [Par 86] are available. There is still a wide gap between the numerical measurements and the effectively proved properties.

We want to investigate the Banach algebra $\mathcal{X}$ of all essentially bounded $M(2,\mathbb{R})$-cocycles over a given dynamical system. We are interested in the subset $\mathcal{P}$ of cocycles, where the upper and lower Lyapunov exponents are different almost everywhere. We show that $\mathcal{P}$ is dense in $\mathcal{X}$ in the L^∞ topology, if the underlying dynamical system is aperiodic. In $\mathcal{X}$ lies the Banach manifold $\mathcal{A}$ of $SL(2,\mathbb{R})$-cocycles which forms a multiplicative group. Also here, if the dynamical system is aperiodic, $\mathcal{A} \cap \mathcal{P}$ is dense in $\mathcal{A}$.

We find further that an arbitrary small perturbation located on sets with arbitrary small measure can bring us into $\mathcal{P}$. This can provide some explanation for the fact that one gets often positive Lyapunov exponents when making numerical experiments.

We will also make a statement about circle-valued cocycles which are here modeled as $SO(2,\mathbb{R})$-cocycles. The density result for positive Lyapunov exponents implies that the subgroup of coboundaries is dense in the abelian group of $SO(2,\mathbb{R})$-cocycles. We show also, that the statements stay true, if $\mathbb{R}$ is replaced by $\mathbb{C}$. Analogously, the subset of coboundaries is dense in the group of $SU(2,\mathbb{C})$-cocycles. We have separated the proof of the real case from the complex case, because for the later, more technical complications are involved, which make the proof less transparent.

In section 2, we introduce the concepts and cite some known results, which are used in section 3 to prove our statements. In section 4, the results and some open problems are shortly discussed .

2 Preparations

2.1 Matrix cocycles over a dynamical system

A *dynamical system* (X, T, m) is a set X with a probability measure m and a measurable invertible map T on X which preserves the measure m. The probability space (X, m) is assumed to be a Lebesgue space. The dynamical system is *aperiodic* if the set of periodic points

$$\{x \in X \mid \exists n \in \mathbb{N} \text{ with } T^n(x) = x\}$$

has measure zero. If the dynamical system is aperiodic, there exists for every $n \in \mathbb{N}, n > 0$ and every $\epsilon > 0$ a measurable set Y such that $Y, T(Y), \ldots, T^{n-1}(Y)$ are

41

pairwise disjoint and such that $m(Y_{rest}) \leq \epsilon$ where $Y_{rest} = X \setminus \bigcup_{k=0}^{n-1} T^k(Y)$. This is *Rohlin's lemma* (for a proof see [Hal 56]) and the set Y is called a $(n, \epsilon)-$ *Rohlin set.*

Denote by $M(2, \mathsf{R})$ the vector space of all real 2×2 matrices and by * matrix transposition. We will study the Banach space

$$\mathcal{X} = L^\infty(X, M(2, \mathsf{R})) = \{A : X \to M(2, \mathsf{R}) \mid A_{ij} \in L^\infty(X)\}$$

with norm

$$|||A||| = |\,||A(\cdot)||\,|_{L^\infty(X)} \,,$$

where $\| \cdot \|$ denotes the usual operator norm for matrices acting on C^2 with the Euclidean norm. Define also

$$\mathcal{A} = L^\infty(X, SL(2, \mathsf{R})) \,,$$

where $SL(2, \mathsf{R})$ is the group of 2×2 matrices with determinant 1. Take on $\mathcal{A}$ the induced topology from $\mathcal{X}$. Denote by $\circ$ matrix multiplication. With the multiplication

$$AB(x) = A(x) \circ B(x)$$

$\mathcal{X}$ is a Banach algebra. Denote by $A(T)$ the mapping $x \mapsto A(T(x))$. For $n > 0$, we write

$$A^n = A(T^{n-1})A(T^{n-2}) \cdots A$$

and $A^0 = 1$ where $1(x)$ is the identity matrix. With this notation, A satisfies the *cocycle-identity*

$$A^{n+m} = A^n(T^m)A^m$$

for $n, m \geq 0$. The mapping $(n, x) \mapsto A^n(x)$ is called a *matrix cocycle* over the dynamical system (X, T, m). With a slight abuse of language, we will just call the elements in $\mathcal{X}$ *matrix cocycles*. The Banach manifold $\mathcal{A} \subset \mathcal{X}$ is a multiplicative group. This group contains the commutative group

$$\mathcal{O} = L^\infty(X, SO(2, \mathsf{R})) \,,$$

where $SO(2, \mathsf{R}) = \{A \in SL(2, \mathsf{R}) \mid A^*A = 1\}$. The group $\mathcal{O}$ is called the group of *circle-valued cocycles* .

2.2 Lyapunov exponents and the multiplicative ergodic theorem

According to the *multiplicative ergodic theorem of Oseledec* (see [Rue 79]), the limit

$$M(A)(x) := \lim_{n \to \infty} ((A^n)^*(x)A^n(x))^{1/2n}$$

42

exists pointwise almost everywhere for an element $A \in \mathcal{X}$. Let

$$\exp(\lambda^-(A,x)) \leq \exp(\lambda^+(A,x))$$

be the eigenvalues of $M(A)(x)$. The numbers $\lambda^{+/-}(A,x)$ are called the *Lyapunov exponents* of A and they are measurable functions on X taking possibly also the value $-\infty$. We define

$$\mathcal{P} = \{A \in \mathcal{X} \mid \lambda^-(A,x) < \lambda^+(A,x), a.e.\}$$

and call the elements in $\mathcal{P}$ *nonuniform partial hyperbolic*. For $A \in \mathcal{P}$, there exists (still according to the multiplicative ergodic theorem) a measurable mapping from X into the space of all one-dimensional subspaces of R^2

$$x \mapsto W(x)$$

which is *coinvariant*

$$A(x)W(x) = W(T(x))$$

and such that for every $w \in W(x), w \neq 0$, we have

$$\lambda^-(A,x) = \lim_{n \to \infty} n^{-1} \log |A^n(x)w|$$

In the case $A \in \mathcal{A}$, we will refer to

$$\lambda(A,x) = \lambda^+(A,x) = -\lambda^-(A,x) \geq 0$$

as the *Lyapunov exponent*. It is for fixed $A \in \mathcal{A}$ a function in $L^\infty(X)$ and if $A \in \mathcal{P} \cap \mathcal{A}$, this function is positive almost everywhere. We shall call a cocycle in $\mathcal{A} \cap \mathcal{P}$ *nonuniform hyperbolic*. For $A \in \mathcal{X}$,

$$\lambda(A) = \lim_{n \to \infty} n^{-1} \int_X \log \|A^n(x)\| \, dm(x)$$

is the *integrated upper Lyapunov exponent*. The existence of this limit (taking possibly the value $-\infty$) can be seen also easily without knowledge of the multiplicative ergodic theorem, because the sequence

$$C_n = \int_X \log \|A^n(x)\| \, dm(x)$$

satisfies $C_{n+m} \leq C_n + C_m$.

The integrated Lyapunov exponent is in the same way also defined for complex-valued matrix cocycles $A \in L^\infty(X, M(2, \mathsf{C}))$. The next lemma gives a formula for the integrated Lyapunov exponent. In [Led 82] , one can find a more general version of this lemma.

43

Lemma 2.1 *If $A \in \mathcal{P} \cap \mathcal{A}$ and $w(x)$ is a unit vector in $W(x)$ then*

$$\lambda(A) = -\int_X \log |A(x)w(x)| \, dm(x) \; .$$

Proof. Call

$$\psi(x) = -\log |A(x)w(x)| \; .$$

From the multiplicative ergodic theorem, we have

$$
\begin{aligned}
\lambda(A) &= -\lim_{n\to\infty} n^{-1} \int_X \log |A^n(x)w(x)| \, dm(x) \\
&= \lim_{n\to\infty} n^{-1} \int_X \sum_{i=0}^{n-1} \psi(T^i(x)) \, dm(x)
\end{aligned}
$$

Birkhoff's ergodic theorem gives now

$$\lambda(A) = \int_X \psi(x) dm(x) = -\int_X \log |A(x)w(x)| dm(x) \; .$$

$\square$

2.3 Induced systems and derived cocycles

If $Z \subset X$ is a measurable set of positive measure, one can define a new dynamical system (Z, T_Z, m_Z) as follows: Denote by n(x) the *return time* for an element $x \in Z$, which is $n(x) = \min\{n \geq 1 | T^n(x) \in Z\}$. Poincaré's recurrence theorem implies, that $n(x)$ is finite for almost all $x \in Z$. Now, $T_Z(x) = T^{n(x)}(x)$ is a measurable transformation of Z, which preserves the probability measure $m_Z = m(Z)^{-1} m$. The system (Z, T_Z, m_Z) is called the *induced system* constructed from (X, T, m) and Z. It is ergodic, if (X,T,m) is ergodic (see [Cor 82]).
The cocycle $A_Z(x) = A^{n(x)}(x)$ is called the *derived cocycle* of A over the system (Z, T_Z, m_Z). If Y is a (n, ϵ)-Rohlin set and $Y_{rest} = X \setminus \bigcup_{k=0}^{n-1} T^k(Y)$, then the return time of a point in $Z = Y \cup Y_{rest}$ is less or equal to n. This implies, that for $A \in \mathcal{X}$, the entries of A_Z are in $L^\infty(Z)$.
In the following lemma 2.2, we cite a formula, which relates the integrated Lyapunov exponent of an induced system $\lambda(A_Z)$ with $\lambda(A)$. This formula is analogous to the *formula of Abramov* (see [Den 76]), which gives the metric entropy of an induced system from the entropy of the system. Lemma 2.2 is also stated in a slightly different form by Wojtkowsky [Woj 85].

Lemma 2.2 *If (X,T,m) is ergodic and $Z \subset X$ has positive measure, then*

$$\lambda(A_Z) \cdot m(Z) = \lambda(A).$$

Proof: Given $Z \subset X$ with $m(Z) > 0$, the return time

$$n(x) = \min\{n > 0 \mid T^n(x) \in Z\}$$

of almost all $x \in Z$ is finite. Define for $x \in Z$

$$N_k(x) = \sum_{i=0}^{k-1} n((T_Z)^i(x)) \, .$$

Because (X, T, m) is ergodic, we have for almost all $x \in Z$

$$
\begin{aligned}
\lambda(A_Z) &= \lim_{n \to \infty} n^{-1} \log \|(A_Z)^n(x)\| \\
&= \lim_{n \to \infty} n^{-1} \log \|A^{N_n(x)}(x)\| \\
&= \lim_{n \to \infty} \frac{N_n(x)}{n} \cdot \lim_{n \to \infty} N_n^{-1}(x) \log \|A^{N_n(x)}(x)\| \\
&= \lim_{n \to \infty} \frac{N_n(x)}{n} \cdot \lim_{k \to \infty} k^{-1} \log \|A^k(x)\| \\
&= \int_Z n(x) \, dm_Z(x) \cdot \lambda(A) \, .
\end{aligned}
$$

In the last equality, we have used Birkhoff's ergodic theorem applied to the system (Z, T_Z, m_Z) to get

$$\lim_{n \to \infty} N_n(x)/n = \int_Z n(x) dm_Z(x)$$

for almost all $x \in Z$. The *recurrence lemma of Kac* [Cor 82] gives

$$\int_Z n(x) dm_Z(x) = m(Z)^{-1} \, .$$

$\square$

We write

$$R(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \, .$$

Given a cocycle $A \in \mathcal{A}$, we are interested in the parametrized cocycle $\beta \mapsto AR(\beta)$ where $\beta \in \mathbb{R}/(2\pi\mathbb{Z})$.

Let Z be a measurable subset of X with positive measure. We denote with χ_Z the characteristic function of Z. We will use later the following little technical lemma:

Lemma 2.3 *For $A \in \mathcal{X}$ and $\beta \in \mathbb{R}$, we have $(AR(\chi_Z \cdot \beta))_Z = A_Z R(\beta)$*

Proof. Take a $x \in Z$ and call $k = n_Z(x)$. By definition

$$A_Z(x) R(\beta) = A(T^{k-1}(x)) \circ \cdots \circ A(x) \circ R(\beta) \, .$$

Because $T(x), T^2(x), \ldots, T^{k-1}(x)$ are not in Z, we have also

$$(AR(\chi_Z \cdot \beta))_Z(x) = A(T^{k-1}(x)) \circ \cdots \circ A(x) \circ R(\beta) \, .$$

$\square$

2.4 A result of M.Herman

We can write A(x) in the *QR decomposition* $A(x) = D(x) \circ R(\phi(x))$ with

$$D(x) = \begin{pmatrix} c(x) & b(x) \\ 0 & c^{-1}(x) \end{pmatrix} ,$$

where c, c^{-1}, and b are in $L^\infty(X)$ and $\phi : X \mapsto \mathbb{R}/(2\pi\mathbb{Z})$ is measurable. Crucial for us is the following result of [Her 83], the proof of which we will repeat here.

Proposition 2.4 (Herman) $\frac{1}{2\pi} \int_0^{2\pi} \lambda(AR(\beta)) \, d\beta \geq \int_X \log \sqrt{((c+c^{-1})^2 + b^2)/4} \, dm$.

Proof. Define the complex number $w = e^{i\beta}$ and the complex cocycle

$$B_w(x) = w \cdot e^{i\phi(x)} \cdot AR(\beta) .$$

Because $|w \cdot e^{i\phi(x)}| = 1$ we have

$$\lambda(AR(\beta)) = \lambda(B_w)$$

One can write

$$B_w(x) = D(x) \circ (G + w^2 \cdot e^{2i\phi(x)}\overline{G}) ,$$

where

$$G = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} .$$

We choose $r > 1$ and extend the definition of B_w from $\{|w| = 1\}$ to $\{|w| < r\}$. The mapping $w \mapsto B_w$ is a holomorphic mapping from $\{w| < r\}$ to the Banach algebra $L^\infty(X, M(2, \mathbb{C}))$. We claim that the mapping

$$w \mapsto \lambda(B_w)$$

is subharmonic. Proof. For each $n \in N$ and almost all $x \in X$, the mapping

$$w \mapsto b_n(w, x) = n^{-1} \log \|B_w^n(x)\|$$

is subharmonic on $\{|w| < r\}$, because

$$w \mapsto B_w^n(x)$$

is analytic there. Define for $k \in N$

$$a_{n,k}(w, x) = \max(b_n(w, x), -k) .$$

From Fubini's theorem follows that also

$$a_{n,k}(w) = \int_X a_{n,k}(w, x) dm(x)$$

46

is subharmonic. The sequence $k \mapsto a_{n,k}$ is decreasing and therefore

$$a_n(w) = \inf_{k \in \mathbb{N}} \{a_{n,k}(w)\} = n^{-1} \int_X \log \|B_w^n(x)\| dm(x)$$

is also subharmonic.
Finally, also

$$\lambda(B_w) = \inf_{n \in \mathbb{N}} \{a_n(w)\}$$

is subharmonic. We conclude

$$\frac{1}{2\pi} \int_0^{2\pi} \lambda(AR(\beta)) \, d\beta = \int_{|w|=1} \lambda(B_w) dw \geq \lambda(B_0)$$

and calculate with

$$L = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}$$

$$\begin{aligned} \lambda(B_0) &= \lambda(DG) \\ &= \lambda(L^{-1}DGL) \\ &= \int_X \log \sqrt{((c+c^{-1})^2 + b^2)/4} \, dm \ . \end{aligned}$$

$\square$

Denote by ν the Lebesgue measure on $\mathbb{R}/(2\pi\mathbb{Z})$. Herman's proposition implies the following corollary:

Corollary 2.5 $\nu\{\beta \in \mathbb{R}/(2\pi\mathbb{Z}) \mid \lambda(AR(\beta)) = 0\} \leq 1/\lambda(A)$.

Proof. Because

$$\lambda(AR(\beta)) \leq \int_X log\sqrt{(c+c^{-1})^2 + b^2} \, dm$$

for all $\beta \in \mathbb{R}/(2\pi\mathbb{Z})$, we have

$$\begin{aligned} \nu\{\beta \in \mathbb{R}/(2\pi\mathbb{Z}) \mid \lambda(AR(\beta)) = 0\} &\leq 1 - \frac{\int_X \log \sqrt{((c+c^{-1})^2 + b^2)/4} \, dm}{\int_X \log \sqrt{((c+c^{-1})^2 + b^2)} \, dm} \\ &= \log(2)/\int_X \log \sqrt{((c+c^{-1})^2 + b^2)} \, dm \\ &\leq \log(2)/\lambda(A) \leq 1/\lambda(A) \ . \end{aligned}$$

$\square$

3 Results

3.1 Density of nonuniform hyperbolicity

Theorem 3.1 *If the dynamical system (X, T, m) is aperiodic, then $\mathcal{P} \cap \mathcal{A}$ is dense in $\mathcal{A}$.*

Proof. We assume without loss of generality that the dynamical system is ergodic: In the general case, we can decompose the system in its *ergodic components* (See [Den 76]). The union of ergodic fibers, which are aperiodic, has measure 1. So, it is enough, to prove the statement for an ergodic aperiodic dynamical system.

Take an arbitrary $A \in \mathcal{A}$ and an $\epsilon > 0$. We show that there exists $B \in \mathcal{P} \cap \mathcal{A}$, with $|||B - A||| \leq \epsilon$. Choose first a constant $\mu > 1$ such that

$$2|||A||| \cdot |\mu - \mu^{-1}| \leq \epsilon/3 . \tag{1}$$

Take next $n \in \mathbb{N}$, $n > 0$ so big that

$$8\pi |||A||| \cdot \frac{\mu + \mu^{-1}}{n \cdot \log(\mu)} \leq \epsilon/3 . \tag{2}$$

Take now a $(n, 1/n)$-Rohlin set Y and call $Z = Y \cup Y_{rest}$, where

$$Y_{rest} = (X \setminus \bigcup_{k=0}^{n-1} T^k(Y)) .$$

Note that

$$m(Z) = m(Y) + m(Y_{rest}) \leq 2/n .$$

We can easily find a cocycle $C \in \mathcal{A}$, with

$$|||C - A||| \leq \epsilon/3 \tag{3}$$

such that $C_Z(z) \notin SO(2, \mathbb{R})$ for $z \in Z$ and so that additionally

$$|||C||| \leq 2|||A||| . \tag{4}$$

Looking at the induced system (Z, T_Z, m_Z) and the derived cocycle C_Z and applying Proposition 2.4 we see that there exists $\beta_0 \in \mathbb{R}/(2\pi\mathbb{Z})$, such that

$$\lambda(C_Z R(\beta_0)) > 0 .$$

Using lemma 2.3, we find that the cocycle

$$D = C R(\chi_Z \beta_0)$$

satisfies $D_Z = C_Z R(\beta_0)$ and so

$$\lambda(D_Z) = \lambda(C_Z R(\beta_0)) > 0 .$$

Lemma 2.2 gives
$$\lambda(D) = \lambda(D_Z) \cdot m(Z) > 0 \ .$$
Because the dynamical system is assumed to be ergodic, the Lyapunov exponents of D are nonzero almost everywhere and there exists according to the multiplicative ergodic theorem a function $x \mapsto W(x)$, which is coinvariant: $A(x)W(x) = W(T(x))$. Call $u(x) \in [0, \pi)$ the modulo π unique angle a unit vector $w(x) \in W(x)$ makes with the first basis vector in R^2. This means that the rotation $R(u(x))$ turns the first basis vector into the space $W(x)$. We use the notation

$$\mathrm{Diag}(\mu^{-1}) = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix} ..$$

The cocycle
$$E = R(u(T))\mathrm{Diag}(\mu^{-1})R(u(T))^{-1}D$$
has the same coinvariant direction $W(x)$ as D and if we take for $x \in X$ a unit vector $w(x) \in W(x)$, we can write using lemma 2.1

$$\begin{aligned} \lambda(E) &= -\int_X \log |E(x)w(x)| dm(x) \\ &= -\int_X \log |\mu^{-1}D(x)w(x)| dm(x) \\ &= -\int_X \log |D(x)w(x)| dm(x) + \log(\mu) \\ &= \lambda(D) + \log(\mu) \ . \end{aligned}$$

It follows with lemma 2.2 and $m(Z) \le 2/n$ that

$$\begin{aligned} \lambda(E_Z) &= \lambda(D_Z) + \frac{\log(\mu)}{m(Z)} \\ &\ge \lambda(D_Z) + \log(\mu) \cdot \frac{n}{2} \\ &\ge \log(\mu) \cdot \frac{n}{2} \ . \end{aligned}$$

Corollary 2.5 applied to the cocycle E_Z over the system (Z, T_Z, m_Z) implies that

$$\nu\{\beta \in \mathsf{R}/(2\pi\mathsf{Z}) | \lambda(E_Z R(\beta)) = 0\} \le \frac{2}{n \cdot \log(\mu)} \ .$$

We find therefore a $\beta_1 \in \mathsf{R}/(2\pi\mathsf{Z})$ with

$$\beta_1 \le \frac{4\pi}{n \cdot \log(\mu)} \tag{5}$$

such that $\lambda(E_Z R(\beta_1 - \beta_0)) > 0$. But then

$$B = E R(\chi_Z(\beta_1 - \beta_0)) \in \mathcal{P} \cap \mathcal{A}$$

49

because of lemma 2.2 and because lemma 2.3 implies that $B_Z = E_Z R(\beta_1 - \beta_0)$. We claim that $|||B - A||| \leq \epsilon$. To see this, we define

$$F = R(u(T))\text{Diag}(\mu^{-1})R(u(T))^{-1}C .$$

The norm of F can be estimated as

$$|||F||| \leq |||C||| \cdot (\mu + \mu^{-1}) . \tag{6}$$

Recall the definition of B

$$B = R(u(T))\text{Diag}(\mu^{-1})R(u(T))^{-1}CR(\chi_Z\beta_1) .$$

Using the inequalities 6, 4, 5 and 2, one gets

$$
\begin{aligned}
|||B - F||| &\leq |||F||| \cdot |\beta_1| \\
&\leq |||C||| \cdot (\mu + \mu^{-1}) \cdot |\beta_1| \\
&\leq 2|||A||| \cdot (\mu + \mu^{-1}) \cdot |\beta_1| \\
&\leq 8\pi|||A||| \cdot \frac{\mu + \mu^{-1}}{n \cdot \log(\mu)} \leq \epsilon/3 .
\end{aligned}
$$

Further, with the inequalities 1 and 4, we have the estimate

$$
\begin{aligned}
|||F - C||| &\leq |||C||| \cdot |||1 - R(u)\text{Diag}(\mu^{-1})R(u)^{-1}||| \\
&= |||C||| \cdot |||R(u)[1 - \text{Diag}(\mu^{-1})]R(u)^{-1}||| \\
&= |||C||| \cdot |||\text{Diag}(1 - \mu^{-1})||| \\
&\leq |||C||| \cdot |\mu - \mu^{-1}| \\
&\leq 2|||A||| \cdot |\mu - \mu^{-1}| \leq \epsilon/3 .
\end{aligned}
$$

From these two estimates and the inequality 3, the claim

$$|||B - A||| \leq \epsilon .$$

follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2   Density of nonuniform partial hyperbolicity

Corollary 3.2 *If the dynamical system (X,T,m) is aperiodic, then $\mathcal{P}$ is dense in $\mathcal{X}$.*

Proof. Given $\epsilon \geq 0$ and $A \in \mathcal{X}$, we will show that there exists $B \in \mathcal{P}$, such that $|||A - B||| \leq \epsilon$. Note first that the set

$$\{A \in \mathcal{X} \mid \exists \delta > 0, \det(A(x)) \geq \delta , a.e.\}$$

is dense in $\mathcal{X}$. Take an element C out of this set with

$$||| A - C ||| < \epsilon/2 \; .$$

Then $D = C/\det(C) \in \mathcal{A}$. We can apply theorem 3.1 to get $E \in \mathcal{A} \cap \mathcal{P}$, with

$$||| D - E ||| \leq \frac{\epsilon}{2} \cdot |\det(C)|_{L^\infty (X)} \; .$$

Call $B = E \cdot \det(C)$. Then

$$
\begin{aligned}
||| C - B ||| &= ||| D \cdot \det(C) - E \cdot \det(C) ||| \\
&\leq ||| D - E ||| \cdot |\det(C)|_{L^\infty (X)} \\
&\leq \epsilon/2
\end{aligned}
$$

and we got an element $B \in \mathcal{P}$ with

$$||| A - B ||| \leq ||| A - C ||| + ||| C - B ||| \leq \epsilon \; .$$

$\square$

3.3   Perturbations located on sets of small measure

The proof of theorem 3.1 gives even more information about how one can perturb a given cocycle $A \in \mathcal{A}$, to get into $\mathcal{A} \cap \mathcal{P}$: the proof shows that there exist cocycles H_1 and H_2 arbitrarily near to 1 such that $B = H_1 A H_2 \in \mathcal{P} \cap \mathcal{A}$.
The Lyapunov exponents of

$$C = H_1^{-1} B H_1(T^{-1}) = A H_2 H_1(T^{-1})$$

are the same as the Lyapunov exponents of B, because for every $n \in \mathsf{N}$

$$H_1(T^n)^{-1} B^n H_1(T^{-1}) = C^n \; .$$

This implies

Corollary 3.3 *If the dynamical system (X, T, m) is aperiodic there exists for each cocycle $A \in \mathcal{A}$ a cocycle $H \in \mathcal{A}$ arbitrary near the identity such that*

$$AH \in \mathcal{P} \cap \mathcal{A} \; .$$

In the same way, for $A \in \mathcal{X}$, there exist cocycles $H \in \mathcal{X}$ arbitrarily near the identity such that $AH \in \mathcal{P}$.

We think that the next corollary could be an explanation for the fact that one measures so often positive Lyapunov exponents in numerical experiments.

Corollary 3.4 *Assume (X, T, m) is aperiodic. Given $\epsilon > 0$. For $A \in \mathcal{A}$, there exists $B \in \mathcal{P} \cap \mathcal{A}$ (for $A \in \mathcal{X}$, there exists a $B \in \mathcal{P}$) such that $|||B - A||| \leq \epsilon$ and*

$$m\{x \in X \mid B(x) \neq A(x)\} \leq \epsilon \,.$$

Proof. Again, we can assume that the dynamical system (X, T, m) is aperiodic ergodic, because if the result is true for all aperiodic ergodic fibers of a given system, it is also true for the system itself.

Given $A \in \mathcal{A}$ (the case $A \in \mathcal{X}$ follows the same lines) and given $\epsilon \geq 0$. Take $n \in N$ with $2/n \leq \epsilon$ and choose a $(n, 1/n)$-Rohlin set $Y \subset X$ and define

$$Z = Y \cup (X \setminus \bigcup_{k=0}^{n-1} T^k(Y)) \,.$$

The above corollary 3.3 applied to the induced cocycle A_Z assures the existence of a cocycle H over the dynamical system (Z, T_Z, m_Z) such that

$$A_Z H \in \mathcal{P}$$

and

$$|||A_Z H - A_Z||| \leq \epsilon \,.$$

If we extend the cocycle H to X by setting it to 1 outside Z and define $B = AH$ we see by Lemma 2.2 that $B \in \mathcal{P}$, because

$$B_Z = A_Z H \in \mathcal{P} \,.$$

The cocycle B is different from A only on Z . We have $m(Z) \leq \epsilon$ and also

$$|||B - A||| \leq \epsilon \,.$$

$\square$

3.4 An application to circle valued cocycles

We say that $A, B \in \mathcal{A}$ are *cohomologous*, if there exists a $C \in \mathcal{A}$, such that

$$A = C(T) A C^{-1} \,.$$

What are the cohomology classes in $\mathcal{A}$? Cohomologous cocycles have the same Lyapunov exponents. So the Lyapunov exponents could help to distinguish different cohomology classes. One could ask further what are the cohomology classes in the set

$$\{A \in \mathcal{A} \mid \lambda(A) = \lambda_0\} \,,$$

where λ_0 is a given non-negative number. We will try to deal with this question. Cocycles cohomologous to 1 are called *coboundaries*. How big is the set of coboundaries? We have to restrict the problem still more in order to make a statement. By a *circle-valued cocycle*, we mean an element in

$$\mathcal{O} = L^\infty(X, SO(2, \mathbf{R})) \ .$$

The Banach manifold $\mathcal{O}$ is an abelian subgroup of $\mathcal{A}$. We say, $A \in \mathcal{O}$ is a *coboundary in* $\mathcal{O}$, if there exists a $B \in \mathcal{O}$ such that $A = B(T)B^{-1}$. It is not difficult to see (we will give a proof in the next chapter), that $A \in \mathcal{O}$ is a coboundary in $\mathcal{A}$ if and only if it is a coboundary in $\mathcal{O}$.

There are no methods known to decide in general if a given circle-valued cocycle is a coboundary or not. Even in the simplest cases, this remains a largely unsolved problem. There exist some analytic conditions in [Bag 88]. The problem has been studied intensively in the case, when the underlying dynamical system (X, T, m) is an irrational rotation of the circle [Mer 85] and where the cocycle takes only two values, because this has application in representations of Lie groups and in number theory. The concept is also important in the construction of ergodic skew products.

We call $\mathcal{C}$ the set of coboundaries in $\mathcal{O}$. They form a subgroup of $\mathcal{O}$. The abelian group $\mathcal{H}^1(T, SO(2, \mathbf{R})) = \mathcal{O}/\mathcal{C}$ is called the *first cohomology group* of the dynamical system. We don't know how to determine this group. What we can tell is said in the following theorem

Theorem 3.5 *If the dynamical system (X, T, m) is aperiodic, $\mathcal{C}$ is dense in $\mathcal{O}$.*

Proof. Because $\mathcal{O} \subset \mathcal{A}$ and $\mathcal{P}$ is dense in $\mathcal{A}$, we can find for every $A \in \mathcal{O}$ a sequence $A_n \to A$ with $A_n \in \mathcal{P}$. As in the proof of theorem 3.1, we can find for almost all $x \in X$ a rotation $R(u_n(x))$, which turns the first bases vector of R^2 into a coinvariant space $W_n(x)$ of A_n. We can then write

$$A_n(x) = R(u_n(T(x)))C_n(x)R(u_n(x))^{-1} \ ,$$

where $C_n(x)$ is upper tridiagonal. Because

$$A_n \to A \in \mathcal{O} \ ,$$

we must have $C_n \to 1$. Call

$$B_n(x) = R(u_n(T(x)))R(u_n(x))^{-1} \ .$$

Then $B_n \to A$ and $B_n \in \mathcal{C}$. $\qquad\square$

In a more general setup when the group $SO(2, \mathbf{R})$ is replaced by a general locally compact second countable abelian group, Parthasarathy and Schmidt [Part 77] have shown that a generic set of cocycles are coboundaries. They take the topology of convergence in measure which is a weaker topology then the uniform topology we have taken. Their result doesn't cover Theorem 3.5.

53

4 Complex Matrix cocycles

4.1 Density of nonuniform hyperbolicity for complex matrix cocycles

We consider now the Banach manifold

$$\mathcal{A}_C = L^\infty(X, SL(2, \mathbb{C}))$$

in the complex Banach algebra

$$\mathcal{X}_\mathbb{C} = L^\infty(X, M(2, \mathbb{C}))$$

and ask if also here

$$\mathcal{P}_\mathbb{C} = \{ A \in \mathcal{A}_\mathbb{C} \mid \lim_{n \to \infty} n^{-1} \log \| A^n(x) \| > 0 \ a.e. \}$$

is dense in $\mathcal{A}$, if the dynamical system is aperiodic.

The multiplicative ergodic theorem is true in general for matrix cocycles with entries in a local field ([Rag 79]). One can also deduce the multiplicative ergodic theorem over the complex field from the real case ([Rue 79]). The statement we need here is the following:
For $A \in \mathcal{X}_\mathbb{C}$, the limit

$$M(A) = \lim_{n \to \infty} ((A^n)^* A^n)^{1/2n}$$

exists pointwise almost everywhere. The Lyapunov exponents $\lambda^{+/-}(A, x)$ are defined again by the eigenvalues

$$\exp(\lambda^-(A, x)) < \exp(\lambda^+(A, x))$$

of $M(A)(x)$. For every A in the set of nonuniform partial hyperbolic cocycles

$$\mathcal{P}_\mathbb{C} = \{ A \in \mathcal{X}_\mathbb{C} \mid \lambda_-(A, x) < \lambda_+(A, x) \ a.e. \} \ ,$$

there exists a measurable mapping $x \mapsto W(x)$ from X into $\mathbb{C}^2$, which is coinvariant. Furthermore

$$\lambda^-(A, x) = \lim_{n \to \infty} n^{-1} \log \| A^n(x) w \|$$

for every $w \in W(x), w \neq 0$. The results 2.1, 2.2 and 2.3 are valid also word by word. The result of Herman has to be formulated a little bit differently:

Proposition 4.1 *Assume A can be written as $A = \text{Diag}(c)R$, where $R \in \mathcal{O}$. Then*

$$\frac{1}{2\pi} \int_0^{2\pi} \lambda(AR(\beta)) d\beta \geq \int_X \log\left(\frac{|c + c^{-1}|}{2}\right) dm \ .$$

54

The proof follows word by word the proof of proposition 2.4. Note, that because c is now allowed to be complex, the right hand side can become negative or even $-\infty$. We can also deduce from this a corollary:

Corollary 4.2 *Assume*
$$|c + c^{-1}| \geq (|c| + |c^{-1}|)/2,$$
then
$$\nu\{\beta \in \mathbb{R}/(2\pi\mathbb{Z}) \mid \lambda(AR(\beta)) = 0\} \leq 2/\lambda(A).$$

Proof. Because
$$\lambda(AR(\beta)) \leq \int_X \log(|c| + |c^{-1}|)\, dm$$
for all $\beta \in \mathbb{R}/(2\pi\mathbb{Z})$, we have

$$
\begin{aligned}
\nu\{\beta \in \mathbb{R}/(2\pi\mathbb{Z}) | \lambda(AR(\beta)) = 0\} &\leq 1 - \frac{\int_X \log(\frac{|c+c^{-1}|}{2})\, dm}{\int_X \log(|c| + |c^{-1}|)\, dm} \\
&\leq 1 - \frac{\int_X \log(\frac{|c|+|c^{-1}|}{4})\, dm}{\int_X \log(|c| + |c^{-1}|)\, dm} \\
&= \log(4)/\int_X \log(|c| + |c^{-1}|)\, dm \\
&\leq \log(4)/\lambda(A) \leq 2/\lambda(A).
\end{aligned}
$$

$\square$

We need still an other concept: dual to the construction of an induced dynamical system is the If (X, T, m) is a dynamical system *integral extension of a dynamical system* and $f \in L^1(X)$ is a positive integer-valued function, then one can define a new dynamical system (X^f, T^f, m^f), where

$$X^f = \{(x, i) \mid x \in X \text{ and } 1 \leq i \leq f(x)\}$$

and the measure m^f is defined as follows: for any measurable subset (Y, i) in X^f, one puts

$$m^f((Y, i)) = \frac{m(Y)}{\int f\, dm}.$$

This measure is preserved by the transformation

$$T^f(x, i) = \begin{cases} (x, i+1) & \text{if } i+1 \leq f(x) \\ (T(x), 1) & \text{if } i+1 > f(x) \end{cases}.$$

The space X^f can be visualized as a tower, whose foundation is X and which has $f(x)$ floors over each point $x \in X$. Under the action of T^f, a point (x, i) is lifted vertically up one floor, if this is possible and else lowered down to the ground floor,

where it takes the position of the point $(T(x), 1)$. The space X is identified with $(X, 1)$.

It is easy to see, that taking the induced system of the integral extension gives the old system back

$$(X^f_X, T^f_X, m^f_X) = (X, T, m) .$$

On the other hand, if (X_Y, T_Y, m_Y) is an induced system and

$$f(x) = \min\{n \geq 1 \mid T^n(x) \in Y\}$$

is the return time of a point $x \in Y$, then

$$(X_Y{}^f, T_Y{}^f, m_Y{}^f) = (X, T, m).$$

Also (X, T, m) is ergodic, if and only if (X^f, T^f, m^f) is ergodic.

We say, a cocycle A^f over the integral extension (X^f, T^f, m^f) is an *integral cocycle* of $A \in \mathcal{A}$, if

$$A^f_X = A$$

and $[A^f]_{ij} \in L^\infty(X^f)$. There are of course several possibilities to choose an integral cocycle. It follows from lemma 2.2, that

$$\lambda(A^f) = \frac{\lambda(A)}{\int_X f \, dm} .$$

We call $\mathcal{X}^f, \mathcal{A}^f, \mathcal{P}^f$ the spaces over (X^f, T^f, m^f), which play the role of $\mathcal{X}, \mathcal{A}, \mathcal{P}$ over (X,T,m). We also denote with $||| \cdot |||$ the norm in $\mathcal{X}^f$.

We define $\mathcal{U} = L^\infty(X, SU(2, \mathbb{C}))$, where

$$SU(2, \mathbb{C}) = \{A \in SL(2, \mathbb{C}) \mid A^*A = 1\}$$

is the space of special unitary matrices. We will need still the following lemma:

Lemma 4.3 *For every $A \in \mathcal{A}_\mathbb{C}$, there exists $U \in \mathcal{U}$, such that $U(T)AU^{-1} = DR$, where D is diagonal and $R \in \mathcal{O}$.*

Proof. Every $A \in \mathcal{A}_\mathbb{C}$ can be written as $A = U_1 D_1 U_2$, where $U_i \in \mathcal{U}$ and D_1 is diagonal. So, A is cohomologous to $B = D_1 U_3$ with $U_3 = U_2 U_1^{-1}(T^{-1})$.

Every $U \in \mathcal{U}$ can be written as

$$U = \text{Diag}(d)R(\phi)\text{Diag}(e)$$

with $d, e, \phi \in L^\infty(X)$.

We apply this to write U_3 as $U_3 = D_2 R D_3$, where $D_i \in \mathcal{U}$ are diagonal and $R \in \mathcal{O}$. The cocycle B is cohomologous to

$$C = D_3^{-1} D_1 D_2 R .$$

56

$\square$

We have now the tools to prove the result analogous to Theorem 3.1. The proof is very similar to the real case and there are only some technical changes which come from the fact that Herman's result is no more directly applicable.

Theorem 4.4 *If the dynamical system (X, T, m) is aperiodic, then $\mathcal{P}_\mathbf{c} \cap \mathcal{A}_\mathbf{c}$ is dense in $\mathcal{A}_\mathbf{c}$.*

Proof. Again, we assume (X, T, m) to be ergodic and deduce the general case from this by an ergodic decomposition. There will be no confusion, if we leave away the index $\mathbf{c}$ in $\mathcal{A}_\mathbf{c}$ and its subspaces during this proof.

Given $A \in \mathcal{A}$. It is enough to show, that for every $\epsilon > 0$ there exists $U \in \mathcal{U}$ and an integral extension C^f of
$$C = U(T)AU^{-1}$$
with $f(x) \leq 2$, such that
$$\|C^f(x)\| \leq 2\||A|\|^2$$
and such that for some $B^f \in \mathcal{P}^f$ we have $\||B^f - C^f\|| \leq \epsilon$. The reason for this is, that we can define then $B = B^f{}_X$, so that also $U(T)^{-1}BU$ is in $\mathcal{P}$ and close to A independent of the extension and U

$$
\begin{aligned}
\||U(T)^{-1}BU - A\|| &= \||U(T)(U(T)^{-1}BU - A)U^{-1}\|| = \||B - C\|| \\
&\leq 2\||B^f - C^f\|| \cdot \||A|\|^2 \\
&\leq 2\||A|\|^2 \cdot \epsilon .
\end{aligned}
$$

Choose first a constant $\mu > 1$ such that
$$2\||A|\|^2 \cdot |\mu - \mu^{-1}| \leq \epsilon/2 .$$

Take next $n \in N, n > 0$ so big, that
$$16\pi\||A|\|^2 \cdot \frac{\mu + \mu^{-1}}{n \cdot \log(\mu)} \leq \epsilon/2 .$$

Take now a $(n, 1/n)$-Rohlin set Y and call $Z = Y \cup Y_{rest}$. We look at the derived cocycle A_Z over the induced system (Z, T_Z, m_Z). From lemma 4.3 follows, that there exists $U \in \mathcal{U}_Z$ such that $U(T_Z)A_ZU^{-1} = \text{Diag}(c_1)R$ where $R \in \mathcal{O}$. We have
$$C_Z = (U(T)AU^{-1})_Z = U(T_Z)A_ZU^{-1} = \text{Diag}(c_1)R .$$

Define
$$
h(x) = \begin{cases} 2 & 1 \leq |c_1(x)| \leq 2 \\ 2^{-1} & 2^{-1} \leq |c_1(x)| < 1 \\ 1 & \text{else} . \end{cases}
$$

With $f(x) = 1 + \chi_Z(x)$ we construct the integral cocycle

$$C^f(x,1) = \begin{cases} \text{Diag}(h(x))C(x) & x \in Z \\ C(x) & \text{otherwise} \end{cases}$$

$$C^f(x,2) = \text{Diag}(h(x)^{-1}),$$

which satisfies $C^f_X = C$. Define $Z^f = (Z,1) \cup (Z,2)$. Then $(C^f)_{Z^f} = \text{Diag}(c)R$ with

$$|c + c^{-1}| \geq (|c| + |c|^{-1})/2 .$$

We can also estimate

$$||C^f(x)|| \leq 2|||A||| .$$

The aim is now, to find $B^f \in \mathcal{P}^f_C$, with $|||B^f - C^f||| \leq \epsilon$, which will finish the proof because then

$$|||B - C||| \leq 2\epsilon .$$

From now on, we skip the upper index f. Also this will give no confusion, because we don't need any more the old dynamical system (X, T, m). The only systems are now (X^f, T^f, m^f), which we will denote now with (X,T,m) and the induced system (Z, T_Z, m_Z) which would have had before the name $(X^f_{Z^f}, T^f_{Z^f}, m^f_{Z^f})$. The proof follows now closely the old proof in the real case.

We have seen, that the derived cocycle C_Z can be written as $C_Z = \text{Diag}(c)R$, where c satisfies

$$|c + c^{-1}| \geq (|c| + |c|^{-1})/2$$

because

$$|c| \notin [\frac{1}{2}, 2] .$$

Applying proposition 4.1, we see that there exists $\beta_0 \in \mathsf{R}/(2\pi \mathsf{Z})$, such that

$$\lambda(C_Z R(\beta_0)) > 0 .$$

The cocycle

$$D = CR(\chi_Z \beta_0)$$

satisfies $D_Z = C_Z R(\beta_0)$ and so $\lambda(D) > 0$. Because then the Lyapunov exponents are different from zero almost everywhere, there exists a mapping $x \mapsto W(x)$ from X to the 1-dimensional complex subspaces of $\mathbf{C}^2$, which is coinvariant: $A(x)W(x) = W(T(x))$.

Call U the element in $\mathcal{U}$ which turns the first basis vector into the space $W(x)$. The cocycle

$$E = U(T)\text{Diag}(\mu^{-1})U(T)^{-1}D$$

has the same coinvariant direction $W(x)$ as D and if we take for $x \in X$ a unit vector $w(x) \in W(x)$, we can write

$$\lambda(E) = -\int_X \log|E(x)w(x)| \, dm(x) = \lambda(D) + \log(\mu) .$$

58

It follows, that

$$\lambda(E_Z) \geq \log(\mu) \cdot \frac{n}{2} \ .$$

Corollary 4.2 applied to the cocycle E_Z over the system (Z, T_Z, m_Z) implies that we can find a $\beta_1 \in R/(2\pi Z)$ with

$$\beta_1 \leq \frac{8\pi}{n \cdot \log(\mu)}$$

such that $\lambda(E_Z R(\beta_1 - \beta_0)) > 0$. But then

$$B = ER(\chi_Z(\beta_1 - \beta_0)) \in \mathcal{P}_C \cap \mathcal{A}_C$$

because $B_Z = E_Z R(\beta_1 - \beta_0)$.
We claim, that $|||B - C||| \leq \epsilon$. To see this we define

$$F = U(T)\text{Diag}(\mu^{-1})U(T)^{-1}C \ .$$

The norm of F can be estimated as $|||F||| \leq |||C||| \cdot (\mu + \mu^{-1})$. Recall the definition of B

$$B = U(T))\text{Diag}(\mu^{-1})U(T)^{-1}CR(\chi_Z\beta_1) \ .$$

We get

$$
\begin{aligned}
|||B - F||| &\leq |||F||| \cdot |\beta_1| \leq |||C||| \cdot (\mu + \mu^{-1}) \cdot |\beta_1| \\
&\leq 2|||A||| \cdot (\mu + \mu^{-1}) \cdot |\beta_1| \leq 16\pi|||A||| \cdot \frac{\mu + \mu^{-1}}{n \cdot \log(\mu)} \leq \epsilon/2 \ .
\end{aligned}
$$

Further,

$$
\begin{aligned}
|||F - C||| &\leq |||C||| \cdot |||1 - U\text{Diag}(\mu^{-1})U^{-1}||| = |||C||| \cdot |||U[1 - \text{Diag}(\mu^{-1})]U^{-1}||| \\
&= |||C||| \cdot |||\text{Diag}(1 - \mu^{-1})||| \leq |||C||| \cdot |\mu - \mu^{-1}| \\
&\leq 2|||A||| \cdot |\mu - \mu^{-1}| \leq \epsilon/2 \ .
\end{aligned}
$$

From these two estimates, the claim

$$|||B - C||| \leq \epsilon$$

follows. $\qquad\qquad\square$

4.2 Corollaries for complex cocycles

In the same way then in the real case, we can deduce some corollaries. Because the proofs are identical, we will leave them out.

Corollary 4.5 *If the dynamical system (X, T, m) is aperiodic, $\mathcal{P}_C$ is dense in $\mathcal{X}_C$.*

Corollary 4.6 *If (X, T, m) is aperiodic, the density results hold also with the metric*

$$d(A, B) = |||A - B||| + m\{x \in X | A(x) \neq B(x)\} .$$

The set of coboundaries in $\mathcal{U}$ don't form a group because $\mathcal{U}$ is not Abelian. But also here we get

Corollary 4.7 *If (X, T, m) is aperiodic, the coboundaries are dense in $\mathcal{U}$.*

Because $SU(2, \mathbf{C})$ is a $2 : 1$ covering of $SO(3, \mathbf{R})$, we get immediately

Corollary 4.8 *If (X, T, m) is aperiodic, the coboundaries are dense in*

$$L^\infty(X, SO(3, \mathbf{R})) .$$

Remark. Questions about coboundaries seem to be more subtle in the case of smooth cocycles. Nerurkar [Ner 88] shows that the set of not-coboundaries in the set of of C^r circle-valued cocycles is residual if the transformation T has a sufficient quick periodic approximation. Elliasson [Eli 91] looked at real analytic $o(3, \mathbf{R})$-valued cocycles over an ergodic Kronecker flow on the torus T^d where the rotation vector is Diophantine and showed that generically the skew product flow has a unique invariant measure which implies that not-coboundaries are residual in the space of cocycles.

5   Cocycles from Schrödinger operators

We would like also to calculate the Lyapunov exponent λ on the subset

$$\mathcal{B} = \left\{ A \in \mathcal{A} \mid A(x) = A(V, x) = \begin{pmatrix} V(x) & -1 \\ 1 & 0 \end{pmatrix}, V \in L^\infty(X) \right\}$$

of $\mathcal{A}$, which can be identified with the Banach space $L^\infty(X)$. To a *random Jacobi operator* $L = \sigma + \sigma^* + V$ belongs the cocycle $A(V) \in \mathcal{B}$. The Lyapunov exponent of the *transfer cocycles* $A(V + E)$, with energy parameter $E \in \mathbf{R}$ gives information about the spectrum of L. If $\lambda(A(V + E)) > 0$ for almost all $E \in \mathbf{R}$, then the spectrum has no absolutely continuous part.

We would like to know the size of

$$\mathcal{P} \cap \mathcal{B} = \{A(V) \mid \lambda(A(V)) > 0\}$$

or

$$(\mathcal{P} \cap \mathcal{B})^+ = \{A(V) \mid \lambda(A(V + E)) > 0, \text{for almost all } E \in \mathbf{R}\} .$$

One expects that for general $V \in L^\infty(X)$

$$A(gV) \in \mathcal{P} \cap \mathcal{B}$$

60

or even

$$A(gV) \in (\mathcal{P} \cap \mathcal{B})^+$$

for large enough $|g|$.

Proposition 5.1 *If (X, T, m) is aperiodic then $\mathcal{P} \cap \mathcal{B}$ is dense in $\mathcal{B}$.*

The idea of the proof is to use a induced dynamical system and to apply the density result for $\mathcal{P}$. The proof will need the following lemma.

Lemma 5.2 *For every $k > 0$ there exists a measurable set $Z \subset X$ of positive measure such that the return time of a point $z \in Z$ to Z is in the set $\{k, k+1, \ldots, 2k\}$.*

Proof. Take a $(k, 1/2)$ Rohlin set Y and build the *Kakutani tower* (also called *Kakutani skyscraper*)

$$X = \bigcup_i X_i \,,$$

where $X_1 = Y$ and

$$X_{n+1} = T(X_n) \setminus Y \,.$$

Each point in the set

$$Z = \bigcup_{j=1}^{\infty} T^{-k+1}(X_{kj})$$

has the return time in the set $\{k, \ldots, 2k\}$. $\square$

Proof of the proposition.
Fix a measurable set Z such that every point $z \in Z$ has return time in $\{4, \ldots, 8\}$.
Define $X_i = T^{i-1}(Z)$ for $i = 1, \ldots, 4$.
Given $A = A(V) \in \mathcal{B}$ and $\epsilon > 0$. We want to find $\hat{A} \in \mathcal{B}$ with

$$|||\hat{A} - A||| \le \epsilon$$

and $\hat{A} \in \mathcal{P}$. Write

$$A_Z(x) = A(T^3 x) A(T^2 x) A(Tx)) A(x) R(x) \,.$$

A direct calculation for

$$
\begin{aligned}
A_Z R^{-1} &= \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \\
&= A(T^3 x) A(T^2 x) A(Tx)) A(x) \\
&= A(V_4(x)) A(V_3(x)) A(V_2(x)) A(V_1(x))
\end{aligned}
$$

gives

$$
\begin{aligned}
d(x) &= 1 - V_2(x)V_3(x) \\
b(x) &= V_1(x) + V_3(x) - V_1(x)V_2(x)V_3(x) = V_3(x) + V_1(x)d(x) \\
c(x) &= -V_2(x) - V_4(x) + V_2(x)V_3(x)V_4(x) = -V_2(x) + V_4(x)d(x) \\
a(x) &= \frac{1 - b(x)c(x)}{d(x)}.
\end{aligned}
$$

By replacing $V_i(x)$ with some suitable $\tilde{V}_i(x)$ near $V_i(x)$, we can find for some $\epsilon_1 > 0$ a cocycle $\tilde{A} \in \mathcal{B}$ with

$$
|||\tilde{A} - A||| \leq \frac{\epsilon}{2}
$$

such that

$$
\tilde{A}_Z R^{-1}(x) = \begin{pmatrix} \tilde{a}(x) & \tilde{b}(x) \\ \tilde{c}(x) & \tilde{d}(x) \end{pmatrix}
$$

satisfies $|\tilde{d}(x)| \geq \epsilon_1 > 0$ and $|\tilde{d}(x) - 1| \geq \epsilon_1 > 0$. We apply now the density result of $\mathcal{P}$ in $\mathcal{A}$. For every $\delta > 0$ there exists

$$
\begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} R \in \mathcal{A}_Z \cap \mathcal{P}_Z
$$

such that

$$
\left|\left|\left| \begin{pmatrix} \tilde{a}(x) - \hat{a}(x) & \tilde{b}(x) - \hat{b}(x) \\ \tilde{c}(x) - \hat{c}(x) & \tilde{d}(x) - \hat{d}(x) \end{pmatrix} \right|\right|\right| \leq \delta.
$$

We define a cocycle $\hat{A} \in \mathcal{B}$ satisfying $\hat{A}(x) = A(\hat{V}_i)(x)$ for $x \in X_i$, where for $x \in \bigcup_{i=1}^4 X_i$

$$
\begin{aligned}
\hat{V}_3(x) &= \tilde{V}_3(x) \\
\hat{V}_2(x) &= (1 - \hat{d}(x))/\tilde{V}_3(x) \\
\hat{V}_4(x) &= (\hat{c} + \hat{V}_2)/\hat{d}(x) \\
\hat{V}_1(x) &= (\hat{b} - \hat{V}_3)/\hat{d}(x)
\end{aligned}
$$

and $\hat{A}(x) = \tilde{A}(x)$ for $x \in X \setminus \bigcup_{i=1}^4 X_i$. Because $\tilde{V}_3(x)$ and $\hat{d}(x)$ are bounded away from 0, this map $\hat{b}, \hat{c}, \hat{d} \mapsto (\hat{V}_1, \hat{V}_2, \hat{V}_3, \hat{V}_4)$ is continuous near $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$. By choosing δ small enough, we obtain for almost all $x \in Z$

$$
|\tilde{V}(x) - \hat{V}(x)| \leq \frac{\epsilon}{2}
$$

implying

$$
|||\tilde{A} - \hat{A}||| \leq \frac{\epsilon}{2}.
$$

Because $\hat{A}_Z$ is in $\mathcal{P}$, also $\hat{A} \in \mathcal{P}$. Furthermore

$$
|||A - \hat{A}||| \leq |||A - \tilde{A}||| + |||\tilde{A} - \hat{A}||| \leq \epsilon/2 + \epsilon/2 \leq \epsilon.
$$

$\square$

6 Discussion

6.1 Aperiodicity

The aperiodicity condition for the dynamical system (X, T, m) is necessary in all the results. A periodic ergodic dynamical system is just a cyclic permutation of a finite set X. If the cardinality of X is N, and $A \in \mathcal{A}$ is given, then the Lyapunov exponents of A can be written as

$$\lambda^{+/-}(A, x) = N^{-1} \log |\mu^{+/-}| \,,$$

where $\mu^{+/-}$ are the eigenvalues of A^N. The Lyapunov exponents are different, if and only if $|\text{tr}(A^N)| > 2$. We see, that $\mathcal{P} \cap \mathcal{A}$ is an open set in $\mathcal{A}$ and also $\mathcal{A} \setminus \mathcal{P}$ has nonempty interior.

Given $A \in \mathcal{O}$. If $A = B(T)B^{-1}$ is a coboundary, we have

$$A^N = B(T^N)B^{-1} = 1 \,.$$

The necessary condition $A^N = 1$ for A to be a coboundary is also sufficient: Let

$$A_1, A_2, ..., A_N$$

be the values A takes on the set $X = \{1, 2, ...N\}$ and assume $A^N = 1$. Define $B_1 = 1$ and

$$B_k = A_1 A_2 \cdots A_{k-1}$$

for k=2,..,N. Then $A_k = B_{k+1}B_k^{-1}$ for all $k = 1, ..., N$ and A is a coboundary. So, coboundaries form a $(N - 1)$-dimensional manifold in the N dimensional torus $\mathcal{O} = SO(2, \mathbb{R})^N$.

6.2 Open problems

- We conjecture that $\mathcal{P}$ is generic in $\mathcal{A}$ or that $\mathcal{P}$ contains even an open dense subset.

- Do there exists analogous results for $L^\infty(X, M(d, \mathbb{R}))$ and $L^\infty(X, M(d, \mathbb{C}))$ for $d > 2$?

- Can one find also a density result in $\mathcal{X}^0 = C^0(X, M(2, \mathbb{R}))$ if T is a homeomorphism on a compact metric space?

- Is it even possible to find analogous results in $\mathcal{X}^r = C^r(X, M(2, \mathbb{R}))$ if T is a C^r-diffeomorphism on a C^r-manifold?

- More powerful methods to decide if $A \in \mathcal{P}$ or not are still needed. Are the Lyapunov exponents of the standard mapping positive on a set of positive measure?

- We conjecture that for a general compact connected topological group G, the set of coboundaries are dense in $L^\infty(X, G)$.

References

[Bag 88] L.Baggett. *On Circle valued Cocycles of an ergodic measure preserving transformation.*
Israel J. Math.,61:29-38, 1988.

[Chu 89] V.Chulaevsky. *Lyapunov Exponents and Anderson Localisation for Discrete Schroedinger Operators.*
Preprint FIM ETH Zuerich, 1989

[Chu 91] V.Chulaevsky, Ya.G.Sinai. *The exponential localization and structure of the spectrum for 1D quasi-periodic discrete Schrödinger operators.*
Reviews in Mathem. Phys., 3: 241-284, 1991

[Cor 82] I.P.Cornfeld,S.V.Fomin,Ya.G.Sinai. *Ergodic Theory.*
Springer, New York, 1982

[Cyc 87] H.L.Cycon,R.G.Froese,W.Kirsch,B.Simon. *Schroedinger Operators.*
Texts Monographs Phys., Springer, 1987

[Den 76] M.Denker,C.Grillenberg,K.Sigmund. *Ergodic theory on compact spaces.*
Lecture Notes in Math.,No. 527,Springer 1976

[Eli 91] L.H.Eliasson. *Ergodic Skew-Systems on* $\mathsf{T}^d \times SO(3,\mathsf{R})$.
Preprint Forschungsinstitut für Mathematik ETH Zürich.

[Fro 90] J.Frhlich, T.Spencer, P.Wittwer. *Localization for a Class of One Dimensional Quasi-Periodic Schrödinger Operators.*
Commun. Math. Phys., 132:5-25, 1990

[Fur 63] H.Furstenberg. *Non-commuting random products.*
Trans.Amer.Math.Soc.,108:377-428, 1963

[Hal 56] P.Halmos. *Lectures on Ergodic Theory.*
The mathematical society of Japan, Tokyo, 1956

[Her 83] M.R.Herman. *Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2.*
Comment.Math.Helv., 58:453-502, 1983

[Kat 86] A.Katok,J.-M.Strelcyn. *Invariant manifolds, Entropy and Billards : Smooth maps with Singularities.*
Lecture Notes in Math., No. 1222, Springer 1986

[Led 82] F.Ledrappier. *Quelques Proprietés des Exposants Caracteristiques.*
Lecture Notes in Math., No. 1097,Springer 1982

[Led 84] F.Ledrappier. *Positivity of the exponents for stationary sequences of matrices.*
Lecture Notes in Math., No. 1186,Springer 1984

[Led 91] F.Ledrappier, L.-S. Young. *Stability of Lyapunov exponents.*
Ergod. Th. Dynam. Syst, 11:469-484, 1991

[Gol 92] I.Ya.Goldsheid, E.Sorets. *Lyapunov Exponents of the Schrödinger Equation with Quasi-Periodic Potential on a Strip.*
Commun. Math. Phys., 145:507-513, 1992

[Gol 92a] I.Ya.Goldsheid, E.Sorets. *Lyapunov Exponents of the Schrödinger Equation with Certain Classes of Ergodic Potentials.*
Preprint Nr. 147 Institut für Mathematik Ruhr-Universität Bochum, März 1992

[Mer 85] K.Merrill. *Cohomology of Step functions under irrational Rotations.*
Israel J. Math.,52:320-340,1985.

[Ner 88] M.G.Nerurkar. *On the construction of smooth ergodic skew-products.*
Ergod. Th. Dynam. Syst., 8:311-326, 1988

[Par 86] M.Paramio,J.Sesma. *A continued fraction Procedure to determine invariant directions in twist mappings .*
Phys. Lett. A, 117:333-336,1986

[Part 77] K.R.Parthasarathy, K.Schmidt. *On the cohomology of a hyperfinite action.*
mboxMh.Math., 84:37-48, 1977

[Pes 77] Ya.B.Pesin. *Characteristic Lyapunov Exponents and Smooth Ergodic Theory.*
Russian Math. Surveys, 32:55-114, 1977

[Rag 79] M.S.Raghunathan. *A proof of Oseledec's Multiplicative Ergodic Theorem.*
Israel J. of Math., 32:356-362, 1979

[Rue 79] D.Ruelle. *Ergodic theory of differentiable dynamical systems.*
Publ.math. de l'IHES, 50:27-58, 1979

[Rue 79a] D.Ruelle. *Analycity Properties of the Characteristic Exponents of Random Matrix Products.*
Adv. Math., 32:68-80, 1979

[Sor 91] E.Sorets, T.Spencer. *Positive Lyapunov Exponents for Schrödinger Operators with Quasi-Periodic Potentials.*
Commun. Math. Phys., 142: 543-566, 1991

[Woj 85] M.Wojtkowski. *Invariant families of cones and Lyapunov exponents.*
Ergodic Theory Dynamical Systems, 5:145-161, 1985

[Woj 86] M.Wojtkowski. *Principles for the Design of Billiards with Nonvanishing Lyapunov Exponents.*
Comm. Math. Phys.,105:391-414,1986.

[You 86] L.-S.Young. *Random Perturbations of Matrix Cocycles.*
Ergodic Theory Dynamical Systems,6:627-637,1986.

[You 92] L.-S.Young. *Some Examples of nonuniformly hyperbolic cocycles.*
Preprint, Arizona, 1992

Discontinuity and positivity of Lyapunov exponents

Contents

Abstract

Let T be an aperiodic automorphism of a standard probability space (X, m). Let $\mathcal{P}$ be the subset of $\mathcal{A} = L^\infty(X, SL(2, \mathbb{R}))$ where the upper Lyapunov exponent is positive almost everywhere.

We prove that the set $\mathcal{P} \setminus \text{int}(\mathcal{P})$ is not empty. So, there are always points in $\mathcal{A}$ where the Lyapunov exponents are discontinuous.

We show further that the decision whether a given cocycle is in $\mathcal{P}$ is at least as hard as the following cohomology problem: Can a given measurable set $Z \subset X$ be represented as $Y \Delta T(Y)$ for a measurable set $Y \subset X$?

1   Introduction

We want to investigate the Banach manifold

$$\mathcal{A} = L^\infty(X, SL(2, \mathbb{R}))$$

of all measurable bounded $SL(2, \mathbb{R})$-cocycles over a given aperiodic dynamical system (X, T, m). We are interested in the subset $\mathcal{P}$ of cocycles where the upper Lyapunov exponent

$$\lambda^+(A, x) = \lim_{n \to \infty} n^{-1} \log \|A^n(x)\|$$

is positive almost everywhere.

We have shown [Kni 2] that $\mathcal{P}$ is dense in $\mathcal{A}$. This could give some explanation why one encounters so often positive Lyapunov exponents when making numerical simulations.

Numerical experiments suggest also that the Lyapunov exponents behave irregular in dependence of parameters. We prove in this chapter that the set $\mathcal{P} \setminus \text{int}(\mathcal{P})$ is not empty. On this set the Lyapunov exponent is discontinuous. Discontinuity of Lyapunov exponents has been mentioned at different places (see [You 86]). The only published result we found is in [Joh 84] in the case of $sl(2, \mathbb{R})-$ cocycles over almost periodic flows. proved there that discontinuities can already occur when changing a real parameter of the cocycle. His situation is different from ours in that he has a special flow and special cocycles occurring in the theory of Schrödinger operators, where we have an aperiodic but else arbitrary discrete dynamical system.

The idea for producing examples where λ^+ is discontinuous, is to exchange the expanding and contracting directions of the cocycle. This idea is not new and has been used in [Kif 82] to give examples where the Lyapunov exponent of identically distributed independent random matrices depend discontinuously from the common distribution. Our situation is different, because Kifer changes the dynamical system and not the cocycle. We will see that the exchanging of expanding and contracting directions must be done carefully. It can happen that the exchanging is making stable a part of the unstable directions and unstable a part of the stable directions.

This, we don't want. A cohomology condition for measurable sets will assure that the stable and unstable directions become indistinguishable. This will give zero Lyapunov exponents. For certain cocycles, which we call weak, we can make such an exchanging by small perturbations.

We mention now some results which concern the regularity of the Lyapunov exponent: Hölder continuity (and in some cases even C^∞ smoothness) of the Lyapunov exponent with respect to a real parameter has been shown by le Page [Pag 89] in the case of independent identically distributed random matrices.

Ruelle's ([Rue 79a]) results show that there is an open set in $\mathcal{A}$ where the Lyapunov exponent is real analytic. It is the set

$$\mathcal{S} = \{A \in \mathcal{A} \mid \exists C \in \mathcal{A}, \ \exists \epsilon > 0, \ [C(T)AC^{-1}(x)]_{ij} \geq \epsilon\}$$

which is contained in int($\mathcal{P}$). One could call $\mathcal{S}$ the *uniform hyperbolic* part of $\mathcal{A}$ (or the set with exponential dichotomy [Joh 86]) and $\mathcal{P} \setminus \mathcal{S}$ the nonuniform hyperbolic part. The elements in $\mathcal{P} \setminus \text{int}\mathcal{P}$ which will be constructed here are not uniform hyperbolic. But we will see that we can choose such elements in the closure of $\mathcal{S}$.

Unsuccessful efforts to find more powerful methods to prove positivity of the upper Lyapunov exponent of $SL(2, \mathbb{R})-$ cocycles led us to believe that the question whether A is in $\mathcal{P}$ is difficult and subtle in general. We want to illustrate this by showing that the decision is at least as hard as deciding whether a certain circle-valued cocycle is a coboundary. The circle-valued cocycles considered here have the range $\mathbb{Z}_2 = \{1, -1\}$. The group $\mathcal{E}$ of such cocycles can be identified with the set of measurable subsets of X with group operation Δ. The elements in $Z\Delta T(Z)$ are called coboundaries and form a subgroup. We will prove that the positivity of the Lyapunov exponent of a cocycle can depend on the question whether a certain set is a coboundary or not. This question about coboundaries has been investigated in [Bag 88]. In the special case, when (X, T, m) is an irrational rotation on the circle and the sets considered are intervals, the problem has been treated in ([Vee 69], [Mer 85]). Even in this reduced form, the coboundary problem is still not solved.

2 Preparations

A *dynamical system* (X, T, m) is a set X with a probability measure m and a measure preserving invertible map T on X. We assume that (X, m) is a Lebesgue space and that the dynamical system is ergodic. The later involves no loss of generality because the arguments can be applied to each ergodic fiber of the ergodic decomposition in general. The dynamical system is called *aperiodic* if the set of periodic points

$$\{x \in X \mid \exists n \in N \ \text{with} \ T^n(x) = x\}$$

has measure zero.

Denote by $M(2,\mathbb{R})$ the vector space of all real 2×2 matrices equipped with the usual operator norm $\| \cdot \|$. In the Banach space

$$L^\infty(X, M(2,\mathbb{R})) = \{A : X \to M(2,\mathbb{R}) | \ A_{ij} \in L^\infty(X)\}$$

with norm $\||A\|| = | \ \|A(\cdot)\| \ |_{L^\infty(X)}$ lies the Banach manifold

$$\mathcal{A} = L^\infty(X, SL(2,\mathbb{R})) \, ,$$

where $SL(2, R)$ is the group of 2×2 matrices with determinant 1. Take on $\mathcal{A}$ the induced topology from $L^\infty(X, M(2,\mathbb{R}))$. Denote by o matrix multiplication. With the multiplication $AB(x) = A(x) \circ B(x)$ the space $L^\infty(X, M(2,\mathbb{R}))$ is a Banach algebra. Denote by $A(T)$ the mapping $x \mapsto A(T(x))$. For $n > 0$ we write

$$A^n = A(T^{n-1}) \cdots A(T)A$$

and $A^0 = 1$ where $1(x)$ is the identity matrix. The mapping $(n, x) \mapsto A^n(x)$ is called a *matrix cocycle* over the dynamical system (X, T, m). With a slight abuse of language we will call the elements in $\mathcal{A}$ *matrix cocycles* or simply cocycles.

Denote by * matrix transposition. According to the *multiplicative ergodic theorem of Oseledec* (see [Rue 79]) the limit

$$M(A)(x) := \lim_{n \to \infty} ((A^n)^*(x)A^n(x))^{1/2n}$$

exists almost everywhere for $A \in \mathcal{A}$. Let

$$\exp(\lambda^-(A, x)) \leq \exp(\lambda^+(A, x))$$

be the eigenvalues of $M(A)(x)$. The numbers $\lambda^{+/-}(A, x)$ are called the *Lyapunov exponents* of A. Because T is ergodic we write $\lambda^{+/-}(A)$ for the value, $\lambda^{+/-}(A, x)$ takes almost everywhere. Because $M(A)(x)$ has determinant 1, one has $-\lambda^-(A) = \lambda^+(A)$. We call $\lambda(A) = \lambda^+(A)$ *the Lyapunov exponent* of A and define

$$\mathcal{P} = \{A \in \mathcal{A} \mid \lambda(A) > 0\} \, .$$

For $A \in \mathcal{P}$, there exist two measurable mappings $W^{+/-}$ from X into the projective space $\mathbf{P}^1$ of all one dimensional subspaces of R^2 which satisfy the *co-invariance condition*

$$A(x)W^{+/-}(x) = W^{+/-}(T(x)) \, .$$

$W^{+/-}(x)$ are the eigenspaces of $M(A)(x)$. Given $A \in \mathcal{A}$ we can define the *skew product action* $T \times A$ on the space $X \times P^1$:

$$T \times A : (x, W) \mapsto (T(x), A(x)W) \, .$$

The projection π from $X \times P^1$ onto X defines a projection π^* of probability measures. We say, a probability measure μ on $X \times P^1$ *projects down* to $\pi^*\mu$. Ledrappier [Led 82] has found an addendum of the multiplicative ergodic theorem. We report here only a special case. For $W \in P^1$ we will always denote with w a unit vector in W.

Proposition 2.1 *a) If $A \in \mathcal{P}$, there exist exactly two ergodic $T \times A$ invariant probability measures $\mu^{+/-}$ on $X \times P^1$ which project down to m and one has*

$$\lambda^{+/-}(A) = \int_{X \times P^1} \log |A(x)w| \, d\mu^{+/-}(x,W) \; .$$

The measures $\mu^{+/-}$ have their support on

$$X^{+/-} = \{(x, W^{+/-}(x)) | x \in X\} \; .$$

b) For every ergodic $T \times A$ invariant probability measure μ which projects down to m

$$\lambda(A) = |\int_{X \times P^1} \log |A(x)w| \, d\mu(x,W)| \; .$$

Remark. Part a) of Proposition 2.1 has been stated also in [Her 81] and in the case of cocycles with random noise in [You 86].

Let $Z \subset X$ be a measurable set of positive measure. A new dynamical system (Z, T_Z, m_Z) can be defined as follows: Poincaré's recurrence theorem implies that the *return time*

$$n(x) = \min\{n \geq 1 \mid T^n(x) \in Z\}$$

is finite for almost all $x \in Z$. Now, $T_Z(x) = T^{n(x)}(x)$ is a measurable transformation of Z which preserves the probability measure $m_Z = m(Z)^{-1} \cdot m$. The system (Z, T_Z, m_Z) is called *the induced system* constructed from (X, T, m) and Z. It is ergodic if (X, T, m) is ergodic (see [Cor 82]).
The cocycle $A_Z(x) = A^{n(x)}(x)$ is called the *derived cocycle* of A over the system (Z, T_Z, m_Z). In the following lemma 2.2 we cite a formula which relates the Lyapunov exponent of an induced system $\lambda(A_Z)$ with $\lambda(A)$. This formula is analogue to the formula of Abramov (see [Den 76]) which gives the metric entropy of an induced system from the entropy of the system. Lemma 2.2 is also stated in a slightly different form by Wojtkowsky [Woj 85].

Lemma 2.2 (Wojtkowsky) *If $m(Z) > 0$ then $\lambda(A_Z) \cdot m(Z) = \lambda(A)$.*

Remark. Wojtkowsky gives the formula

$$\lambda(A_Z) = \int_Z n(x) \, dm_Z(x) \cdot \lambda(A).$$

The version given here follows with the recurrence lemma of Kac [Cor 82] which says

$$\int_Z n(x) \, dm_Z(x) = m(Z)^{-1} \; .$$

3 Cocycles with values in $\{1, -1\}$

We denote with $\mathcal{E}$ the set of $\{1, -1\}$−*valued cocycles*

$$\mathcal{E} = \{A \in \mathcal{A} \mid A(x) \in \{1, -1\}\} \ .$$

To each $A \in \mathcal{E}$ we can associate a measurable set

$$\psi(A) = \{x \in X \mid A(x) = -1\} \ .$$

It is easy to see that

$$\psi(A) \Delta \psi(B) = \psi(AB) \ ,$$

where Δ denotes the symmetric difference. The map ψ is invertible. So the group $\mathcal{E}$ is isomorphic to the group of measurable sets in X with group operation Δ. We call a measurable set Y a *coboundary* if there exists a measurable set Z such that $Y = Z \Delta T(Z)$. Also $A \in \mathcal{E}$ is called a *coboundary* if $\psi(A)$ is a coboundary. Call $\mathcal{C}$ the subgroup of coboundaries.
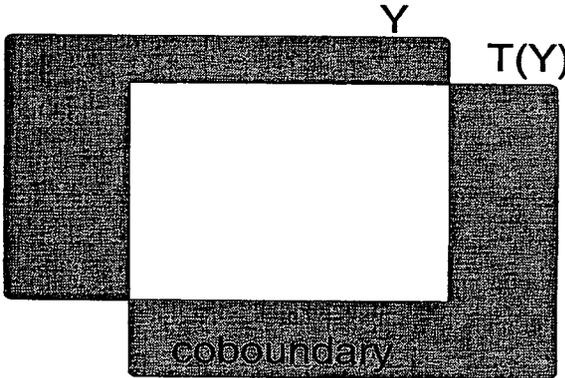
The group

$$\mathcal{H}(T, \mathbb{Z}_2) = \mathcal{E}/\mathcal{C}$$

is the cohomology group of measurable sets. We don't know how to determine it.

We will use the notation $Y^c = X \setminus Y$. Given a cocycle $A \in \mathcal{E}$ we can build a *skew product* $T \times A$ on $X \times \{1, -1\}$ as follows:

$$T \times A : (x, u) \mapsto (T(x), A(x)u) \ .$$

It leaves invariant the product measure $m \times \nu$ where ν is the measure $\nu(\{1\}) = \nu(\{-1\}) = 1/2$ on $\{1, -1\}$. One can see the skew product action $T \times A$ as follows: Take two copies of the dynamical system (X, T, m). The dynamics is then given on both copies as usual. Only when hitting the set $\psi(A)$ one jumps to the other system.

A necessary and sufficient condition under which the ergodicity of (X, T, m) implies the ergodicity of $(X \times \{1, -1\}, T \times A, m \times \nu)$ is given in the following result of Stepin [Ste 71]:

Proposition 3.1 (Stepin) *For $A \in \mathcal{E}$ the skew product $T \times A$ is ergodic if and only if A is not a coboundary.*

Proof. Assume $A = \psi^{-1}(Y)$ and $Y = Z \Delta T(Z)$. The set

$$Q = Z \times \{1\} \cup Z^c \times \{-1\}$$

is $T \times A$ invariant. Therefore $T \times A$ is not ergodic.

On the other hand, assume $A = \psi^{-1}(Y)$ and there exists a set $Q \subset X \times \{1, -1\}$ satisfying $0 < (m \times \nu)(Q) < 1$, which is $T \times A$ invariant. Let Z be defined by the equation

$$Q \cap (X \times \{1\}) = Z \times \{1\} .$$

One checks that $Y = Z \Delta T(Z)$. So, A is a coboundary. $\qquad \square$

Remark. There exists a generalization of the above result (as formulated in [Lem 89]). Take a compact abelian group G with Haar measure ν. A measurable map $A : X \to G$ is called a $G - cocycle$. Such a cocycle defines a skew product $T \times A$ on $X \times G$:

$$(T \times A)(x, g) = (T(x), A(x)g)$$

which preserves the measure $m \times \nu$. The result is that $T \times A$ is ergodic if and only if for any nontrivial character $\chi \in \hat{G}$ the circle-valued cocycle $x \mapsto \chi(A(x))$ is not a coboundary. The proof given in [Anz 51] in the case where G is the circle can be modified easily to prove the general result. As a special case, if G is the cyclic group of order 2, one gets the above result of Stepin.

Lemma 3.2 *A measurable set Y with $m(Y) > 0$ is a coboundary if and only if $(T_Y)^2$ is not ergodic.*

Proof. Assume first that Y is a coboundary $Y = Z \Delta T(Z)$. Call $Z_1 = Z \setminus T(Z)$ and $Z_2 = T(Z) \setminus Z$. Then $T_Y(Z_1) = Z_2$ and $T_Y(Z_2) = Z_1$ imply $(T_Y)^2(Z_1) = Z_1$. Therefore $(T_Y)^2$ is not ergodic because $0 < m(Z_1) < 1/2$.

If on the other hand $(T_Y)^2$ is not ergodic then $\exists Z \subset Y$ with $(T_Y)^2(Z) = Z$ and $0 < m(Z) < m(Y)$. We claim that $Y = Z \Delta T_Y(Z)$. Because

$$Z \cap T_Y(Z) = (T_Y)^2(Z) \cap T_Y(Z) = T_Y(T_Y(Z) \cap Z) ,$$

73

the ergodicity of T_Y implies that $Y = Z \cap T_Y(Z)$ or $m(Z \cap T_Y(Z)) = 0$. The first case implies $Y = Z$ which is not possible because of the assumption $m(Z) < m(Y)$. So $m(Z \cap T_Y(Z)) = 0$. The same argument with $Z' = Y \setminus Z$ implies

$$m(Z' \cap T_Y(Z')) = m(Y \setminus (Z \cup T_Y(Z))) = 0 \ .$$

From $Y = Z \cup T_Y(Z)$ and $m(Z \cap T_Y(Z)) = 0$ we get $Y = Z \Delta T_Y(Z)$. If $n(x)$ denotes the return time of a point $x \in Y$ to Y we define

$$U = \{T^k(x) | x \in Z, k = 0, \ldots, n(x) - 1\} \ .$$

Then

$$U \Delta T(U) = Z \Delta T_Y(Z) = Y$$

and Y is a coboundary. $\qquad\square$

We define on $\mathcal{E}$ the metric

$$d(A, B) = m(\{x \in X \mid A(x) \neq B(x)\}) = m(\psi(A) \Delta \psi(B))$$

which makes $\mathcal{E}$ to a topological group.

Proposition 3.3 *If the dynamical system is aperiodic then the set of coboundaries as well as its complement are both dense in $\mathcal{E}$ with respect to the metric d.*

Proof. It is known that the set of $A = \psi^{-1}(Y)$ such that $(T_Y)^2$ is not ergodic is dense in $\mathcal{E}$ (See [Fri 70] p. 125). Applying lemma 3.2 gives that coboundaries are dense.
It is known that the set of $A = \psi^{-1}(Y)$ such that T_Y is weakly mixing is dense in $\mathcal{E}$([Fri 70] p. 126). If T_Y is weakly mixing also $(T_Y)^2$ is weakly mixing ([Fur 81] p.83) and $(T_Y)^2$ must be ergodic. Apply again lemma 3.2. $\qquad\square$

Remarks.

• In proposition 3.3 has entered the assumption that the probability space (X, m) is a Lebesgue space. There exists an automorphism of a probability space such that each measurable set Y is a coboundary. (See [Akc 65].)

• Proposition 3.3 gives some indication that the decision whether a set is a coboundary or not might be subtle, especially when trying to deal with the question numerically.

• Let us mention that for an ergodic periodic dynamical system (X, T, m) a set $Z \subset X$ is a coboundary if and only if the cardinality of Z is even. This follows quickly from the above lemma 3.2. Proposition 3.3 is no more true in the periodic

case.

• Of course the construction of coboundaries is easy: take a measurable set $Z \subset X$ and form the coboundary $Y = Z \Delta T(Z)$. On the other hand, we don't know of an easy construction of sets which are not coboundaries.

Lemma 3.4 *Assume* $Z \subset Y \subset X$ *with* $m(Z) > 0$. *Then,* Z *is a coboundary for* T *if and only if* Z *is a coboundary for* T_Y.

Proof. Because $(T_Y)_Z = T_Z$ we have also $((T_Y)_Z)^2 = (T_Z)^2$. The claim follows with lemma 3.2. □

We will use the following corollary of the Proposition 3.3:

Corollary 3.5 *For every* $Y \subset X$ *with* $m(Y) > 0$ *there exists* $Z \subset Y$ *which is not a coboundary.*

Proof. Look at the dynamical system (Y, T_Y, m_Y). If (X, T, m) is aperiodic the proposition 3.3 assures that there exists $Z \subset Y$ such that Z is not a coboundary for T_Y. This means with lemma 3.4 that Z is not a coboundary for T. If (X, T, m) is periodic, choose $Z \subset Y$ which consists of one element. This Z is not a coboundary because $(T_Z)^2$ is trivially ergodic. □

4 Continuity and discontinuity of the Lyapunov exponent

Computer experiments indicate that the Lyapunov exponent λ is discontinuous. But from the topological point of view we have a big set, where λ is continuous. Recall that a subset of a topological space is called *generic* if it contains a countable intersection of open dense sets. The complement of a generic set is called *meager*.

Theorem 4.1 *The set*

$$\{A \in \mathcal{A} \mid \lambda \text{ is continuous in } A\}$$

is generic in $\mathcal{A}$.
$\mathcal{P} \setminus int\mathcal{P}$ *is meager.*

Proof. We can write

$$\lambda(A) = \lim_{n \to \infty} \lambda_n(A)$$

with

$$\lambda_n(A) = n^{-1} \int \log \|A^n\| \, dm .$$

So, λ is the pointwise limit of continuous functions λ_n. A theorem of Baire (see [Hah 32] p.221 or the Appendix of this chapter) states that the set of continuity points of such a function is generic.

The set $\mathcal{P} \backslash \mathrm{int}(\mathcal{P})$ is a subset of all the discontinuity points. It is therefore meager. $\square$

<u>Definition</u>. We say a cocycle $A \in \mathcal{P}$ *is weak on* $Y \subset X$ if the following three conditions are satisfied:

a) the return time to Y^c is unbounded,

b) $A(x) = 1$ for $x \in Y$,

c) $(1,0) \in W^+(x)$ and $(0,1) \in W^-(x)$ for $x \in Y$.

We call A *weak*, if $A \in \mathcal{P}$ and there exists $Y \subset X$ with $0 < m(Y) < 1$ such that A is weak on Y.

Lemma 4.2 *There exist weak cocycles if* (X, T, m) *is aperiodic.*

Proof. If the dynamical system is aperiodic, there exists for every $n \in N, n > 0$ and every $\epsilon > 0$ a measurable set Z such that $Z, T(Z), \ldots, T^{n-1}(Z)$ are pairwise disjoint and such that $m(\bigcup_{k=0}^{n-1} T^k(Z)) \geq 1 - \epsilon$. This is *Rohlin's lemma* (for a proof see [Hal 56]) and the set Z is called a $(n, \epsilon)-$ *Rohlin set.*

Define the set

$$ Y = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} T^k(Z_n) , $$

where Z_n is a $(n2^n, 1/2)$-Rohlin set. Then $m(Y) \leq 1/2$ and the return time to Y^c is not bounded. Take a diagonal cocycle $D(x) = \mathrm{Diag}(\mu(x), \mu^{-1}(x))$ with $\mu(x) = 1$ for $x \in Y$ and $\mu(x) = 2$ for $x \in Y^c$. This cocycle D is weak. $\square$

The main result in this section is:

Theorem 4.3 $\mathcal{P} \backslash \mathrm{int}(\mathcal{P})$ *is not empty if and only if* (X, T, m) *is aperiodic.*

For the proof we will need another lemma. Call

$$ R(\phi) = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} $$

and denote with $\mathcal{X}_Z$ the *characteristic function* of a measurable set $Z \subset X$.

Lemma 4.4 *If* $A \in \mathcal{P}$ *is weak on* Y *and* $Z \subset Y$ *is not a coboundary then the cocycle*

$$ B(x) = R(\pi/2 \cdot \mathcal{X}_Z(x)) A(x) $$

is in $\mathcal{A} \backslash \mathcal{P}$.

Proof. Given a cocycle A which is weak on $Y \subset X$. The two sets

$$X^{+/-} = \{(x, W^{+/-}(x)) | x \in X\} \subset X \times P^1$$

are invariant under the skew product action $T \times A$. We call $T^{+/-}$ the action of $T \times A$ restricted to $X^{+/-}$ and $\mu^{+/-}$ the two ergodic $T \times A$ invariant measures projecting down to m. The dynamical systems $(X^{+/-}, T^{+/-}, \mu^{+/-})$ are isomorphic to (X, T, m). Define

$$B(x) = R(\pi/2 \cdot \mathcal{X}_Z(x))A(x) ,$$

where $Z \subset Y$ is not a coboundary. The set $X^+ \cup X^-$ is invariant under $T \times B$ and $(\mu^+ + \mu^-)/2$ is an invariant measure of $T \times B$ which projects down to m. The system $(X^+ \cup X^-, T \times B, (\mu^+ + \mu^-)/2)$ is isomorphic to $(X \times \{1, -1\}, T \times \psi^{-1}(Z), m \times \nu)$ which we have met in the last section. Stepin's result implies that the measure $(\mu^+ + \mu^-)/2$ is an ergodic $T \times B$ invariant measure on $X \times P^1$. This gives then with proposition 2.1b)

$$\begin{aligned}
\lambda(B) &= |\int \log|A(x)w|d\mu^+(x, W) + \int \log|A(x)w|d\mu^-(x, W)|/2 \\
&= |\lambda^+(A) + \lambda^-(A)|/2 = 0 .
\end{aligned}$$

$\square$

Proof of theorem 4.3. Assume (X,T,m) is aperiodic. It is enough to show: if A is weak then $A \in \mathcal{P} \setminus \text{int}(\mathcal{P})$. With Lemma 4.2 follows then that $\mathcal{P} \setminus \text{int}\mathcal{P}$ is not empty. Let $A \in \mathcal{P}$ be weak on Y and let $\epsilon > 0$ be given. We will construct a $B \in \mathcal{A}$ such that $\lambda(B) = 0$ and $|||B - A||| \le \epsilon$. Take n so big, so that $|||A||| \cdot \pi/2n \le \epsilon$. Choose $V \subset Y^c$, such that $T(V), \ldots, T^n(V)$ are disjoint from Y^c and $m(V) > 0$. This is possible because the return time to Y^c is not bounded. Then there exists with corollary 3.5 a set $Z \subset V$ which is not a coboundary. Define

$$U = X \setminus \bigcup_{k=1}^{n} T^k(Z)$$

and look at the induced system (U, T_U, m_U). Then A_U is weak over $Y \cap U$ and with lemma 3.4 follows that Z is not a coboundary for T_U, because it is not a coboundary for T. Application of lemma 4.4 gives that

$$C = R(\pi/2 \cdot \mathcal{X}_Z)A_U$$

has zero Lyapunov exponent. Define the cocycle

$$B(x) = R(\pi/(2n) \cdot \mathcal{X}_{U^c}(x))A(x) .$$

We check that

$$B_U = C .$$

This gives with lemma 2.2 and $\lambda(C) = 0$ also $\lambda(B) = 0$. Further

$$|||B - A||| \leq |||A||| \cdot \pi/2n \leq \epsilon .$$

We have shown that a weak cocycle is in $\mathcal{P} \setminus \mathrm{int}\mathcal{P}$.
If (X, T, m) is periodic then the Lyapunov exponent is continuous and so $\mathcal{P} = \mathrm{int}\mathcal{P}$.

$\square$

Remarks.

• We say $A, B \in \mathcal{A}$ are *cohomologous in* $\mathcal{A}$ if there exists $C \in \mathcal{A}$, such that $C(T)AC^{-1} = A$. Cohomologous cocycles have the same Lyapunov exponents and if A is conjugated to a weak cocycle then it is also in $P \setminus \mathrm{int}\mathcal{P}$.
• It was surprising for us to find diagonal cocycles in $\mathcal{P} \setminus \mathrm{int}\mathcal{P}$. We expected that the arbitrary closeness of stable and instable directions are responsible for the discontinuity of the Lyapunov exponent. This can also be the case as the following remark indicates.

• Assume $A \in \mathcal{P} \setminus \mathrm{int}(\mathcal{P})$ and $A_n \to A$ with $\lambda(A_n) = 0$. Because $\mathcal{P}$ is dense in $\mathcal{A}$ (see [Kni 2]) we can find $B_n \to A$ with $|||B_n - A_n||| \leq 1/n$ and $B_n \in \mathcal{P}$. If μ_n denotes a $T \times B_n$ invariant probability measure projecting down to m then μ_n converges weakly to $(\mu^+ + \mu^-)/2$ where $\mu^{+/-}$ are the $T \times A$ invariant ergodic measures projecting down to m. In some sense the stable and unstable directions of B_n come closer and closer together as n is increasing.

5 Difficulty of the decision whether the Lyapunov exponent is positive

There are only a few methods to decide whether $A \in \mathcal{P}$ or not. The only method which works for general dynamical systems is Wojtkowsky's cone method [Woj 85]. But there are many examples where one measures positive Lyapunov exponent numerically without being able to prove it. This suggests that the general problem is difficult. The next theorem could be one of the reasons for the subtlety.

Theorem 5.1 *Given a measurable set $Y \subset X$ with $0 < m(Y) < 1$. There exists $A \in \mathcal{A}$, such that $B = R(\pi/2 \cdot \mathcal{X}_Y)A \in \mathcal{P}$ if and only if Y is a coboundary.*

Proof. Given $Y \subset X$ with $0 < m(Y) < 1$ we build the *Kakutani skyscraper* over Y, which is a partition $X = \bigcup_{i \geq 1} Y_i$, where $Y_1 = Y$ and $Y_{n+1} = T(Y_n) \setminus Y$.
We have $m(Y_2) > 0$ because $m(Y) < 1$. Define $U = Y_2$ and the diagonal cocycle $A(x) = \mathrm{Diag}(2, 2^{-1})$ for $x \in U$ and $A(x) = 1$ else. The Lyapunov exponent of A is

$$\lambda(A) = \log(2) \cdot m(U) > 0 .$$

Clearly the vector $(1,0)$ is in $W^+(x)$ and the vector $(0,1)$ is in $W^-(x)$. We denote with $\mu^{+/-}$ the two ergodic $T \times A$ invariant measures on $X \times P^1$ which project down to m and have their support on

$$X^{+/-} = \{(x, W^{+/-}(x)) | x \in X\} \ .$$

If Y is not a coboundary, we conclude like in the proof of Lemma 4.4 that $(\mu^+ + \mu^-)/2$ is an ergodic $T \times B$ invariant measure projecting down to m and so $\lambda(B) = 0$.
If Y is a coboundary, that is if $Y = Z \Delta T(Z)$, there is a $T \times A$ invariant set

$$Q = \{(x, W^+(x)) \mid x \in Z\} \cup \{(x, W^-(x)) \mid x \in Z^c\} \ .$$

This set Q carries an ergodic $T \times A$ invariant measure μ which projects onto the measure m. Because $U = Y_2$ is disjoint from Y either $U \subset Z \cap T(Z)$ or U is disjoint from $Z \cup T(Z)$. This implies $U \subset Z$ or $U \subset Z^c$ and we have either

$$\{(x, W^+(x)) \mid x \in U\} \subset Q$$

or

$$\{(x, W^-(x)) \mid x \in U\} \subset Q \ .$$

Because $A(x)$ is different from the identity matrix only on U and is there $\mathrm{Diag}(2, 2^{-1})$ we have

$$\begin{aligned}
\lambda(B) &= |\int_Q \log|A(x)w| \ d\mu(x, W)| \\
&= \log(2) \cdot m(U) = \lambda(A) > 0 \ .
\end{aligned}$$

$\square$

If we would have an algorithm to find out whether a given cocycle $A \in \mathcal{A}$ is in $\mathcal{P}$ or not we could also find out for a measurable set $Z \subset X$ if Z is a coboundary or not. So, the cohomology problem in $\mathcal{E}$ exhibits already a difficulty for calculating or estimating the Lyapunov exponents.

6 Open questions

Let us mention to the end some open questions:

• Assume T is a homeomorphism of a compact metric space X leaving a Borel probability measure m invariant. Is the upper Lyapunov exponent continuous on

$$C(X, SL(2, \mathbb{R})) = \{A : X \to SL(2, \mathbb{R}) \mid A \text{ continuous}\} \ ?$$

• We believe that the cohomology problem in $\mathcal{E}$ is difficult. Is there a difficult mathematical problem which is embeddable in the cohomology problem for measurable

sets?

• For which $r \geq 0$ is there a nonempty set in $\mathcal{A}$ such that the Lyapunov exponent is there r times but not $r + 1$ times differentiable?

• Can discontinuities of the Lyapunov exponent λ occur on special curves through $\mathcal{A}$? In the theory of random Jacobi matrices [Cyc 87] one would like to know about regularity properties of λ on the curve

$$E \mapsto A_E = \begin{pmatrix} V + E & -1 \\ 1 & 0 \end{pmatrix} ,$$

where $V \in L^\infty(X, R)$. Johnson [Joh 84] has examples for discontinuities in the case of almost periodic Schrödinger operators. An other interesting curve would be the circle

$$\beta \mapsto AR(\beta) = A \begin{pmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{pmatrix} .$$

Can we always find $A \in \mathcal{A}$, where $\beta \mapsto \lambda(AR(\beta))$ is not continuous?

• Is every $A \in \mathcal{P} \setminus \mathrm{int}(\mathcal{P})$ cohomologous to a weak cocycle ?

7 Appendix: A remark about cohomology

Let $\mathcal{E}$ denote the set of Z_2 valued cocycles, $\mathcal{O}$ the set of circle valued cocycles and $\mathcal{A}$ the set of $SL(2, R)$ cocycles over a dynamical system (X, T, m). The group $\mathcal{E}$ as the center of the multiplicative group $\mathcal{A}$ and at $\mathcal{O}$ as a maximal abelian subgroup of $\mathcal{A}$. We will prove, that $A \in \mathcal{E}$ is not a coboundary in $\mathcal{E}$, then A is neither a coboundary in $\mathcal{O}$ nor a coboundary in $\mathcal{A}$. This implies that if we find cocycles, which are not coboundaries in $\mathcal{E}$ then we have found also cocycles which are not coboundaries in $\mathcal{A}$.

Lemma 7.1 *(a) $A \in \mathcal{E}$ is a coboundary in $\mathcal{E}$ if and only if it is a coboundary in $\mathcal{O}$.*
(b) If $A_1, A_2 \in \mathcal{O}$ and $\exists C \in \mathcal{A}$ such that $A_1 = C(T)A_2C^{-1}$, then there exists $C \in \mathcal{O}$ with $A_1 = C(T)A_2C^{-1}$.
(c) $A \in \mathcal{O}$ is a coboundary in $\mathcal{O}$ if and only if it is a coboundary in $\mathcal{A}$.
(d) $A \in \mathcal{E}$ is coboundary in $\mathcal{E}$ if and only if it is a coboundary in $\mathcal{A}$.

Proof. We assume without loss of generality, that the dynamical system is ergodic. In the general case, we can decompose the system in its ergodic components and if the statement is true for every ergodic fiber, then it is also true for the system itself.

(a) Given $A \in \mathcal{E}$. We have only to show, that $\exists B \in \mathcal{O}$

$$A = B(T)B^{-1}$$

implies $\exists B \in \mathcal{E}$

$$A = B(T)B^{-1} .$$

The other direction is trivial. From the relation $A = B(T)B^{-1}$, we get $B(T) = \pm B$. Because the system is ergodic, $|B_{ij}(x)|$ is constant for all $i, j \in \{1, 2\}$. This means, that

$$B(x) = C(x)R(\phi(x)) ,$$

where $\phi(x)$ is a constant and $C(x) \in \{-1, 1\}$. We can choose $\phi = 0$ by replacing $B(x)$ by $B(x)R(\phi)^{-1}$. We have thus found $C \in \mathcal{E}$ with $A = C(T)C^{-1}$.

(b) Given $A_1, A_2 \in \mathcal{O}$ and $C \in \mathcal{A}$ such that $A_1 = C(T)A_2 C^{-1}$. We write $C = R_1 D R_2$, where $D = \mathrm{Diag}(\mu, \mu^{-1})$ is diagonal and $R_i \in \mathcal{O}$. Then

$$R_2(T)A_2 R_2^{-1} D = D(T)R_1(T)^{-1} A_1 R_1 ,$$

and we can write

$$B_2 D = D(T)B_1$$

for $B_i = R_i(T)A_i R_i^{-1}$. If $B_i(x)$ is a rotation about an angle $\phi_i(x)$, we have

$$\begin{pmatrix} \cos(\phi_2)\mu & \sin(\phi_2)\mu^{-1} \\ -\sin(\phi_2)\mu & \cos(\phi_2)\mu^{-1} \end{pmatrix} = \begin{pmatrix} \cos(\phi_1)\mu(T) & \sin(\phi_1\mu(T)) \\ -\sin(\phi_1)\mu(T)^{-1} & \sin(\phi_1\mu(T)^{-1}) \end{pmatrix} .$$

From these four equations, we can derive

$$\begin{aligned} \cos^2(\phi_1) &= \cos^2(\phi_2), \\ \sin^2(\phi_1) &= \sin^2(\phi_2) \end{aligned}$$

and

$$\cos(\phi_1)/\cos(\phi_2) = \mu/\mu(T) > 0$$

which implies $\mu = \mu(T)$ and $\phi_1 = \phi_2$. If $\phi_1 \neq 0$. We deduce from $\sin(\phi_1)\mu^{-1} = \sin(\phi_2)\mu(T)$, that $\mu = 1$. If $\phi_1 = 0$,

$$1 = B_1 = R_1(T)A_1 R_1^{-1} = R_2 A_2 R_2^{-1} = B_2 = 1$$

and $C' = R_1 R_2$ gives

$$C'(T)A_2 C'^{-1} = A_1 .$$

(c) is a special case of (b).

(d) follows from (a) and (c) together. $\qquad\qquad\square$

This has the following consequence for the cohomology groups:

Corollary 7.2 $\mathcal{H}(T, \mathbb{Z}_2)$ *is a subgroup of* $\mathcal{H}(T, SO(2, \mathbb{R}))$, *which is a subset of* $\mathcal{H}(T, SL(2, \mathbb{R}))$.

Proof. There is a natural inclusions $\mathcal{E} \subset \mathcal{O} \subset \mathcal{A}$. Denote in these groups the coboundaries with $\mathcal{C}_{\mathcal{E}}, \mathcal{C}_{\mathcal{O}}, \mathcal{C}_{\mathcal{A}}$. The cohomology groups are defined as $\mathcal{H}(T, \mathbb{Z}_2) = \mathcal{E}/\mathcal{C}_{\mathcal{E}}, \mathcal{H}(T, SO(2, \mathbb{R})) = \mathcal{O}/\mathcal{C}_{\mathcal{O}}$ and we have the set of cohomology classes $\mathcal{H}(T, SL(2, \mathbb{R})) = \mathcal{A}/\mathcal{C}_{\mathcal{A}}$. There is a trivial inclusion $\mathcal{C}_{\mathcal{E}} \subset \mathcal{C}_{\mathcal{O}} \subset \mathcal{C}_{\mathcal{A}}$. The above lemma 7.1 showed that $\mathcal{C}_{\mathcal{E}} = \mathcal{C}_{\mathcal{O}} \cap \mathcal{E}$, and $\mathcal{C}_{\mathcal{O}} = \mathcal{C}_{\mathcal{A}} \cap \mathcal{O}$. It follows that $\mathcal{H}(T, \mathbb{Z}_2) = \mathcal{E}/\mathcal{C}_{\mathcal{E}}$ is a subgroup of $(\mathcal{O})/\mathcal{C}_{\mathcal{O}}$ and this is a subset of the set of cohomology classes in $\mathcal{A}$. □

8 Appendix: The theorem of Baire

We add a proof of Bair's theorem. The theorem can be found in [Hah 32].

Theorem 8.1 (Baire) *Given a complete metric space (X, d) and a sequence f_n of real valued continuous functions on X, which converge pointwise to a function f. Then, the set of continuity points of f is generic.*

Proof. Define for $i, j \geq 1$

$$A_{ij} = \{x \in X \mid |f(x) - f_i(x)| \leq 1/j\}$$

and

$$A_j = \bigcup_{i=1}^{\infty} \text{int}(A_{ij}) \,.$$

Then $A = \bigcap_{j=1}^{\infty} A_j$ is the set of continuity points of f.
Define further

$$B_{ij} = \{x \in X \mid |f_k(x) - f_l(x)| \leq 1/j, \text{ for } k, l \geq i\}$$

The continuity of f_k implies, that B_{ij} is closed. The assumption, that the sequence f_n is converging pointwise gives

$$\bigcup_{i=1}^{\infty} B_{ij} = X \,.$$

Because for each $i \geq 1$, we have $\text{int}(B_{ij}) \subset A_j$, we have also

$$\bigcup_{i=1}^{\infty} \text{int}(B_{ij}) \subset A_j \,.$$

Because $B_{ij} \setminus \text{int}(B_{ij})$ is nowhere dense, we conclude, that $X \setminus A_j$ is meager. So, also

$$X \setminus A = \bigcup_{j=1}^{\infty} (X \setminus A_j)$$

is meager. □

References

[Akc 65] M.A.Akcoglu,R.V.Chacon. *Generalized eigenvalues of automorphisms.*
Proc.Amer.Math.Soc., 16:676-680, 1965.

[Anz 51] H.Anzai. *Ergodic Skew Product Transformations on the Torus.*
Osaka Math. J., 3:83-99, 1951.

[Bag 88] L.Baggett. *On circle valued cocycles of an ergodic measure preserving
transformation.*
Israel J. Math., 61:29-38, 1988.

[Cor 82] I.P.Cornfeld,S.V.Fomin,Ya.G.Sinai. *Ergodic Theory.*
Springer, New York, 1982

[Cyc 87] H.L.Cycon,R.G.Froese,W.Kirsch,B.Simon. *Schroedinger Operators.*
Texts and Monographs in Physics, Springer, 1987

[Den 76] M.Denker,C.Grillenberg,K.Sigmund. *Ergodic Theory on Compact Spaces.*
Lecture Notes in Math., No. 527, Springer 1976

[Fri 70] N.A.Friedman. *Introduction to Ergodic Theory.*
Van Nostrand-Reinhold, Princeton, New York, 1970

[Fur 81] H.Fürstenberg. *Recurrence in Ergodic Theory and Combinatorial Number
Theory.* Princeton University Press, Princeton, New Jersey, 1981

[Hah 32] H.Hahn. *Reelle Funktionen, 1.Teil, Punktfunktionen.* Akademische Ver-
lagsgesellschaft, Leipzig, 1932

[Hal 56] P.Halmos. *Lectures on Ergodic Theory.*
The mathematical society of Japan, 1956

[Her 81] M.R.Herman. *Construction d'un difféomorphisme minimal d'entropie
topologique non nulle.*
Ergodic Theory & Dynamical Systems,1:65-76, 1981

[Joh 84] R.A.Johnson. *Lyapunov numbers for the almost periodic Schroedinger
equation.*
Illinois J. Math., 28:397-419, 1984

[Joh 86] R.A.Johnson. *Exponential Dichotomy, Rotation Number and Linear Dif-
ferential Operators with Bounded Coefficients.*
J. Differential Equations, 61:54-78, 1986

[Kif 82] Y.Kifer. *Perturbations of Random Matrix Products.*
Z.Wahrscheinlichkeitstheorie verw.Gebiete, 61:83-95, 1982

[Kni 2] O.Knill. *Positive Lyapunov exponents for a dense set of bounded measurable $SL(2,R)-$ cocycles.*
Ergod. Th. Dynam. Sys., 12, 319-331, 1992

[Led 82] F.Ledrappier. *Quelques Proprietés des Exposants Caracteristiques.*
Lecture Notes in Math., No. 1097, Springer, 1982

[Lem 89] M.Lemańczyk. *On the Weak Isomorphism of Stricly Ergodic Homeomorphisms.*
Monatsh. Math., 108:39-46, 1989

[Mer 85] K.D.Merrill. *Cohomology of step functions under irrational rotations.*
Israel J. Math., 52:320-340, 1985.

[Pag 89] E. Le Page. *Regularité du plus grand exposant caractéristique des produits de matrices aléatoires indépendantes et applications.*
Ann. Inst. Henri Poincaré , 25:109-142, 1989

[Rue 79] D.Ruelle. *Ergodic theory of differentiable dynamical systems.*
Publ.math. de l'IHES, 50:27-58, 1979

[Rue 79a] D.Ruelle. *Analycity Properties of the Characteristic Exponents of Random Matrix Products.*
Advances in Mathematics, 32:68-80, 1979

[Ste 71] A.M.Stepin. *Cohomologies of automorphism groups of a Lebesgue space.*
Functional Anal. Appl., 5:167-168, 1971

[Vee 69] W.Veech. *Strict ergodicity in zero dimensional dynamical systems and the Kronecker-Weyl theorem mod 2.*
Trans.Amer.Math.Soc.,140:1-33, 1969

[Woj 85] M.Wojtkowski. *Invariant families of cones and Lyapunov exponents.*
Ergodic Theory & Dynamical Systems, 5:145-161, 1985.

[You 86] L.-S.Young. *Random perturbations of matrix cocycles.*
Ergodic Theory & Dynamical Systems, 6:627-637, 1986.

Isospectral deformations of random Jacobi operators

Contents

Abstract

We show the integrability of infinite dimensional Hamiltonian systems obtained by making isospectral deformations of random Jacobi operators over an abstract dynamical system.

The time 1 map of these so called random Toda flows can be expressed by a QR decomposition.

1 Introduction

Toda systems have been studied extensively since their discovery by Toda in 1967. Since then, several approaches for their integration have been found and many generalizations have been invented.

Examples are:
• The *tied* or *aperiodic Toda* lattice describes isospectral deformations of finite dimensional aperiodic tridiagonal Jacobi matrices

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdot & \cdot & 0 \\ a_1 & b_2 & a_2 & \cdot & \cdot & \cdot \\ 0 & a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{N-2} & 0 \\ \cdot & \cdot & \cdot & a_{N-2} & b_{N-1} & a_{N-1} \\ 0 & \cdot & \cdot & 0 & a_{N-1} & b_N \end{pmatrix}.$$

The integration is performed by taking a spectral measure as the new coordinate. From this measure, the matrix can be recovered. The measure moves linearly by the Toda flow. For a generic Hamiltonian, the matrices converge to diagonal matrices for $t \to \pm\infty$. The integration of the first flow, which has an interpretation of particles on the line, has first been performed in [Mos 75]. For the other flows see [Dei 83]. There are *Lie algebraic generalizations* of this Toda lattice [Bog 76], [Kos 79], [Sym 80] and interpretations as a *geodesic flow* [Per 90] or a *constrained harmonic motion* [Dei 80].

• The *half- infinite Toda lattice* is an infinite dimensional generalization of the tied lattice. It describes isospectral deformations of tridiagonal operators

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdot \\ a_1 & b_2 & a_2 & \cdot \\ 0 & a_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

on $l^2(\mathbb{N})$. The integration is technically more difficult and has been performed in [Dei 85],[Ber 86] [Li 87]. It resembles the integration of the tied lattice. Again, the operators converge in general to diagonal operators [Dei 85].

• The *periodic Toda lattice* consists of isospectral deformations of periodic Jacobi matrices

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdot & 0 & & a_N \\ a_1 & b_2 & a_2 & \cdot & & \cdot & 0 \\ 0 & a_2 & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{N-2} & & 0 \\ 0 & \cdot & \cdot & a_{N-2} & b_{N-1} & & a_{N-1} \\ a_N & 0 & \cdot & 0 & a_{N-1} & & b_N \end{pmatrix} .$$

The first flow describes a periodic chain of particles interacting with an exponential potential. The explicit integration uses methods of algebraic geometry [Kac 75] [vMo 76] [vMo 78]. The flow is conjugated to a motion of an auxiliary spectrum. *Jacobi's map* transforms this motion into a linear flow on the *Jacobi variety* of the hyperelliptic curve attached to the matrix. The system is periodic or quasiperiodic.

• A *rapidly decreasing case* of the Toda lattice is a situation with free particle boundary conditions at infinity. The integration is done by an *inverse scattering transform* [Fla 74], [Fad 86]. It is an isospectral deformation of doubly infinite Jacobi matrices decaying at infinity. A recent detailed investigation of the long time behavior of such a Toda system can be found in [Kam 93].

In this article, we discuss the Toda lattice with a new boundary condition: a so-called *random boundary condition*. It is a generalization of both the periodic and the aperiodic system and consists of deformations of random Jacobi operators. This generalization is a special case of an abstract generalization found by Bogoyavlensky [Bog 88],[Bog 90] p.172), who suggested to consider such differential equations in an associative algebra.

Another suggestion for investigating such random systems is offered in ([Car 90] p. 436), where it is motivated by the problem of finding a complete *inverse spectral theory* for random Jacobi operators, a project initiated by Carmona and Kotani([Car 87]).

The random Toda lattice is a discrete version of the *Korteweg de Vries equation* defined over a flow. A special case of such a KdV equation leads to an isospectral deformation of almost periodic Schrödinger operators. In the work of Johnson and Moser [Joh 82] the *Floquet exponent* for such Schrödinger operators is introduced and the Floquet exponent is shown to be an invariant of the KdV deformation. It is natural to ask for analogous results in the discrete case.

The chapter is organized as follows:

In the *second section*, we define isospectral Toda deformations of random Jacobi operators. Random Jacobi operators are selfadjoint elements $L = a\tau + (a\tau)^* + b$ in the crossed product $\mathcal{X}$ of the commutative Banach algebra $L^\infty(X)$ with an abstract dynamical system (X, T, m). The multiplication in $\mathcal{X}$ is just the convolution

multiplication of power series $\sum L_n \tau^n$ in the variable τ with the additional rule that $\tau^k L_n = L_n(T^k)\tau^k$ for all $k, n \in \mathbb{Z}$. The random Jacobi operators form a Banach space $\mathcal{L} \subset \mathcal{X}$. For almost all $x \in X$, one gets *stochastic Jacobi matrices* $[L(x)]_{mn} = L_{n-m}(T^m x)$. We are interested in deformations of Jacobi operators which are given by a differential equation $\dot{L} = [B_H(L), L]$, where $B_H(L)$ is skew symmetric in $\mathcal{X}$ and depends on a Hamiltonian H. These Toda flows generalize periodic and aperiodic finite dimensional Toda flows. The periodic case is obtained when the cardinality of the set X is finite. If $a(x) = 0$ on a set of positive measure one gets the aperiodic case. There is a trace in the C^* algebra $\mathcal{X}$ and for each continuous function $f : \mathbb{C} \to \mathbb{C}$ there is an integral $\mathrm{tr}(f(L))$ of the Toda flow. Another integral of the deformations is the mass $\exp(\int_X \log(a)\, dm(x))$. This integral cannot be written in the form $\mathrm{tr}(f(L))$.

In the *third section* we give an integration of the random Toda lattice in the following sense: There is a mapping ϕ from $\mathcal{L}$ to an infinite dimensional vector space $\mathcal{G}$, in which the flow is linear. The mapping ϕ has a left inverse ψ and the time one maps $\mathrm{Exp}_H, \overline{\mathrm{Exp}}_H$ of the flow in the old and new coordinates are related by $\mathrm{Exp}_H(L) = \psi \circ \overline{\mathrm{Exp}}_H \circ \phi(L)$. The idea is to approximate the random Toda flow by finite dimensional aperiodic Toda lattices which are known to be integrable. This approximation is due to a lemma which says, roughly speaking, that a differential equation $\dot{x} = f(x)$ in a Banach space gives a flow which is also continuous in a weaker topology, if f is continuous with respect to this weaker one.
There can be transient behavior for the random Toda lattice: The random Toda flow splits into infinitely many aperiodic finite dimensional flows, provided that $a(x)$ is zero on a set of positive measure and the underlying dynamical system is ergodic.

In the *fourth section*, we show that QR decompositions generalize to the infinite dimensional case. Such a generalization is known for half infinite Jacobi operators [Dei 85]. The QR decomposition for invertible matrices can also be used to express the time 1 map for each flow.

In the *fifth section* it is shown that random Jacobi operators L have a determinant $\det(L - E)$ which is an integral of motion for the Toda flows. The *Floquet exponent* $w(E)$ satisfying $\det(L - E) = \exp(-w(E))$ is related by the Thouless formula to the Lyapunov exponent and to the rotation number of the transfer cocycle of L. These functions as well as the Taylor coefficients of $w(E)$ calculated at a point E_0 outside the spectrum of L are also integrals. Random Jacobi operators appear in a natural way for twist diffeomorphisms. They are the second variation of a Percival functional.

At last, in the *sixth section*, we look at generalizations of random Toda flows, for example at isospectral deformations in the crossed product of any Banach algebra with a dynamical system. In the same way as the random Toda lattice, random

singular value decomposition flows can be defined.

2 Random Jacobi operators and random Toda flows

2.1 Random Jacobi operators

A *dynamical system* (X, T, m) is an automorphism T of a probability space (X, m). Consider the set of sequences $K_n \in L^\infty(X)$, where $K_n \neq 0$ only for finitely many $n \in \mathbb{Z}$. This forms an algebra with the multiplication

$$(KM)_n(x) = \sum_{k+m=n} K_k(x) M_m(T^k x) \ .$$

The algebra carries an involution given by

$$(K^*)_n(x) = \overline{K_{-n}}(T^n x) \ .$$

We denote by $\mathcal{X}$ the completion of this algebra with respect to the norm

$$|||K||| = |\ ||K(x)||\ |_\infty \ ,$$

where $K(x)$ is the bounded operator in $l^2(\mathbb{Z})$ given by the infinite matrix

$$[K(x)]_{mn} = K_{n-m}(T^m x) \ .$$

The multiplication and involution in $\mathcal{X}$ is defined such that

$$K \in \mathcal{X} \mapsto K(x) \in \mathcal{B}(l^2(\mathbb{Z}))$$

is an algebra homomorphism:

$$KL(x) = K(x) L(x), \ K^*(x) = K(x)^* .$$

The algebra $\mathcal{X}$ is a C^*- algebra called the *crossed product* of $L^\infty(X)$ with the dynamical system (X, T, m). Elements in $\mathcal{X}$ are called *random operators*. For $K \in \mathcal{X}$ we define the *trace* by

$$\mathrm{tr}(K) = \int_X K_0 \, dm \ .$$

For all $K, M \in \mathcal{X}$,

$$
\begin{aligned}
\mathrm{tr}(KM) &= \int \sum_n K_n M_{-n}(T^n) \, dm = \int \sum_n K_{-n} M_n(T^{-n}) \, dm \\
&= \int \sum_n M_n K_{-n}(T^n) \, dm = \mathrm{tr}(MK) \ .
\end{aligned}
$$

It follows that for any invertible $U \in \mathcal{X}$ and every $K \in \mathcal{X}$,

$$\operatorname{tr}(UKU^{-1}) = \operatorname{tr}(K) .$$

In order to simplify the writing and the algebraic manipulations, we will write elements $K \in \mathcal{X}$ in the form

$$K = \sum_n K_n \tau^n$$

and think of τ just as a symbol. The multiplication in $\mathcal{X}$ is the multiplication of power series with the additional rule $\tau^k K_n = K_n(T^k)\tau^k$ for shifting the τ's to the right and the requirement that $\tau^* = \tau^{-1}$. If we interpret τ as a shift operator $f \mapsto f(T)$ in $L^2(X)$ and K_n as a multiplication operator, we have a representation of $\mathcal{X}$ in $\mathcal{B}(L^2(X))$:

$$Kf = \sum_n K_n f(T^n) .$$

If the dynamical system (X, T, m) is ergodic, there exists for each $K \in \mathcal{X}$ a set of full measure such that for x in this set, the operators $K(x)$ have the same spectrum $\Sigma(K(x))$ denoted by $\Sigma(K)$. This is a version of *Pastur's theorem*. The proof ([Cyc 87] p. 168) which is written for a parallel case proves it. In general, when no ergodicity is assumed, define

$$\Sigma(K) = \{E \in \mathbb{C} \mid m(\{x \in X \mid E \in \Sigma(K(x))\}) > 0\} .$$

A selfadjoint element $L \in \mathcal{X}$ of the form

$$L = a\tau + (a\tau)^* + b$$

is called a *random Jacobi operator* if $a, b \in L^\infty(X, \mathbb{R})$. Denote by $\mathcal{L}$ the real Banach space of all random Jacobi operators in $\mathcal{X}$. We call

$$M(L) := \exp(\int_X \log(a) \, dm) \in \mathbb{R} \cup \{-\infty\}$$

the *mass* of the operator L. (The branch of the logarithm in the definition of the mass is chosen so that $\log(-1) = i \cdot \pi$, and $\log(1) = 0$.)

Remark. C^* algebra techniques for random Jacobi matrices were promoted in [Bel 85]. The crossed product construction goes back to Murray and von Neumann and plays an important role in the theory of C^* algebras (non-commutative topology) and von Neumann algebras (non-commutative measure theory).

2.2 Random Toda flows

We define the projections

$$K = \sum_{n=-\infty}^{\infty} K_n \tau^n \mapsto K^\pm = \sum_{\pm n > 0} K_n \tau^n$$

which yield the decomposition $K = K^- + K_0 + K^+$. For a *Hamiltonian*

$$H \in C^\omega(\mathcal{L}) := \{L \mapsto \operatorname{tr}(h(L)) \mid h \text{ entire}, \ h(\mathbb{R}) \subset \mathbb{R}\} \,,$$

the differential equation

$$\dot{L} = [B_H(L), L] \,,$$

with $B_H(L) = h'(L)^+ - h'(L)^-$ defines a flow which we call *a random Toda flow* or *a random Toda lattice.*

Remark. The name "random Toda lattice" was proposed by Carmona,Kotani [Car 87]. They say there: *We hope to be able to discuss the problems of some "random Toda lattices" in the near future.*

Theorem 2.1 *For each $H(L) = \operatorname{tr}(h(L)) \in C^\omega(\mathcal{L})$, the flow*

$$\dot{L} = [h'(L)^+ - h'(L)^-, L] = [B_H(L), L]$$

defined on $\mathcal{L}$ is Hamiltonian, isospectral and every $G \in C^\omega(\mathcal{L})$ as well as the mass $\exp(\int \log(a) \, dm)$ are integrals. The flows are globally defined and commute pairwise.

Proof.

Invariance of the spectrum and local integrals:
In $\mathcal{X}$, the differential equation $\dot{Q} = -B_H Q$ with $Q(0) = 1$ has a unitary solution $Q(t)$, because B_H is skew symmetric. The formula $Q(t)L(t)Q(t)^* = L(0)$ shows that the flows leave invariant the spectrum $\Sigma(L)$.
For each $g \in C(\Sigma(L))$ we have

$$G(L(t)) = \operatorname{tr}(Q^*(t)g(L(0))Q(t)) = \operatorname{tr}(g(L(0))) = G(L(0))$$

giving the *local integrals* $G(L) = \operatorname{tr}(g(L))$ for $g \in C(\Sigma(L))$.

Global existence:
The local existence of the Toda flows follows from *Cauchy's existence theorem* (see [Die 68]) and the fact that

$$f_H : \mathcal{L} \to \mathcal{L}, L \mapsto f_H(L) = [B_H(L), L]$$

is Fréchet differentiable: denote with B_R the ball with radius R in the Banach space $\mathcal{L}$ and with $Df_H(L)$ the Fréchet derivative of $f_H : \mathcal{L} \to \mathcal{L}$. The operator norm in $\mathcal{B}(\mathcal{L})$ is written as $||| \cdot |||_1$. Claim: f_H is differentiable and $\forall R > 0 \ \exists C_{H,R} > 0$ such that for all $L \in B_R$

$$|||f_H(L)||| \leq C_{H,R}, \ |||Df_H(L)|||_1 \leq C_{H,R} \,.$$

Proof. For $L \in B_R$ we have necessarily $|a|_\infty, |b|_\infty < R$ and we obtain the rough estimate $|(L^n)_i|_\infty \le 3^n R^n$, leading with $h(z) = \sum_n h_n z^n, \bar{h}(z) = \sum_n |h_n| z^n$ to $|||h(L)^\pm||| \le 3R\bar{h}'(3R)$ and

$$|||f_H(L)||| \le 12R^2 \bar{h}'(3R) .$$

Similarly we obtain with

$$D(h(L)^+ - h(L)^-)U = (h'(L)U)^+ - (h'(L)U)^-$$

the estimate

$$|||Df_H(L)|||_1 \le 12R^2(\bar{h}''(3R) + \bar{h}'(3R)) =: C_{H,R} .$$

□

Since $\Sigma(L)$ is invariant by the flow, the norm $|||L(t)|||$ is constant. This assures global existence of the flow.

Hamiltonian character of the flow:

We will show that $\mathcal{L}$ is a *Poisson manifold* and that the Toda lattice with the Hamiltonian H can be written in $C^\omega(\mathcal{L})$ as $\dot{F} = \{F, H\}$.

Define the projection $\Delta : \mathcal{X} \to \mathcal{L}$ by

$$K = \sum_{n=-\infty}^{\infty} K_n \tau^n \mapsto K^\Delta = K_{-1}\tau^{-1} + K_0 + K_1\tau .$$

Given $G = \mathrm{tr}(g(L)) \in C^\omega(\mathcal{L})$, we denote with

$$\nabla G = g'(L)^\Delta \in \mathcal{L}$$

the *functional derivative* of $G : \mathcal{L} \to \mathsf{R}$. The Fréchet derivative DG satisfies $DG(L)U = \mathrm{tr}(\nabla G(L)U)$ for all $U \in \mathcal{L}$. Claim. The following formula of Moerbeke ([vMo 76]) holds:

$$[h'(L)^+ - h'(L)^-, L] = [L^+ - L^-, h'(L)^\Delta] .$$

Proof. By linearity in h, it is enough to check this formula for $h'(L) = L^n$: For $|k| > 2$ we have

$$[(L^n)^+ - (L^n)^-, L]_k = [L^n, L]_k = 0 .$$

We use the notation $L^n = \sum_{k=-n}^{n} l_k \tau^k$ to verify also

$$\begin{aligned}
[(L^n)^+ - (L^n)^-, L]_0 &= [l_1\tau, (a\tau)^*] - [l_{-1}\tau^*, a\tau] \\
&= [a\tau, l_{-1}\tau^*] - [(a\tau)^*, l_1\tau] \\
&= [L^+ - L^-, (L^n)^\Delta]_0
\end{aligned}$$

93

and

$$[(L^n)^+ - (L^n)^-, L]_1 = [l_1\tau, b] + [l_2\tau^2, (a\tau)^*]$$
$$= -[l_0, a\tau] = [a\tau, l_0]$$
$$= [L^+ - L^-, (L^n)^\Delta]_1 \, ,$$

where we used the identity

$$0 = [L^n, L]_1 = [l_1\tau, b] + [l_2\tau^2, (a\tau)^*] + [l_0, a\tau] \, .$$

With the *Poisson bracket*

$$\{F, G\} := 2 \cdot \mathrm{tr}(\nabla F(L^+ - L^-)\nabla G) = \mathrm{tr}(\nabla F[L^+ - L^-, \nabla G]) \, ,$$

$C^\omega(\mathcal{L})$ is a Lie algebra. An observable $G = \mathrm{tr}(g(L)) \in C^\omega(\mathcal{L})$ evolves according to

$$\dot{G} = \frac{d}{dt}\mathrm{tr}(g(L)) = \mathrm{tr}(Dg(L)\dot{L}) = \mathrm{tr}(\nabla G\dot{L})$$
$$= \mathrm{tr}(\nabla G(L)[L^+ - L^-, \nabla H(L)]) = \{G, H\} \, .$$

Since every $G \in C^\omega(\mathcal{L})$ is a constant of motion, we have $\{H, G\} = 0$ for all $H, G \in C^\omega(\mathcal{L})$ and so all these Hamiltonian flows commute: with the notation $X_H F = \{F, H\}$ one has using the Jacobi identity

$$[X_H, X_G]F = (X_H X_G - X_G X_H)F = \{\{F, G\}, H\} - \{\{F, H\}, G\}$$
$$= \{\{H, G\}, F\} = X_{\{H,G\}}F = 0 \, .$$

Conservation laws for mass and momentum:
The differential equation $\dot{L} = [B_H(L), L]$ is equivalent to

$$\frac{d}{dt}\log(a) = h'(L)_0(T) - h'(L)_0 \, ,$$
$$\frac{d}{dt}b = 2ah'(L)_1 - 2a(T^{-1})h'(L)_1(T^{-1}) \, .$$

These are discrete conservation laws for the mass integral $\log(M) = \int_X \log(a) \, dm$ and the *momentum integral* $\int_X b \, dm$.
We can see from the above conservation laws that the imaginary part of $\log(a)$ does not move. This implies that the set $Y(t) = \{x \in X \mid a(x) < 0\}$ does not move.

Remark. To approach the common notation for Hamiltonian systems, one could use the notation $J_L K := [L^+ - L^-, K]$ for $K, L \in \mathcal{L}$ and $< K, L >= \mathrm{tr}(KL)$. The Toda flows can then be written as

$$\dot{L} = J_L \nabla H(L)$$

and the Poisson bracket is

$$\{F, G\}_L = < \nabla F, J_L \nabla G > \, .$$

94

One can show that the 2-form $w_L(K, M) = <K, J_L M>$ is degenerate. Like this, $\mathcal{L}$ is not a *symplectic manifold* but only a Poisson manifold. See [Sch 87] for more information about infinite dimensional Hamiltonian systems.

Example. (The first Toda lattice). For $H(L) = \text{tr}(\frac{L^2}{2})$ one obtains the differential equation $\dot{L} = [L^+ - L^-, L]$. Expressed in the coordinates a, b, this gives

$$\begin{aligned} \dot{a} &= a(b(T) - b) \,, \\ \dot{b} &= 2a^2 - 2a^2(T^{-1}) \,. \end{aligned}$$

For fixed $x \in X$ we write $a_n = a(T^n x), b_n = b(T^n x)$; this leads to

$$\begin{aligned} \dot{a}_n &= a_n(b_{n+1} - b_n) \,, \\ \dot{b}_n &= 2(a_n^2 - a_{n-1}^2) \,, \end{aligned}$$

and reduces to the periodic Toda lattice in the case when $|X|$ is finite.

Example. (The second Toda lattice). For $H(L) = \text{tr}(\frac{L^3}{3})$ one obtains the differential equation $\dot{L} = [(L^2)^+ - (L^2)^-, L]$. Expressed in the coordinates a, b, this gives

$$\begin{aligned} \dot{a} &= a(b^2(T) - b^2 + a^2(T) - a^2(T^{-1})) \,, \\ \dot{b} &= 2a^2(b(T) + b) - 2a^2(T^{-1})(b + b(T^{-1})) \,. \end{aligned}$$

2.3 Visualisation of the first Toda flow

We can visualise the motion of $\{a_n, b_n\}$ as the evolution of infinitely many points

$$(\log(a) + ib)(T^n x) = \log(a_n) + ib_n \in \mathbb{C} \,.$$

Assume now $X = \mathsf{T}^1$ and $a, b : X \to \mathbb{R}$ are smooth. We get then a curve

$$x \in \mathsf{T}^1 \mapsto (\log(a) + ib)(x) \in \mathbb{C}$$

in the complex plane. The motion of the Toda flow induces then a motion of this curve which keeps on being smooth for all times. Because the Toda flow has the integrals

$$\int_X \log(a) \, dm, \text{ and } \int_X b \, dm$$

the center of mass of the configuration is conserved.

We calculated numerically an example of such an evolution with the following Mathematica program

```
RKStep[f_List, y_List,y0_List, dt_]:=Block[{k1,k2,k3,k4 },
  k1=dt N[ f /. Thread[y -> y0]      ];k2=dt N[ f /. Thread[y -> y0+k1/2]];
  k3=dt N[ f /. Thread[y -> y0+k2/2]];k4=dt N[ f /. Thread[y -> y0+k3 ]];
  y0+(k1+2*k2+2*k3+k4)/6];
RKLastPoint[f_List,y_List,y0_List,{t1_, dt_}]:=Block[{yy},
  yy=y0;Do[yy=RKStep[f,y,yy,N[dt]],{i,1,Round[N[t1/dt]]}];yy ];

Variab[n_]:=Join[Table[a[i],{i,1,n}],Table[b[i],{i,n}]];
Diffeq[n_]:=Join[Table[a[i] b[Mod[i,n]+1]-b[i] a[i],{i,n}],
                 Table[2 a[i] a[i]-2 a[Mod[i-2,n]+1] a[Mod[i-2,n]+1],{i,n}]];
TodaLat[cc0_,t1_,dt_]:=Block[{l=Length[cc0]/2,k},
  cc=Variab[l];hh=Diffeq[l]; RKLastPoint[hh,cc,cc0,{t1,dt}]  ];
Init[n_]:=Block[{s=10*Pi/n},
  N[Join[Table[Exp[Cos[j*s]],{j,n}],Table[Sin[j*s],{j,n}]]]];
Pict[c_]:=Block[{l=Length[c]/2}, ListPlot[Table[{Log[c[[i]]],c[[i+1]]},{i,1}],
     PlotRange->{{-3,3},{-3,3}},DisplayFunction->Identity,Axes->False]];
Film[NumbPart_,NumbPict_,TimeInt_]:=Block[{c=Init[NumbPart],Movie={}},
  Do[Movie=Append[Movie,Pict[c]];c=TodaLat[c,TimeInt,0.01],{i,NumbPict}];Movie];
Display["!psfix -land -stretch > todafilm.ps",
  Show[GraphicsArray[Partition[Film[101,16,0.05],4]],
  DisplayFunction->$DisplayFunction,Frame->True,
  PlotLabel->FontForm["The Toda lattice with 101 particles",{"Helvetica",12}]]]
```

integrating a system of 101 differential equations.

The Toda lattice with 101 particles
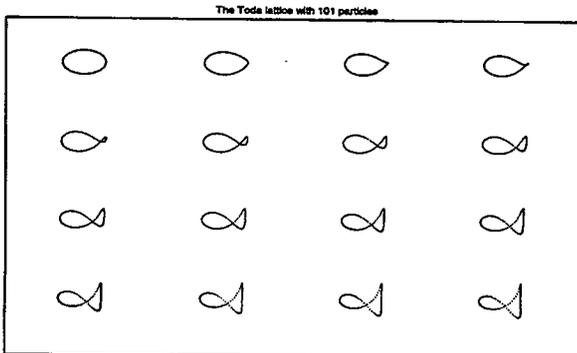
2.4 The flow in Flaschka coordinates

The Toda flows can sometimes be interpreted as a Hamiltonian flow in the new coordinates q, p (see Flaschka [Fla 74]) defined by:

$$4a^2 = e^{q(T)-q}, \ 2b = p \ .$$

To introduce these coordinates $q, p \in L^\infty(X)$, the function a^2 must be a multiplicative coboundary: there must exist $f \in L^\infty(X)$ such that

$$a^2 = f(T)f^{-1} \ .$$

Not every a satisfies this. A necessary condition is for example that $\int_X \log(a) \, dm = 0$. The Toda differential equations transform into the Hamilton equations

$$\dot{q} = H_p \ ,$$
$$\dot{p} = -H_q \ ,$$

where $H(q, p) = H(L) = \text{tr}(h(L))$. The first flow, given by $h(E) = \frac{E^2}{2}$, describes an infinite chain of particles with position $q_n = q(T^n x)$ moving according to

$$\frac{d^2}{dt^2} q_n = e^{q_{n+1}-q_n} - e^{q_n-q_{n-1}} \ .$$

On the Banach space $L^\infty(X) \times L^\infty(X)$ there is a symplectic structure. With

$$z = (q, p) \in L^\infty(X) \times L^\infty(X) \ ,$$

the flows can be written as

$$\dot{z} = J\nabla H(z) \ .$$

3 Integration of the random Toda lattice

If (X, T, m) is periodic, the integration of the random Toda lattice is known and the solutions can be given explicitly in terms of *Theta functions*. In the case of positive mass one has a collection of periodic Toda lattices and in the case of zero mass one has a collection aperiodic Toda lattices. In the following, we examine the case of an aperiodic dynamical system. Nevertheless the periodic case can be viewed as a special case of this also as we explain at the end of this paragraph.

Denote by $\text{Exp}_H : \mathcal{L} \to \mathcal{L}$ the time 1 map of the flow given by H. We want to find new coordinates in an infinite dimensional vector space $\mathcal{G}$ where the time 1 map $\overline{\text{Exp}}$ is easy to calculate.

Theorem 3.1 *Assume (X, T, m) is aperiodic.*

a) There exists an injective map $\phi : \mathcal{L} \to \mathcal{G} = (L^\infty(X) \times L^\infty(X))^{\mathbb{N}}$,

$$a, b \in L^\infty(X) \mapsto (\lambda, r) = \{(\lambda_i, r_i)\}_{i \in \mathbb{N}}$$

which has a surjective left inverse ψ defined on the flow invariant subset

$$\mathcal{H} = \{\overline{\mathrm{Exp}}_H(\phi(\mathcal{L})) \mid H \in C^\omega(\mathcal{L})\} \subset \mathcal{G} \ .$$

The flow given by the differential equation $\dot{L} = [B_H(L), L]$ has in the new coordinates the form

$$\overline{\mathrm{Exp}}_{tH}(\lambda, r) = (\lambda, e^{th'(\lambda)} r) = \{(\lambda_i, e^{th'(\lambda_i)} r_i)\}_{i \in \mathbb{N}}$$

and is conjugated to the flow in $\mathcal{L}$:

$$\mathrm{Exp}_H(L) = \psi \circ \overline{\mathrm{Exp}}_H \circ \phi(L) \ .$$

b) Assume (X, T, m) is ergodic. If $L \in \mathcal{L}$ is given such that $Y = \{a(x) = 0\}$ has positive measure, then there exists a generic set in $C^\omega(\mathcal{L})$, such that for H in this set, $L(t)(x)$ converges in the weak operator topology to a diagonal operator for almost all $x \in X$.

3.1 Integration of the Toda flows in the case when $\{a(x) = 0\}$ has positive measure.

Proof.

a) Call T_Y the *induced mapping* on $Y = \{a(x) = 0\}$,

$$T_Y(y) = T^{n(y)}(y) \ ,$$

where

$$n(y) = \min\{n > 0 \mid T^n y \in Y\} \ .$$

By Poincaré recurrence, $n(y)$ is finite for almost all $y \in Y$. For fixed $y \in Y$, we write $a_n = a(T^n y), b_n = b(T^n y)$ and call the aperiodic Jacobi matrix

$$L_y = \begin{pmatrix} b_1 & a_1 & 0 & \cdot & & \cdot & & 0 \\ a_1 & b_2 & a_2 & \cdot & & & & \cdot \\ 0 & a_2 & \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & a_{(n(y)-2)} & & 0 \\ \cdot & \cdot & \cdot & a_{(n(y)-2)} & & b_{(n(y)-1)} & a_{(n(y)-1)} \\ 0 & \cdot & \cdot & 0 & & a_{(n(y)-1)} & & b_{n(y)} \end{pmatrix}$$

a *block*. It evolves independently from the other part of the matrix $L(y)$. (In the case $n(x) = 1$, the matrix L_y is a 1×1 matrix containing just a 0.) We have in this way a measurable matrix-valued function

$$y \in Y \mapsto L_y \ .$$

The spectrum $\lambda_y^{(1)} < \ldots < \lambda_y^{(n(y))}$ of L_y is simple. We can label each $x \in X$ with a value $\lambda(x)$ by defining

$$\lambda(x) = \lambda_{y(x)}^{(m(x))} ,$$

where $m(x) = \min\{n > 0 \mid T^{-n}x \in Y\}$ and $y(x) = T^{-m(x)}x$. The mapping $x \mapsto \lambda(x)$ is measurable and because $\lambda(x) \leq |||L|||$ for almost every $x \in X$, we have $\lambda \in L^\infty(X)$. Call

$$dE_y = \sum_{i=1}^{n(y)} \lambda(T^i y) P(T^i y) ,$$

the matrix-valued spectral measure of $L_y = \int \lambda dE_y(\lambda)$ where $P(x)$ is the projection on the eigenspace of $\lambda(x)$. Define for $y \in Y$ the probability measure

$$\mu_y = [dE_y]_{11} .$$

If $\delta(z)$ denotes the Dirac measure at $z \in \mathbb{R}$, the measure can be written as

$$d\mu_y = \sum_{i=1}^{n(y)} \frac{r(T^i y)}{R_y(r)} \cdot \delta(\lambda(T^i y)) ,$$

where

$$R_y(r) = \sum_{k=1}^{n(y)} r(T^k y) .$$

The function $r \in L^\infty(X)$ is uniquely defined when we require $R_y(r) = 1$ for $y \in Y$. We have so a map

$$\tilde{\phi} : (a,b) \mapsto (\lambda, r) \in \tilde{\mathcal{G}} = L^\infty(X) \times L^\infty(X) .$$

Define

$$\phi(L) = (\tilde{\phi}_L, \tilde{\phi}_L, \ldots) \in \mathcal{G} .$$

(Infinitely many coordinates in $\mathcal{G}$ will be necessary only in the case $a(x) > 0$, a.e. which is treated later.) In these new coordinates the flow is linear: λ does not change and the weights $r(T^i y)$ of the measure $d\mu_y$ evolve according to

$$\dot{r}(T^i y) = h'(\lambda(T^i y)) \cdot r(T^i y), \ i = 1, \ldots, n(y) .$$

(See [Mos 75] for the first flow and [Dei 85] for all the Toda flows. We will give the proof in an appendix.) Therefore

$$(\lambda(t), r(t)) = (\lambda, e^{h'(\lambda)t} \cdot r) .$$

From $\overline{\text{Exp}}_H(\lambda, r) \in \mathcal{H}$, the operator $\text{Exp}_H(L) \in \mathcal{L}$ can be reconstructed: Take a point $y \in Y$. The values

$$\lambda(T^i y), r(T^i y) \ i = 1, \ldots, n(y)$$

99

determine the measure $d\mu_y$ and from this measure, $a(T^i y), b(T^i y)$ for $i = 1, \ldots, n(y)$ can be recovered. (See also the appendix.) We have thus an inverse ψ on $\mathcal{H} \subset \mathcal{G}$ and by construction

$$\mathrm{Exp}_H(L) = \psi \circ \overline{\mathrm{Exp}}_H \circ \phi(L) \ .$$

b) The operator $L(x)$ has countably many eigenvalues $\lambda \in \{\lambda_i\}_{i \in \mathbb{Z}} =: \sigma_p(L(x))$, and because of ergodicity, $\Sigma_p(L(x)) = \Sigma_p(L)$ for almost all $x \in X$. There exists a generic set of entire functions h which are injective on $\Sigma(L)$ because h is injective on $\Sigma(L)$ if and only if it is in the countable intersection

$$\bigcap_{E',E'' \in \Sigma_p(L)} \{h \mid h(E') \neq h(E'')\}$$

of open dense sets. Denote by Σ_y be the spectrum of the block L_y. If h' is injective on $\Sigma(L)$, it is injective on Σ_y and $L_y(t)$ converges with the Hamiltonian $H(L) = \mathrm{tr}(h(L))$ to a diagonal matrix for $|t| \to \infty$ [Dei 89]. Because this happens for all blocks L_y with $y \in \{a(x) = 0\}$, the operator $L(x)$ converges to a diagonal operator in the weak operator topology. □

3.2 Weak continuity of a flow in a Banach space

The rest of the proof of Theorem 3.1 uses an approximation argument which is based on the following technical result:

Given a Banach space $(\mathcal{M}, \|\cdot\|)$ endowed with a second topology which is weaker than the norm topology and metrizable on each ball $B_R = \{\|z\| < R\} \subset \mathcal{M}$. Call Df the Fréchet derivative of a differentiable map $f : \mathcal{M} \to \mathcal{M}$.

Proposition 3.2 *Assume that a differentiable function $f : \mathcal{M} \to \mathcal{M}$ satisfies*
(i) *f restricted to the ball $B_R \subset \mathcal{M}$ is continuous in the weaker topology.*
(ii) *There exists a constant $C \in \mathbb{R}$, such that for $z \in B_R$*

$$\|Df(z)\| \leq C, \ \|Df(z)f(z)\| \leq C \ .$$

(iii) *The flow given by $\dot{z} = f(z)$ leaves each ball B_R invariant.*
Then, for each $\tau \in \mathbb{R}$, the time τ map $\phi_\tau : B_R \to B_R$ of the flow given by the differential equation $\dot{z} = f(z)$ is weakly continuous on B_R.

The next simple lemma will also be useful.

Lemma 3.3 *Given a sequence of mappings $g_N : B_R \to B_{R'} \subset \mathcal{M}$ converging uniformly to a mapping g for $N \to \infty$:*

$$\sup_{z \in B_R} \|g_N(z) - g(z)\| \to 0 \ .$$

If all the g_N are weakly continuous on B_R, then g is weakly continuous on B_R.

Proof. Denote by d the metric on $B_R \cup B_{R'} \subset \mathcal{M}$ giving the weak topology. Take a sequence $z_n \in B_R$ which is converging weakly to $z \in B_R$. Given $\epsilon > 0$. Because the norm topology is stronger than the weaker topology

$$d(g(z), g_N(z)) \leq \epsilon/3, \ d(g_N(z_n), g(z_n)) \leq \epsilon/3$$

for all $n \in \mathsf{N}$ if N is big enough. Because g_N is weakly continuous, we have $d(g_N(z), g_N(z_n)) \leq \epsilon/3$ for n big enough and thus

$$d(g(z), g(z_n)) \leq d(g(z), g_N(z)) + d(g_N(z), g_N(z_n)) + d(g_N(z_n), g(z_n)) \leq \epsilon \ .$$

$\square$

Now to the proof of Proposition 3.2: Proof. Assumptions (ii) and (iii) together with *Cauchy's existence theorem* assure that a unique solution of $\dot{z} = f(z)$ exists for all times. Divide the interval $[0, \tau]$ into N intervals,

$$[t_k, t_{k+1}] = [\frac{k\tau}{N}, \frac{(k+1)\tau}{N}], \ k = 0, \ldots, N-1$$

of length $h = \tau/N$ and define recursively the Euler steps

$$z_{k+1} = z_k + hf(z_k), \ z_0 = z(0) \ .$$

Using the Taylor development

$$z(t_{k+1}) = z(t_k) + hf(z(t_k)) + \int_{t_k}^{t_{k+1}} (t - t_k) Df(z(t)) f(z(t)) \ dt \ ,$$

the deviation $e_k := z_k - z(t_k)$ from the orbit after k steps can be estimated as follows: we have $||e_0|| = 0$ and from the estimates (ii), we obtain

$$||e_{k+1}|| \leq ||e_k|| + hC||e_k|| + \frac{C}{2}h^2 = (1 + hC)||e_k|| + \frac{C}{2}h^2 =: A||e_k|| + B \ ,$$

and so $||e_0|| = 0, ||e_1|| \leq B$. Inductively, $||e_k|| \leq (A^{k-1} + A^{k-2} + \cdots + A + 1)B$. This yields to

$$||e_k|| \leq \frac{A^k - 1}{A - 1}B = \frac{(1 + hC)^k - 1}{hC} \cdot \frac{C}{2}h^2 \leq \frac{e^{khC} - 1}{C} \cdot \frac{C}{2}h \leq (e^{C\tau} - 1) \cdot \frac{h}{2}$$

as long as z_k stays in B_R. The error after N *Euler steps* is

$$||z(\tau) - z_N|| = ||e_N|| \leq (e^{C\tau} - 1)\frac{\tau}{2N} \ .$$

This estimate is uniform for $z \in B_R$. (If N is so large that

$$(e^{C\tau} - 1)\frac{\tau}{2N} < R - ||z_0||$$

also the linear interpolation of the points $\{z_k\}_{k=1}^N$ (the Euler polygon) is contained in B_R so that the estimates hold for these points also.) The orbit $t \in [0, \tau_0] \mapsto z(t)$ is now approximated in norm by a piecewise linear Euler polygon. A simple Euler step $z \mapsto z + hf(z)$ is continuous in the weak topology if f is continuous in the weak topology. So also the mapping $\phi_N : z \mapsto z_N$, which is a composition of finitely many Euler steps is weakly continuous. For N big enough, we have

$$\|\phi(y) - \phi_N(y)\| \le (e^{K\tau} - 1)\frac{\tau}{2N}$$

uniformly for $y \in B_R$. Lemma 3.3 shows that ϕ is weakly continuous. □

3.3 Integration of the Toda flows in the case when $a(x) > 0$ almost everywhere.

From Proposition 3.2 we get

Corollary 3.4 *Given $H(L) \in C^\omega(\mathcal{L})$. The flow on $\mathcal{B}(l^2(\mathbb{Z}))$ defined by the differential equation $\dot{L}(x) = [B_H(L(x)), L(x)]$ is continuous in the weak operator topology restricted to each ball $B_R \subset \mathcal{B}(l^2(\mathbb{Z}))$.*

Proof. We can apply Proposition 3.2 to the Banach space $\mathcal{B}(l^2(\mathbb{Z}))$. As a Toda flow is isospectral, it will leave each ball B_R invariant. We check first the weak continuity of the mapping

$$L(x) \mapsto f_H(L(x)) := [B_H(L(x)), L(x)]$$

on B_R. For polynomials h, the weak continuity is evident inductively, since the multiplication $\mathcal{B}(l^2) \times \mathcal{L}(x) \to \mathcal{B}(l^2)$ is jointly weakly continuous: each matrix entry of the product LM is the sum of only three elements. If h is analytic, it can be approximated by polynomials $h_n \to h$. Then $f_{H_n}(L(x)) \to f_H(L(x))$ in norm, uniformly on each ball B_R. With Lemma 3.3 also $L(x) \mapsto f_H(L(x))$ is weakly continuous.

The right-hand side of the differential equation $\dot{L} = f_H(L)$ satisfies the boundedness conditions of Proposition 3.2 We have seen in Section 2, that there exists a constant C dependent only on $\||L|\|$ and h such that

$$\|f_H(L(x))\| \le C, \; \||Df_H(L)|\|_1 \le C \; .$$

□

We prove now Theorem 3.1 in the case $a(x) > 0$ almost everywhere:
Proof.
Construction of sets with arbitrary large return time:
Rohlin's lemma (see [Cor 82]) implies that there exists for each $N \in \mathbb{N}$ a measurable set $Z_N \subset X$ of positive measure such that

$$T^{-N}(Z_N), \ldots, Z_N, T(Z_N), \ldots, T^N(Z_N)$$

102

are pairwise disjoint and such that

$$Y_N = X \setminus \bigcup_{i=-N}^{N} T^i(Z_N)$$

has measure $m(Y_N) \leq 1/N$. The countable set

$$\mathcal{Y} = \{Y_1, Y_2, Y_3, \ldots\}$$

has the property that for almost all $x \in X$ and all $N \in \mathbb{N}$ we can find $Y = Y_{k(N,x)} \in \mathcal{Y}$ such that $T^n(x) \notin Y$ for $n = -N, \ldots, N$. (This can be seen as follows. The sets $U_N := \bigcup_{i=-(N-\sqrt{N})}^{N-\sqrt{N}} T^i(Z_N)$ satisfy $m(U_N) \to 1$ for $N \to \infty$. For almost every point $x \in U_N$ we get $T^i x \notin Y_N$ for all $|i| \leq \sqrt{N}$.)

A countable set of random Jacobi operators with zero mass
Given $L \in \mathcal{L}$ with $m\{a(x) = 0\} = 0$. Define for each $Y \in \mathcal{Y}$ the new random Jacobi operator

$$L_Y = (1_{Y^c})a\tau + ((1_{Y^c})a\tau)^* + b ,$$

where 1_{Y^c} is the characteristic function of the set $Y^c = X \setminus Y$. The random Toda flow for L_Y can be integrated according to the already proved case because $Y = \{(L_Y)_1(x) = 0\}$ has positive measure.

Construction of ϕ:
There exists a mapping

$$\tilde{\phi}_Y : L \mapsto L_Y \mapsto \tilde{\phi}(L_Y) \in L^\infty(X)^2$$

which linearizes the flow. Define

$$\phi = \{\tilde{\phi}_Y\}_{Y \in \mathcal{Y}} = (\tilde{\phi}_{Y_1}, \tilde{\phi}_{Y_2}, \ldots) : \mathcal{L} \to \mathcal{G} .$$

Construction of the left inverse ψ:
Take $\overline{\mathrm{Exp}}_H(\lambda, r) \in \mathcal{H}$. For almost all $x \in X$ there is a sequence $N \mapsto k(N, x)$ such that

$$T^i(x) \notin Y_{k(N,x)}$$

for $i = -N, \ldots, N$. Proposition 3.2 implies that for $N \to \infty$

$$\psi_{Y_N}(\overline{\mathrm{Exp}}_H(\lambda, r))(x)$$

converges to $\mathrm{Exp}_H(L)(x)$ in the weak operator topology, where $\psi_i(\lambda_i', r_i')$ is the operator L_{Y_i}' calculated from the spectral data (λ_i', r_i') satisfying $\psi_i \circ \tilde{\phi}(L_{Y_i}') = L_{Y_i}'$. Define

$$\psi(\overline{\mathrm{Exp}}_H(\lambda, r))(x) = \lim_{N \to \infty} \psi_{Y_{k(N,x)}}(\overline{\mathrm{Exp}}_H(\lambda, r))(x) ,$$

103

where the limit is taken in the weak operator topology.

Conjugation of the flow:

Assume $\phi(L) = (\lambda, r)$. In the weak operator topology we have

$$\begin{aligned}
\mathrm{Exp}_H(L)(x) &= \lim_{N \to \infty} \psi_{Y_{k(N,x)}}(\overline{\mathrm{Exp}}_H(\lambda, r))(x) \\
&= \psi(\overline{\mathrm{Exp}}_H(\lambda, r))(x) = \psi \circ \overline{\mathrm{Exp}}_H \circ \phi(L)(x)
\end{aligned}$$

and so

$$\mathrm{Exp}_H(L) = \psi \circ \overline{\mathrm{Exp}}_H \circ \phi(L) \ .$$

$\square$

Remark. The linearisation for general random Toda flows can also be used to prove the *integrability of the periodic Toda lattice*. For this we take an ergodic dynamical system (X, T, m) such that T^N is not ergodic and leaves invariant a set Y of measure $1/N$. We take functions a, b which are constant on Y. Each isospectral deformation for $L = a\tau + (a\tau)^* + b$ is equivalent to a periodic Toda lattice because the functions a, b keep on being constant on Y. We can now do the above linearisation.

4 QR decomposition

For a real $n \times n$ matrix M, there exists a decomposition $M = QR$, where Q is orthogonal and R is upper triangular. For aperiodic Jacobi matrices, Symes [Sym 82] found that the QR decomposition of $\exp(tL)$ integrates the first Toda flow $L(t)$. This was worked out further in [Dei 83] [Dei 89] leading to the observation that the time 1 map of the Hamiltonian $H(L) = \mathrm{tr}(L \cdot \log(L) - L)$ is just one step in the QR algorithm, an algorithm which is used to diagonalize a matrix numerically. More generally, the following fact is known:

Proposition 4.1 *Given* $h \in C^2(\mathbb{R})$. *In the matrix algebra* $M(d, \mathbb{R})$, *the solution of*

$$\dot{L} = [B_H(L), L], \ L(0) = L_0$$

is given by $L(t) = Q^* L_0 Q$, *where* Q *is obtained by the QR decomposition*

$$\exp(th'(L_0)) = QR \ .$$

Proof. We can write

$$e^{th'(L_0)} = QR$$

in a unique way, because $e^{th'(L_0)}$ is invertible. The dependence of Q, R on t is differentiable. Differentiation gives

$$h'(L_0)QR = \dot{Q}R + Q\dot{R}$$

and after multiplication from the right with R^{-1} and multiplication from the left with Q^*, we get

$$Q^* h'(L_0)Q = Q^* \dot{Q} + \dot{R}R^{-1} \ ,$$

where $Q^*\dot{Q}$ is skew symmetric and $\dot{R}R^{-1}$ is upper triangular. Call $\tilde{L}(t) = Q^*(t)L_0Q(t)$ with $Q(0) = 1$. It follows

$$h'(\tilde{L}(t)) = Q^*h'(L_0)Q = Q^*\dot{Q} + \dot{R}R^{-1} \ .$$

We can compare this with the unique decomposition

$$h'(\tilde{L}(t)) \qquad = \qquad -h'(\tilde{L}(t))^+ + h'(\tilde{L}(t))^-$$
$$+2h'(\tilde{L}(t))^+ + h'(\tilde{L}(t))_0$$

into a skew symmetric and an upper triangular part to get

$$Q^*\dot{Q} = -h'(\tilde{L}(t))^+ + h'(\tilde{L}(t))^- = -B_H(\tilde{L}) \ .$$

Now

$$\begin{aligned}
\frac{d}{dt}\tilde{L}(t) &= \dot{Q}^*(t)L_0Q(t) + Q^*(t)L_0\dot{Q}(t) \\
&= (\dot{Q}^*(t)Q(t))Q^*(t)L_0(t)Q(t) + Q^*(t)L_0Q(t)(Q^*(t)\dot{Q}(t)) \\
&= (\dot{Q}^*(t)Q(t))\tilde{L}(t) + \tilde{L}(t)(Q^*(t)\dot{Q}(t)) \\
&= [h'(\tilde{L}(t))^+ - h'(\tilde{L}(t))^-, \tilde{L}(t)] = [B_H(\tilde{L}(t)), \tilde{L}(t)] \ .
\end{aligned}$$

Because $\tilde{L}(t), L(t)$ satisfy the same differential equation as well as the same initial conditions $L(0) = \tilde{L}(0)$, they must coincide. $\qquad\square$
This can be generalized to the random case:

Theorem 4.2 *For $L \in \mathcal{L}$ and $H(L) = \mathrm{tr}(h(L)) \in C^\omega(\mathcal{L})$, there exists a unitary $Q \in \mathcal{X}$ and $R = \sum_{n\geq 0} R_n\tau^n \in \tau^{-1}\mathcal{X}^+$ such that $\exp(h'(L)) = QR$.*
If $L(t)$ satisfies $\dot{L} = [B_H(L), L]$, one obtains $L(t) = Q^*L(0)Q$ with a QR decomposition $\exp(th'(L_0)) = QR$.

Call $\mathcal{T}$ the Banach space of selfadjoint real tridiagonal matrices in $\mathcal{B}(l^2)$ and

$$\begin{aligned}
\tilde{\mathcal{T}} &= \{ p(L) \mid p \text{ polynomial}, L \in \mathcal{T} \} \ , \\
\overline{\mathcal{T}} &= \{ h(L) \mid h \text{ entire}, L \in \mathcal{T} \} \ .
\end{aligned}$$

We call the weak operator topology on $\mathcal{B}(l^2)$ in the following also the *weak topology*. Denote by B_R the ball with radius R in the Banach space $\mathcal{B}(l^2)$ and $\mathcal{T}_R = \mathcal{T} \cap B_R, \tilde{\mathcal{T}}_R = \tilde{\mathcal{T}} \cap B_R, \overline{\mathcal{T}}_R = \overline{\mathcal{T}} \cap B_R$. The weak topology is metrizable on each ball B_R. We denote this metric by d.

Lemma 4.3 *a) Given an entire function f and $R > 0$. The mapping*

$$\mathcal{T}_R \rightarrow \mathcal{B}(l^2), \ L \mapsto f(L)$$

is weakly continuous.
b) $B_R \times \overline{\mathcal{T}}_R \rightarrow \mathcal{B}(l^2), (L, K) \mapsto L \cdot K$ is weakly continuous.

Proof.

a) Because multiplication $\mathcal{T} \times \mathcal{B}(l^2) \to \mathcal{B}(l^2)$ is weakly continuous, one obtains inductively that $L \mapsto p(L)$ is weakly continuous for every polynomial p. Applying Lemma 3.3 gives that $L \mapsto f(L)$ is weakly continuous for an entire function f.

b) The multiplication $B_R \times \tilde{\mathcal{T}}_R \to \mathcal{B}(l^2)$ is weakly continuous. We can approximate $L \in \tilde{\mathcal{T}}$ in norm by elements in $\tilde{\mathcal{T}}$ and this approximation can be made uniform in the ball B_R. Use again Lemma 3.3. $\qquad\square$

We prove now Theorem 4.2:

Proof. Fix $x \in X$. We can approximate $L(x)$ in the weak operator topology by tridiagonal aperiodic $N \times N$ Jacobi matrices $L^{(N)}$. For such matrices we can form

$$\exp(h'(L^{(N)}(x))) = Q^{(N)}(x)R^{(N)}(x)$$

where $Q^{(N)}(x)$ is orthogonal and $R^{(N)}(x)$ is tridiagonal and we know also that $Q^{(N)}(x)^* L^{(N)}(x)Q^{(N)}(x)$ is the time 1 map of the Hamiltonian flow

$$\dot{L}^{(N)}(x) = [B_H(L^{(N)}(x)), L^{(N)}(x)] = f_H(L^{(N)}) \ .$$

We deform $L(x)$ with the same Hamiltonian flow

$$\dot{L}(x) = [B_H(L(x)), L(x)] = f_H(L)$$

to get $\mathrm{Exp}_H L(x) = Q^*(x)L(x)Q(x)$. **Claim.** $Q^{(N)}(x) \to Q(x)$ in the weak operator topology.

Proof. Consider in addition to the above differential equations for $L^{(N)}$ and L also the differential equations

$$\begin{aligned} \dot{Q}^{(N)}(x) &= -B_H(L^{(N)}(x))Q^{(N)}(x) =: g_H(L^{(N)}, Q^{(N)}) \ , \\ \dot{Q}(x) &= -B_H(L(x))Q(x) =: g_H(L, Q). \end{aligned}$$

We can apply Proposition 3.2 to the system

$$\frac{d}{dt}(L, Q) = (f_H(L), g_H(L, Q))$$

in $\mathcal{B}(l^2(\mathbb{Z}))^2$ in order to show that $Q^{(N)}(x) \to Q(x)$ in the weak operator topology. The assumptions of Proposition 3.2 are readily checked: Because $L \mapsto B_H(L)$ is weakly continuous, we can apply Lemma 4.3 b) to conclude that g is also weakly continuous. There are for $L \in B_R \subset \mathcal{B}(l^2(\mathbb{Z}))$ also the estimates $\|g(L(x), Q(x))\| \leq C_{H,R}$ and $\||Dg(L, Q)\||_2 \leq 2C_{H,R}$ where $\||\cdot\||_2$ is the norm in $\mathcal{L} \times \mathcal{X}$. Because B_H is skew symmetric, the norm of $Q(t)$ is a constant. $\qquad\square$

From Lemma 4.3 a) we have

$$e^{h'(L^{(N)}(x))} \to e^{h'(L(x))} \ .$$

With this, the just proved claim and Lemma 4.3 b) one gets

$$(Q^{(N)})^*(x)e^{h'(L^{(N)}(x))} = R^{(N)}(x) \to Q^*(x)e^{h'(L(x))} = R(x)$$

106

in the weak operator topology. It follows that $R(x)$ is also upper triangular and $e^{h'(L(x))} = Q(x)R(x)$ with $Q^*(x)L_0(x)Q(x) = \mathrm{Exp}_H L(x)$. We have now constructed $Q(x)$ and $R(x)$ pointwise for $x \in X$. There is an upper bound for $[R(x)]_{ij}$ due to the fact that the Toda flows are isospectral. By Lebesgue dominated convergence theorem applied to each function $[Q(x)]_{ij}, [R(x)]_{ij}$ we get also random operators $Q, R \in \mathcal{X}$ which satisfy $QR = \exp(h'(L_0))$ with $Q^*L_0Q = \mathrm{Exp}_H L$. $\qquad \square$

Remarks.
• Theorem 4.2 is not yet very helpful in order to understand more about the qualitative behavior Toda flows. It has been pointed out to us by a referee that the QR decomposition in $\mathcal{X}$ is not unique: For example, $\tau = \tau \cdot 1 = 1 \cdot \tau$ because τ is unitary and upper trigonal. We have used the Toda flows to construct a QR decomposition for certain elements in $\mathcal{X}$ and we don't know how to find and perform directly the right QR decomposition which leads to an integration of the Toda flow.

• Having the right QR decomposition for infinite matrices, one could calculate the time τ map of the periodic Toda lattice by a QR decomposition of an infinite but periodic matrix. As the solutions of the periodic Toda lattice can be expressed by Theta functions (see for example [Tod 81]), it would be interesting to know whether the QR decomposition is a reasonable way to calculate Theta functions numerically.

5 Density of states, Lyapunov exponent and Rotation number, Floquet exponent, determinant. Entropy and index of monotone twist maps.

5.1 The Floquet exponent as an integral of the Toda flow

The functional calculus for a normal element K in the C^* algebra $\mathcal{X}$ defines $f(K)$ for a function $f \in C(\Sigma(K))$. The mapping

$$f \mapsto \mathrm{tr}(f(K))$$

is a bounded linear functional on $C(\Sigma(K))$, and by *Riesz representation theorem*, there exists a measure dk on $\Sigma(K)$ with

$$\mathrm{tr}(f(K)) = \int_{\Sigma(K)} f(E) dk(E) .$$

This measure dk is called the *density of states* of K. Because of

$$1 = \int_X 1 \, dm = \mathrm{tr}(1) = \int_{\Sigma(K)} dk(E) ,$$

the measure dk is a probability measure. For selfadjoint elements $K \in \mathcal{X}$, the density of states dk has its support on R. The integral

$$k(E) = \int_{-\infty}^{E} dk(E')$$

is called the *integrated density of states* of K. To an operator $L \in \mathcal{L}$, we attach a complex valued function by means of

$$w(E) := -\text{tr}(\log(L - E)) \, ,$$

which is a priori defined only for $\text{Im}(E) > 0$. Here the branch of the logarithm is chosen so that $\log(1) = 0$. The function w is called the *Johnson-Moser* function or *Floquet exponent*. This gives also a *determinant*

$$e^{-w(E)} = e^{\text{tr}(\log(L-E))} = \det(L - E) \, .$$

(It is a general fact for von Neumann algebras $\mathcal{X}$ that the existence of a finite trace allows the definition of a determinant on the set of invertible elements in $\mathcal{X}$ [Dix 81] p. 119). For the *transfer cocycle*

$$A_E(x) := a^{-1}(T^{-1}x) \begin{pmatrix} E - b(x) & -a^2(T^{-1}x) \\ 1 & 0 \end{pmatrix}$$

of $L = a\tau + (a\tau)^* + b$ the *Lyapunov exponent* is

$$\lambda(A_E) = \lim_{n \to \infty} n^{-1} \int_X \log \|A_E^n(x)\| \, dm(x) \, ,$$

where $A_E^n(x) = A_E(T^{n-1}x) \ldots A(Tx)A_E(x)$ and the *rotation number* is given by $\rho(A_E) = \pi k(E)$. The rotation number can be defined by the cocycle A_E alone [Del 83].

Theorem 5.1 *In the case when $a(x) > 0$ almost everywhere, the Floquet exponent*

$$w(E) = -\text{tr}(\log(L - E))$$

as well as for $E \in$ R the Lyapunov exponent $\lambda(A_E)$ and the rotation number $\rho(A_E)$ are integrals of the Toda flows.

Proof. The *Thouless formula* relates the mass M and the Floquet exponent $w(E)$ with the Lyapunov exponent $\lambda(A_E)$ and the rotation number $\rho(A_E)$: if L has positive mass then

$$-\lambda(A_E) + i\rho(A_E) = w(E) + \log(M) \, .$$

For the proof see [Car 90]. (We added two proofs in the appendix.) In the case of zero mass, both sides are $-\infty$.

In the case $\text{Im}(E) > 0$, the function $g(z) = -\log(z - E)$ is continuous on the real axis and $w(E) = \text{tr}(g(L))$ is an integral of the Toda flows. The function $E \mapsto w(E)$ is a Herglotz function. Because $w(E)$ is time independent for E in the upper half plane, it is also time independent on the real axis. $\qquad\square$

5.2 Monotone twist maps

Random Jacobi operators appear in a natural way when embedding an abstract dynamical system in a monotone twist map. Assume we have given a *generating function* $l \in C^2(\mathbb{R}^2)$ and $r > 0$, such that

$$
\begin{aligned}
l_{12}(q,q') &= \frac{\partial}{\partial q}\frac{\partial}{\partial q'}l(q,q') \geq r \\
l(q,q') &= l(q+1,q'+1).
\end{aligned}
$$

If we define

$$
\begin{aligned}
p(q,q') &= l_1(q,q') = \frac{\partial}{\partial q}l(q,q') \, , \\
p'(q,q') &= -l_2(q,q') = -\frac{\partial}{\partial q'}l(q,q') \, ,
\end{aligned}
$$

q' can be expressed as a function of q and p. The mapping

$$
S : (q,p) \mapsto (q',p')
$$

is called a *monotone twist map*. It leaves invariant the Lebesgue measure $dqdp$ on the cylinder $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Given an abstract dynamical system (X,T,m). In order to have a critical point q of the *Percival functional*

$$
\mathcal{L}(q) = \int_X l(q,q(T)) \, dm \, .
$$

on the Banach manifold $L^\infty(X,\mathbb{T})$, we must have

$$
\delta\mathcal{L}(q) = l_1(q,q(T)) + l_2(q(T^{-1}),q) = 0 \, .
$$

If there exists a q which satisfies this Euler equation, we have embedded a factor of the given dynamical system inside the twist map. (A factor $(\tilde{X},\tilde{T},\tilde{m})$ is just a homomorphic image of (X,T,m): there exists a measurable not necessarily invertible mapping $\phi : X \to \tilde{X}$ with $\phi T = \tilde{T}\phi$.) The second variation of $\mathcal{L}$ is the random Jacobi operator

$$
L(q) = \delta^2\mathcal{L}(q) = a\tau + (a\tau)^* + b \in \mathcal{L}
$$

where

$$
\begin{aligned}
a(x) &= l_{12}(q(x),q(Tx)) \, , \\
b(x) &= l_{11}(q(x),q(Tx)) + l_{22}(q(T^{-1}x),q(x)) \, .
\end{aligned}
$$

The twist condition

$$
l_{12}(q,q') \geq r > 0
$$

implies that L has positive mass. The random Jacobi operator L obtained as a second variation of the Percival functional and the Floquet exponent w are carrying

information about the embedded system. It follows for example with a result of Mather [Mat 68] that the embedded system is a hyperbolic set if and only if L is invertible, because in the resolvent set of L, the cocycle A_E is uniformly hyperbolic. In this case there is by the implicit function theorem also a neighborhood of generating functions such that the Percival functional has a critical point near the given critical point.

Examples.
• Assume X is a bounded subset on the cylinder $\mathsf{T} \times \mathsf{R}$ which is invariant under the twist map S and has positive but finite Lebesgue measure. Call T the restriction of S on X and m the normalized measure on X induced from the Lebesgue measure. There is a critical point $q(x) = \pi_1 x$ where π_1 is the projection on the angle coordinate of the cylinder. The real part of $\log(M) - w(0)$ is the entropy of the twist map restricted to X. The imaginary part of $w(0)$ is an *index*.
For generating functions of the type

$$l(x, x') = \frac{(x' - x)^2}{2} + V(x) \,,$$

the twist map S can be defined on the torus $X = \mathsf{T}^2$. The system (X, T, m) with $T = S$ and $m = dx dy$ can then be taken as the abstract dynamical system and the Floquet exponent $w(E)$ gives

$$w(0) = -\text{entropy} + \mathrm{i} \cdot \text{index} \,.$$

because $M = 1$ in this case.

• A finite dynamical system is just a cyclic permutation T of a finite set X. The Jacobi operator is then periodic of period $|X|$. Finding critical points of the functional $\mathcal{L}$ is equivalent to finding periodic points. Periodic points of period $|X|$ always exist. The rotation number of the Jacobi operator is related to the *Morse index* of the critical point [Mat 84]. When the Jacobi operator is restricted to the finite dimensional Hilbert space of $|X|-$ periodic sequences in $l^2(\mathsf{Z})$, it is just a periodic Jacobi matrix.

• If (X, T, m) is an ergodic automorphism of the circle, nontrivial critical points correspond to invariant circles or Mather sets. Mather's result [Mat 82] proves the existence of nontrivial critical points. There are known examples, where the corresponding random Jacobi operator is invertible because a Mather set can be hyperbolic (see [Gor 85]).

• If X is a closed bounded S invariant set of the cylinder and T is the restriction of S onto X there exists a T invariant probability measure on X. Again, $a(x) = \pi_1(x)$ is a critical point of the functional.
An example. If the twist map has a *transverse homoclinic point* x_0 coming from

a hyperbolic fixed point, then X can be chosen as the closure of $\{S^n(x_0) \mid n \in \mathbb{Z}\}$. Again one has a hyperbolic system and the corresponding random Jacobi operator is invertible.

Remark. Toda deformations and twist maps are still unrelated. Our motivation to study random Toda systems was the hope to make deformations of cocycles appearing in twist maps like the Standard map in order to gain more information about the Lyapunov exponents. It is thinkable that there are deformations which lead to cocycles, where Wojtkowsky's cone criterion [Woj 85] (a necessary and sufficient condition for positive Lyapunov exponents) is applicable to prove positive Lyapunov exponents.

6 Generalizations and questions

6.1 Toda lattices over non-commutative dynamical systems

Let $\mathcal{A}$ be any C^* algebra and $T : \mathcal{A} \to \mathcal{A}, a \mapsto a(T)$ be an automorphism of this algebra. Assume $\mathcal{A}$ has a trace satisfying

$$\text{trace}(ab) = \text{trace}(ba) \,.$$

The crossed product $\mathcal{X}$ of the algebra $\mathcal{A}$ with the dynamical system is again a C^* algebra. Elements in $\mathcal{X}$ can be written as $K = \sum_n K_n \tau^n$ where $K_n \in \mathcal{A}$. On $\mathcal{X}$ there is also a trace defined by

$$\text{tr}(K) = \text{trace}(K_0) \,.$$

Define the Banach space $\mathcal{L} = \{L \in \mathcal{X} \mid L_n = 0 \ |n| > 1, L = L^*\}$. For $H \in C^\omega(\mathcal{L})$, the differential equation

$$\dot{L} = [h'(L)^+ - h'(L)^-, L] = [B_H(L), L]$$

is a Hamiltonian flow in the subspace $\mathcal{L} \subset \mathcal{X}$. It has the Hamiltonian $H(L) = \text{tr}(h(L))$. Let $Q(t)$ be defined by the differential equation

$$\dot{Q} = -B_H(L)Q$$

with initial conditions $Q(0) = 1$. In $\mathcal{X}$, the equality $L(t) = Q(t)^* L(0) Q(t)$ shows that the flow is isospectral. It has the integral $\text{tr}(f(L))$ for each $f \in C(\mathbb{R})$. Given a representation $x : \mathcal{A} \to \mathcal{B}(H)$, which means that $x(a)$ is a bounded linear operator on the Hilbert space $\mathcal{H}$ for all $a \in \mathcal{A}$. If we use the notation $x(a) = a(x)$ and $x(a(T^n)) = a(T^n x)$, each element $K \in \mathcal{X}$ has a representation $K(x)$ in the Hilbert space $l^2(\mathbb{Z}, \mathcal{H})$ defined by the matrix

$$[K(x)]_{mn} = K_{n-m}(T^m x) \,.$$

111

The matrix $K(x)$ is a Jacobi matrix where the entries are linear operators on $\mathcal{H}$.
Examples.
• If $\mathcal{A} = L^\infty(X)$ we are in the case discussed already because an automorphism T comes from a dynamical system (X, T, m).

• Assume X is a compact topological space and $\mathcal{A} = C(X)$. We have then a deformation in

$$\mathcal{L}_c := \{L = a\tau + (a\tau)^* + b \mid a, b \in C(X, \mathbf{R})\} \subset \mathcal{L} \, .$$

Also if X is a compact manifold and $\mathcal{A} = C^r(X, \mathbf{R})$, the space

$$\mathcal{L}_r := \{L = a\tau + (a\tau)^* + b \mid a, b \in C^r(X, \mathbf{R})\} \subset \mathcal{L}_c \subset \mathcal{L}$$

for $r = 1, 2, \ldots$ is kept invariant by the Toda flows.

• $\mathcal{A} = L^\infty(X, M(l, \mathbf{R}))$ gives a quite natural *non-commutative generalization of the Toda lattice*. Such as system could also be called *quantum Toda lattice*. A suggestion to study such systems is in [Chu 86]. In the case $|X| < \infty$, this gives new finite dimensional systems which are candidates for being integrable. The flow can be written as an isospectral deformation of operators called *stochastic Jacobi matrices on the strip* [Kot 88]. They arise as second variations of higher dimensional twist maps like the *Fröschle map* (see [Koo 86]). A non-abelian Toda lattice for half infinite matrices is proposed in [Ber 86b].

6.2 More general Hamiltonians

We have chosen Hamiltonians which are defined by entire functions $h \in C^\omega$. Like this, we could use Cauchy's existence theorem for differential equation in a Banach space. One has to be careful in doing generalizations by choosing functions h which are only analytic in a neighborhood of the spectrum $\Sigma(L)$: it can happen that, for a bounded operator K on $l^2(\mathbf{Z})$, the operator $K^+ - K^-$ is no longer bounded [Dei 85]. Nevertheless one can consider functions $h' \in L^\infty(\Sigma(L))$. The functional calculus defines then $h'(L) \in \mathcal{X}$. Even if $B_H(L) = h'(L)^+ - h'(L)^-$ is unbounded, one can obtain in this way Toda flows in a *weak sense* ([Dei 85]): For all $u, v \in \{u \in l^2(\mathbf{Z}) \mid \exists n_0 > 0 \; u_n = 0, \; \forall |n| > n_0\}$

$$\frac{d}{dt} < u, Lv > = - < B_H(L)u, Lv > - < Lu, B_H(L)v > \, .$$

Remark. The Toda orbit obtained by weak flows would be bigger than the orbit obtained by analytic Hamiltonians. It would be interesting to know whether it is possible to reach like this every operator K in with the same spectrum and mass then L in a strong neighborhood of L.

6.3 Deformation of complex Jacobi operators and deformations with complex time

We considered only real Jacobi operators and deformations where time is real. If $a, b \in L^{\infty}(X, \mathbf{C})$ then

$$L = a\tau + \tau^* a + b \in \mathcal{L}_{\mathbf{C}}$$

is no more selfadjoint in general. Even if we make isospectral deformations, the norm can blow up. There are actually isospectral operators to a given operator which have arbitrary big norm. Given an arbitrary entire function h. The differential equation

$$\dot{L} = [h'(L)^+ - h'(L)^-, L]$$

in the *complex* Banach space $\mathcal{L}_{\mathbf{C}}$ has locally a unique analytic solution $t \mapsto L(t)$ for t in a disc $\mathbf{D}_r(0) \subset \mathbf{C}$.

If we restrict to real time, one can define deformations of complex finite dimensional Jacobi operators (see [Chu 84a]): Also for the random version, the differential equation

$$\dot{L} = [B_H(L), L]$$

with

$$B_H(L) = h'(L)^+ - (h'(L)^+)^* + \mathrm{i} \cdot \mathrm{Im}(h'(L))_0$$

defines an isospectral deformation in $\mathcal{L}_{\mathbf{C}}$. Exactly in the same way as before, one can decompose $\exp(t \cdot h'(L)) = QR$ into a *unitary* Q and upper triangular R such that $L(t) = Q^*(t)L(0)Q(t)$. The flow does not extend to a flow with complex time.

6.4 Random Singular-Value-Decomposition flows

Toda like deformations have been generalized to arbitrary matrices [Dei 89]. Given any real $n \times n$ matrix A and a real entire functions h, one can define the so called *Singular-Value-Decomposition* (SVD) flow (see [Dei 91])

$$\dot{A} = B_H(AA^*)A - AB_H(A^*A) .$$

Under the map $A \mapsto L = A^*A$ the flow goes over in the Toda like flow

$$\dot{L} = [h'(L)^+ - h'(L)^-, L] .$$

Both flows have a unique global solution and preserve the singular values of A respectively the spectrum of L. The flow can be integrated explicitly as follows: With the QR decompositions

$$\begin{aligned} e^{th'(A^*A)} &= Q_1(t)R_1(t) , \\ e^{th'(AA^*)} &= Q_2(t)R_2(t) , \end{aligned}$$

one gets $A(t) = Q_2^* A Q_1$.

Given now any element $A \in \mathcal{X}$ where $\mathcal{X}$ in the C^* crossed product of the Banach algebra $L^\infty(X)$ with a dynamical system. The deformation

$$\dot{L} = B_H(AA^*)A - AB_H(A^*A)$$

is a random version of the singular decomposition flow. The qualitative behavior could be investigated in the same way by approximation in the weak operator topology by finite dimensional SVD flows.

6.5 Deformation of operators with no boundary conditions

Why is it useful to study deformations of operators with *random boundary conditions*? The Toda deformations can also be done for any tridiagonal bounded operator L on $l^2(\mathbb{Z})$ and the flow given by the differential equation

$$\dot{L} = [B_H(L), L]$$

can be approximated in the weak operator topology by finite dimensional Toda flows. The advantage of looking at random operators is the *existence of a finite trace* which is the ergodic average of the diagonal of the operator. Most integrals can be expressed by this trace. Such integrals are invariant by the shift

$$L_{ij} \mapsto L_{i+1,j+1} .$$

Mathematically the Random Toda flows are deformations in a von Neumann algebra of finite type while the general Toda flows are deformations in a von Neumann algebra of infinite type.

Considering random operators instead of general operators gives an important *symmetry*, in that shift-invariant "macroscopic quantities" exist.

6.6 Some questions

• The spectral and *inverse spectral problem* for random Jacobi operators is not solved. What does the isospectral set look like? For which dynamical systems do random operators with the same mass M and the same Floquet exponent $w(E)$ form a group? Does the determinant of the resolvent $(L - E)^{-1}$ determine the isospectral set ? What spectra do occur over a given dynamical system? What kind of spectra occur generically in $\mathcal{L}$ for fixed dynamical system (X, T, m)?

• Can one find the explicit solutions of the random Toda lattice? The integration proposed in this work is somehow artificial. The hope is that the solutions can be written in terms of generalized theta functions. What are the properties of the *transcendental hyperelliptic curve*

$$y^2 = \det(L - E) = e^{-w(E)} ,$$

where $w(E)$ is the Floquet exponent? Especially interesting would be some knowledge about infinite dimensional Jacobians. Is there an infinite dimensional generalization of the Jacobi map?

• What is the asymptotic behavior of the random Toda lattice? What happens for $t \to \pm\infty$ in the case when $\{a(x) = 0\}$ has positive measure? To our knowledge, the general asymptotic behavior of the tied finite dimensional Toda lattice analogous to the first flow [Mos 75] is not known. What happens for Hamiltonians outside the generic set where one has convergence to diagonal operators? (See [Chu 84].)
Is there recurrence in the weak operator topology in the case when $a(x) > 0$ almost everywhere?

• Is it possible to deform a random Jacobi operator in a way to make the spectral problem or the problem of calculating the Floquet exponent more easy? Can one deform a twist mapping in a way such that the corresponding random Jacobi operator is deformed in an isospectral way?

• Is any isospectral deformation in the crossed product of any Banach algebra with any dynamical system integrable?

7   Appendix: Proof of the Thouless formula

We want to show here the Thouless formula

$$w(E) + \log(M) = -\lambda(A_E) + ik(E) .$$

The proof of the Thouless formula needs some preparation: Define for the interval $\Lambda = \{N, N+1, \ldots, M-1, M\}$, the matrix

$$[L^\Lambda]_{ij} = [L]_{ij}, \quad i, j \in \Lambda$$

which is a linear operator on $C^{|\Lambda|}$. If E_j^Λ are the eigenvalues of $[L^\Lambda]$ and $\delta(E)$ denotes the Dirac measure located at a point $E \in \mathbb{R}$, we define the probability measures

$$d\rho^\Lambda = \frac{1}{|\Lambda|} \sum_{k \in \Lambda} \delta(E_k^\Lambda)$$

on the real line. For the finite dimensional operators $[L^\Lambda]$, we define the normalized trace

$$\text{tr}([L^\Lambda]) = \frac{1}{|\Lambda|} \sum_{k \in \Lambda} [L^\Lambda]_{kk}$$

so that

$$\text{tr}([L^\Lambda]) \to \text{tr}(L)$$

for $|\Lambda| \to \infty$.

115

Lemma 7.1 (Avron-Simon) $d\rho^\Lambda$ *converges weakly to dk for* $|\Lambda| \to \infty$.

Proof. We have to show that for each continuous function $f : \mathbf{R} \to \mathbf{C}$,

$$\operatorname{tr}(f(L^\Lambda)) = \int f \, d\rho^\Lambda \to \int f \, dk = \operatorname{tr}(f(L))$$

for $|\Lambda| \to \infty$. It is convenient to introduce new probability measures dk^Λ on $\mathbf{R}$ such that for a function $f \in C(spec(L))$ we have

$$\int f \, dk^\Lambda = \operatorname{tr}([f(L)^\Lambda]) \, .$$

Compare

$$\int f \, d\rho^\Lambda = \operatorname{tr}(f([L^\Lambda])) \, .$$

For $|\Lambda| \to \infty$ the measures dk^Λ converge weakly to dk because

$$\begin{aligned}
\int f \, dk^\Lambda &= \operatorname{tr}([f(L)^\Lambda]) = \frac{1}{|\Lambda|} \sum_{k\in\Lambda} [f(L)^\Lambda]_{kk} \\
&= \frac{1}{|\Lambda|} \sum_{k\in\Lambda} [f(L)]_{kk}
\end{aligned}$$

converges by Birkhoff's ergodic theorem to

$$\int_X f(L)_0 \, dm = \int f \, dk \, .$$

It is therefore enough to show that for each continuous function f

$$\operatorname{tr}([f(L)^\Lambda]) - \operatorname{tr}(f([L^\Lambda])) \to 0$$

for $|\Lambda| \to \infty$. It is enough to show this for polynomials $f(E) = E^n$ because linear combinations of such polynomials are dense in the set of continuous functions on $\mathbf{R}$. We calculate

$$\begin{aligned}
\operatorname{tr}([(L^n)^\Lambda]) &= \frac{1}{|\Lambda|} \sum_{k\in\Lambda} \sum_{j_1,\dots,j_{n-1}=-\infty}^{\infty} [L]_{kj_1} \cdots [L]_{j_{n-1}k} \\
\operatorname{tr}([L^\Lambda]^n) &= \frac{1}{|\Lambda|} \sum_{k\in\Lambda} \sum_{j_1,\dots,j_{n-1}\in\Lambda} [L]_{kj_1} \cdots [L]_{j_{n-1}k}
\end{aligned}$$

and so, because $[L]_{ij} = 0$ for $|i - j| > 1$

$$\begin{aligned}
\operatorname{tr}([(L^n)^\Lambda] - [L^\Lambda]^n) &= \frac{1}{|\Lambda|} \sum_{k\in\Lambda} \sum_{j_1,\dots,j_{n-1}\notin\Lambda} [L]_{kj_1} \cdots [L]_{j_{n-1}k} \\
&\leq \frac{1}{|\Lambda|} \||L|\|^n 2 \cdot 3^{n-1} \, .
\end{aligned}$$

(A rough estimate shows, that there are at most $2 \cdot 3^{n-1}$ summands on the right hand side.) The right hand side goes to zero for $|\Lambda| \to \infty$. □

We will now prove the Thouless formula following Avron-Simon .

116

Proposition 7.2 $\lambda(A_E) = \int_{\mathbb{R}} \log|E - E'| \, dk(E') - \log(M)$

Proof. Define $B_E = a(T^{-1})A_E$. The claim follows from

$$\lambda(B_E) = \int \log|E' - E| \, dk(E') .$$

First, we assume $E \in \mathbb{C} \setminus \mathbb{R}$. In this case, the function $f(E') = \log|E - E'|$ is continuous in the support of dk. We see by induction, that for fixed $x \in X$

$$B_E^n = \begin{pmatrix} P_n & Q_{n-1} \\ a_{n-1}P_{n-1} & a_{n-1}Q_{n-2} \end{pmatrix} ,$$

where P_n and Q_n are polynomials of degree n in E with leading coefficient 1. We have $P_n = 0$ if and only if $Lu = Eu$ has a solution u with $u_0 = u_{n+1} = 0$ and $Q_n = 0$ if and only if $Lu = Eu$ has a solution u with $u_1 = u_{n+2} = 0$. Thus,

$$P_n(E) = \prod_{k=1}^{n}(E_j^{(n)} - E), \;\; Q_n(E) = \prod_{k=1}^{n}(\hat{E}_j^{(n)} - E) ,$$

where $E_j^{(n)}$ (rsp. $\hat{E}_j^{(n)}$) are the eigenvalues of L with boundary conditions $u_0 = u_{n+1} = 0$ (rsp. $u_1 = u_{n+2} = 0$). We can write therefore

$$\frac{1}{n}\log|P_n(E)| = \int \log|E' - E| \, d\rho^{\{1,\cdots,n\}}(E') ,$$
$$\frac{1}{n}\log|Q_n(E)| = \int \log|E' - E| \, d\rho^{\{2,\cdots,n+1\}}(E') ,$$

and because $d\rho^{\{1,\cdots,n\}}$ and $d\rho^{\{2,\cdots,n+1\}}$ go weakly to dk for $n \to \infty$ we have

$$\lambda(B_E) = \int \log|E' - E| \, dk(E')$$

for $Im(E) \neq 0$. Both functions

$$E \;\mapsto\; f(E) = \int \log|E - E'| \, dk ,$$
$$E \;\mapsto\; \lambda(B_E)$$

are subharmonic. So, we have for all $E \in \mathbb{R}$

$$\frac{1}{\pi r^2} \int_{|E'-E|\leq 1} \lambda(B_{E'}) \, d(E') = \frac{1}{\pi r^2} \int_{|E-E'|\leq 1} f(E') \, dE'$$

and taking the limit $r \to 0$, we obtain

$$\lambda(B_E) = \int \log|E - E'| \, dk(E')$$

for all $E \in \mathbb{C}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

8 Appendix: A second proof of the Thouless formula

The Thouless formula

$$w(E) + \log(M) = -\lambda(A_E) + ik(E)$$

has a simple proof with the help of a *formula of Green,MacKay,Meiss*, which relates the determinant of the N periodic Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdot & & a_N \\ a_1 & b_2 & 0 & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & \cdot & b_{N-1} & a_{N-1} \\ a_N & 0 & \cdot & a_{N-1} & b_N \end{pmatrix},$$

with the trace of the *monodromy matrix*

$$A^n = A_N \circ A_{N-1} \circ \cdot \circ A_1 ,$$

where A_i is the *transfer matrix*

$$A_i(x) = a_{i-1}^{-1} \begin{pmatrix} -b_i(x) & -a_{i-1}^2 \\ 1 & 0 \end{pmatrix}$$

and the number $\prod_i a_i$. This relation is given by

Lemma 8.1

$$\text{trace}(A^N) - 2 = \frac{\det(L)}{\prod_{i=1}^N a_i}$$

Proof. Call λ, λ^{-1} the eigenvalues of the $SL(2, \mathbb{R})$ matrix A^N.

$$\det(A^N - \xi)$$

is a polynomial in $\xi + \xi^{-1}$ which vanishes for $\xi = \lambda, \lambda^{-1}$. This implies that

$$\det(A^N - \xi) = (\xi + \xi^{-1}) - \text{tr}(A^N) .$$

The zeros of

$$\det(L(\xi)) = \det \begin{pmatrix} b_1 & a_1 & 0 & \cdot & & \frac{a_N}{\xi} \\ a_1 & b_2 & 0 & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & \cdot & b_{N-1} & a_{N-1} \\ a_N\xi & 0 & \cdot & a_{N-1} & b_N \end{pmatrix},$$

are the eigenvalues of A^N. So

$$\det(L(\xi)) = C \cdot ((\xi + \xi^{-1}) - \text{tr}(A^N)) \, ,$$

where C is a constant. Because we have for $\xi \to \infty$

$$\det(L(\xi)) = -(\prod_{i=1}^{N} a_i)\xi + \ldots$$

the constant is $C = -\prod_{i=1}^{N} a_i$ and

$$\det(L(\xi)) = -(\prod_{i=1}^{N} a_i) \cdot (\xi + \xi^{-1} - \text{tr}(A^N)) \, .$$

The claim follows with $\xi = 1$. □

Proof of the Thouless formula. Given a random Jacobi operator L, we approximate L weakly by periodic Jacobi matrices $L^{(N)}$. With the formula in the lemma we get for E in the upper half plane

$$\frac{1}{N} \log(\text{trace}(A_E^N) - 2) = N^{-1} \log \det(L - E) - N^{-1} \log M \, . \tag{1}$$

We have
$$3^{-1}\|A^N\| \le \text{trace}(A^N) - 2 \le 3\|A^N\|$$

if $\|A^N\|$ is big enough. If $\|A^N\|$ stays bounded then the left hand side goes to zero for $N \to\in \infty$ as well as $\frac{1}{N} \log \|A^N\|$. Applying *Kingman's subadditive ergodic theorem* to the left hand side of Equation 1 and *Birkhoff's ergodic theorem* to the right hand side of Equation 1 gives the the Thouless formula

$$\lambda(A_E) = \text{tr}(\log(L - E)) - M \, .$$

The same subharmonicity argument like in the last proof proves the result for general $E \in \mathbb{C}$. □

9   Appendix: The spectrum of elements in $\mathcal{X}$

The following result generalises Pastur's theorem for random Jacobi matrices over an ergodic dynamical system (X, T, m). The proof is the same.

Proposition 9.1 *For every normal $K \in \mathcal{X}$, there exists $\Sigma \subset \mathbb{C}$ such that the spectrum $\sigma(K(x)) = \Sigma$ for almost all $x \in X$. Moreover the discrete spectrum of $K(x)$ is empty for almost all $x \in X$.*

Proof. Given a projection

$$P = P^* = P^2 \in \mathcal{X} .$$

Denote by Trace the usual trace for matrices which is also allowed to be ∞. We have

$$\mathrm{Trace}(P(x)) = \dim \mathrm{Ran}(P(x))$$

and

$$\mathrm{Trace}(P(T^n x)) = \mathrm{Trace}(P(x))$$

for all $n \in \mathbb{Z}$. Therefore

$$
\begin{aligned}
\mathrm{Trace}(P(x)) &= \int_X \mathrm{Trace}(P(x)) \, dm(x) \\
&\geq \sum_{|i| < N} \int_X P(x)_{ii} \, dm(x) \\
&= (2N+1) \int P(x)_{00} \, dm(x) .
\end{aligned}
$$

Since N can be arbitrary, $\mathrm{Trace}(P(x)) = 0$ or $\mathrm{Trace}(P(x)) = \infty$. The ergodicity of T implies that either $\mathrm{Trace}(P(x)) = 0$ for almost all $x \in X$ or $\mathrm{Trace}(P(x)) = \infty$ for almost all $x \in X$. Given a Borel set Δ in the complex plane $\mathbb{C}$, we can define with the functional calculus the projection $E_\Delta = 1_\Delta(K) \in \mathcal{X}$. Let Δ be in $\mathcal{B}$, a countable basis for the topology in $\mathbb{C}$. Then

$$\eta(\Delta) = \begin{cases} \infty & \text{for } \dim \mathrm{Ran}(E_\Delta) = \infty \\ 0 & \text{for } \dim \mathrm{Ran}(E_\Delta) = 0 . \end{cases}$$

The set

$$X_\Delta = \{ x \in X \mid \dim \mathrm{Ran}(E_\Delta) = \eta(\Delta) \}$$

has full measure and so also

$$X_0 = \cap_{\Delta \in \mathcal{B}} X_\Delta .$$

Given two points $y_1, y_2 \in X_0$. We show that the spectrum of $K(y_1)$ and $K(y_2)$ coincide. Assume E is not in the spectrum of $K(y_1)$. There exists $\Delta \in \mathcal{B}$ such that $E \in \Delta$ and $\Delta \cap spec(K(y_1))$ is empty. Because $\dim Ran(E_\Delta(y_1)) = 0$, we have

$$\dim \mathrm{Ran}(E_\Delta(x)) = 0$$

for all $x \in X_0$ and E is therefore not in the spectrum of $K(y_2)$. The claim follows by interchanging the roles of y_1, y_2.

If there would exist E in the discrete spectrum of $K(x)$ for $x \in X_0$ then $0 < \dim \mathrm{Ran}(E_\Delta) < \infty$ for a $\Delta \in \mathcal{B}$ which would contradict the choice of $x \in X_0$. $\qquad\square$

10 Appendix: The integration of aperiodic Toda lattices

The integration of the tied Toda lattice has been performed in [Mos 75] (see also [Mos 75] or [Dei 85]). In [Mos 75] is a complete picture of this scattering problem. Later [Dei 80] one has recognized that the tied Toda lattice (and many of other integrable systems) is a constrained harmonic motion.

We want to repeat here this integration for all the higher Toda flows

$$\dot{L} = [(L^n)^+ - (L^n)^-, L] = [B_n, L], \quad n = 1, \ldots, N - 1$$

and see that the vector fields are linearly independent and give $N - 1$ commuting flows on a $N - 1$ dimensional surface. It seems that the complete analysis of the scattering properties of a general Hamiltonian flow

$$\dot{L} = [B_H(L), L]$$

is not yet done and we don't deal with the question here also. If $H(L) = tr(h(L))$ is so that h' is injective on the spectrum of L then $L(t)$ converges to a diagonal matrix for $|t| \to \infty$. There are probably new features appearing when h' is no more injective on the spectrum. Given L there exists a generic set of Hamiltonians such that h' is injective on Σ.

The space of aperiodic $N \times N$ Jacobi matrices is a $2N - 1$ dimensional vector space. There are N linear independent commuting integrals

$$F_k = (k + 1)^{-1} tr(L^{k+1})$$

of the Toda flows. Fixing these integrals, there stays a $N - 1$ dimensional surface and at each point we have a natural coordinate system given by the flows generated by the Hamiltonians F_k.

Given an aperiodic Jacobi matrix L acting on an N-dimensional vector space with basis $\{e_1, \ldots e_N\}$. The spectral theorem allows to write $L = \int E d\lambda(E)$, where $d\lambda$ is an operator valued measure. We can also write this as

$$L = \sum_{i=1}^{N} \lambda_i P(\phi_i) \,,$$

where $P(\phi_i)$ is the projection on the eigenspace of the eigenvalue λ_i.

Proposition 10.1 a) The mapping

$$L \mapsto d\mu(E) = <e_1, d\lambda(E)e_4> = \sum \frac{r_i}{r} \delta(\lambda_i)$$

121

with $r = \sum_i r_i$ linearizes the Toda flows:
Given $h \in C^\omega(\mathcal{L})$. The flow

$$\dot{L} = [h'(L)^+ - h'(L)^-, L]$$

induces the motion

$$\dot{r}_k(t) = e^{h'(\lambda_k)t} r_k(0) \,,$$
$$\dot{\lambda}_k = 0$$

of the measure $d\mu$.

b) From the measure $d\mu$, the operator L can be calculated back by making Gramm-Schmidt orthonalisation of the polynomials

$$\{1, E, E^2, \ldots, E^{N-1}\}$$

with respect to the scalar product

$$< P, Q > = \int P(E)Q(E)\, d\mu(E)\,.$$

If $\{P_1, \ldots P_n\}$ is this orthonormal basis then

$$b_n = \int E P_n(E) P_n(E)\, d\mu(E)\,,$$
$$a_n = \int E P_n(E) P_{n+1}(E)\, d\mu(E)\,.$$

c) The flows with $h(L) = L^n$, $n \in \{1, \ldots N-1\}$ are all linearly independent and commuting. The isospectral set is homeomorphic to the $N-1$ dimensional simplex

$$\{r \in \mathsf{R}^N \mid \sum_{i=1}^{N} r_i = 1\}\,.$$

If h' is injective on the spectrum, then for $|t| \to \infty$ the matrix $L(t)$ converges to a diagonal matrix.

Proof. a) Given an aperiodic $N \times N$ Jacobi matrix L. By the spectral theorem it can be written as $L = \int E\, dk(E)$ where dk is the spectral measure of L. Define the real-valued spectral measure $d\mu(E) = < e_1, dk(E)e_1 >$, where e_1 is the first basis vector. One has then for example $L_{11} = \int E\, d\mu(E)$. The measure $d\mu$ is a sum of Dirac measures:
$$d\mu = \sum_{k=1}^{N} \frac{r_k}{r} \delta(\lambda_k)\,,$$
where λ_k are the eigenvalues of L and $r = \sum r_k$. The matrix L has N simple eigenvalues if $a_k > 0$ because the equation $Lu = \lambda u$ shows that there can be only a

one dimensional space of eigenvectors belonging to λ (if we have given u_1, the other components are determined by the equation $Lu = \lambda u$).

Call $u = u(\lambda_k)$ the eigenvector to the eigenvalue λ_k which is normalized by $||u|| = 1$ and $u_1(\lambda_k) > 0$. From $Lu = \lambda u$ we get

$$L^n u = ((L^n)^+ + (L^n)^- + (L^n)^0)u = \lambda^n u \ .$$

From $Lu = \lambda u$ and $\dot{L} = [B_n, L]$ we get $L(t) = U(t)L(0)U(t)^*$ with $\dot{U} = B_n U, U(0) = I$ and $u(t) = U(t)u(0)$. It follows

$$\dot{u} = B_n u = ((L^n)^+ - (L^n)^-)u \ .$$

Especially, because $(L^n)^- u_1 = 0$

$$\dot{u}_1 = ((L^n)^+ u)_1 = (\lambda^n - (L^n)^0)u_1 \ .$$

From the spectral representation

$$(L^n)^0 = \int E^n \, d\mu(E) = \sum_j \frac{\lambda_j^n r_j}{r}$$

we obtain

$$\dot{u}_1(\lambda_k) = (\lambda_k)^n - \sum_{j=1}^{N} \frac{\lambda_j r_j}{r} u_1(\lambda_k) \ . \tag{2}$$

From

$$((L - \lambda)^{-1} e_1, e_1) = \sum_k \frac{(u(\lambda_k), e_1)^2}{\lambda - \lambda_k} = \sum_k \frac{r_k^2/r^2}{\lambda - \lambda_k}$$

we get $u_1(\lambda_k) = r_k/r$ and putting this in Equation 2 gives

$$\dot{r}_k = \lambda_k^n r_k \ .$$

Call $\xi_k = u_1(\lambda_k) = r_k/r$. Since the N eigenvectors must span the whole space, the number ξ_k can't be zero (else $u(\lambda_k)$ would be zero). Because the *van der Monde determinant*

$$det \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{N-1} & \lambda_N \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (\lambda_1)^{N-2} & (\lambda_2)^{N-1} & \cdots & (\lambda_{N-1})^{N-2} & (\lambda_N)^{N-2} \\ (\lambda_1)^{N-1} & (\lambda_2)^{N-1} & \cdots & (\lambda_{N-1})^{N-1} & (\lambda_N)^{N-1} \end{pmatrix} = \prod_{i<j}(\lambda_j - \lambda_i) \neq 0 \ ,$$

all the flows given by $h(L) = L^n$, with $n \in \{1, \ldots, N-1\}$ are linearly independent: Given $\dot{r}_k = v_k r_k$ we can find a suitable linear combination of higher Toda flows which realises this flow. There are only $N - 1$ linear independent flows because the zero'th

flow $\dot{r}_k = r_k$ does not give a flow when constrained to $\sum r_k = 1$. We can directly calculate that the flow generated by F_k are commuting: The Poisson bracket

$$\{F_k, F_l\} = 2\text{tr}(L^k(L^+ - L^-)L^l)$$

is vanishing because taking the adjoint doesn't change the trace but the sign. One can also see it differently. The functions F_k are also integrals under the flow generated by F_l and $0 = \dot{F}_k = \{F_l, F_k\}$.

The isospectral set of a given aperiodic Jacobi operator is the simplex of probability measures on the spectrum $\sigma(L)$ which have the support on the whole set.

Given now a Hamiltonian

$$H(L) = \text{tr}(h(L)) \in C^\omega(\mathcal{L})$$

we get by linearity

$$\dot{r}_k = \sum_n h_n(\lambda_k)^n r_k .$$

We have now translated the motion from the (a, b) coordinates into (λ, ξ) coordinates which have a simple evolution:

$$\dot{\lambda}_k = 0, \quad \dot{\xi}_k = h'(\lambda_k)\xi_k .$$

b) From (λ_k, ξ_k) we can calculate back (a, b): To see this define the measure $d\mu = \sum_{k=1}^N \xi_k \delta(\lambda_k)$ and make Gramm Schmidt orthogonalisation of the functions

$$(1, E, E^2, \ldots, E^{N-1})$$

with respect to the scalar product

$$< P, Q > = \int P(E)Q(E) \, d\mu(E) .$$

This gives an orthonormal bases $(P_1, P_2, \ldots, P_N)$ from which one (a, b) can be recovered:

$$b_n = \int E P_n(E) P_n(E) \, d\mu(E), \ a_n = \int E P_n(E) P_{n+1}(E) \, d\mu(E) .$$

Proof. Define for each E an infinite vector $P_n(E)$ which satisfies

$$(LP(E))_n = E P_n(E), \ n = 1, \ldots N ,$$

$$P_n(E) = 0, \ n \leq 0, P_1(E) = 1 .$$

From the fact that

$$\begin{pmatrix} P_{n+1}(E) \\ P_n(E) \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A^n \begin{pmatrix} P_1 \\ P_0 \end{pmatrix}$$

we see that P_n is a polynomial in E of degree $n - 1$. The vectors

$$(f_i(1), f_i(2), \ldots, f_i(N)) := (\sqrt{\xi_i} P_1(\lambda_i), \ldots, \sqrt{\xi_i} P_N(\lambda_i))$$

are eigenvectors of L and $U_{ij} = f_i(j)$ is an orthogonal matrix if we choose the scalar product in order that f_i are orthonormal. We have then

$$
\begin{aligned}
< f_l, f_m > &= \sum_k < f_l(k), f_m(k) > = \sum_k < f_k(l), f_k(m) > \\
&= \sum_k P_l(\lambda_k) P_m(\lambda_k) r_k = \int P_l P_m \, d\mu .
\end{aligned}
$$

We can write
$$L = U^* Diag(\lambda_1, \ldots, \lambda_n) U ,$$

and so
$$b_l = \sum_i \lambda_i f_i^2(l) = \int E P_l P_l \, d\mu(E) ,$$

$$a_l = \sum_i \lambda_i f_i(l) f_i(l+1) \, d\mu = \int E P_l P_{l+1}(E) .$$

Remark. We can look at the map

$$w_i = \sum_l v_l f_i(l) \rightarrow \hat{w}(E) = \sum_l v_l P_l(E)$$

as a kind of *Fourier transformation*. The inverse transformation is

$$\hat{w}(E) \rightarrow \int \hat{w}(E) P_i(E) \, d\mu(E) = w_i .$$

The transformation is unitary because of the equality

$$< v, w > = \sum_i v_i w_i = \int \hat{v}(E) \hat{w}(E) \, d\rho(E) = < \hat{v}, \hat{w} > .$$

c) We consider now the asymptotic development for $t \rightarrow \infty$ and $t \rightarrow -\infty$. In general, if $h'(E)$ takes different values at $E = \lambda_k$, we have the same situation like in the first flow. There is a pairwise exchange of velocities where the pairing depends on the ordering of $h'(\lambda_k)$. The number k which gives the biggest $h'(\lambda_k)$ exchanges with the number which gives the smallest value. In this non-degenerate case also the Jacobi matrices converge for $t \rightarrow \infty$ and $t \rightarrow -\infty$ to diagonal matrices. This argument can also be found in [Dei 85]. $\qquad \square$

11 Appendix: About random KdV flows

We will quickly have a look at the continuous analogue of the random Toda flows the so called *random KdV flows*. In comparison with the discrete case, the set up is technically more difficult partly because one has to deal with unbounded differential operators.

Take a probability space (X, m) and an invertible flow T_t on X leaving invariant the probability measure m. We denote with D the selfadjoint *Lie-derivative* with respect to the flow T_t.

$$Dq = i \cdot \lim_{\epsilon \to 0} \frac{U_t q - q}{\epsilon} \ .$$

Call $C^\infty(X)$ the set of elements in $L^\infty(X, \mathbb{R})$, which are infinitely many times differentiable with respect to the Lie-derivative D. We build the algebra $\mathcal{X}$ of elements

$$K = \sum_{n=0}^{\infty} q_n D^n \ ,$$

where only finitely many $q_n \in C^\infty(X, \mathbb{R})$ are different from zero. The multiplication is given by the usual multiplication of power series with the Leibniz rule

$$D^n q = \sum_{k=0}^{n} \binom{k}{n} q^{(n-k)} D^k \ .$$

The algebra has a representation on $L^2(X)$ if D is the Lie-derivative. Each element is acting there as a symmetric differential operator. We call the algebra the *algebra of real symmetric differential operators* over the dynamical system (X, T_t, m). We have *local functionals* on $C^\infty(\mathcal{X})$ defined by

$$F(K) = \int_X f(K, K', \dots, K^{(k)}) \, dm \ .$$

Such functionals are integrals of the KdV flows we are going to define. Second order differential operators in $\mathcal{X}$ of the form

$$L = D^2 + q$$

are called *Schrödinger operators*. We write $\mathcal{L}$ for the space of these operators. The functionals

$$
\begin{array}{ll}
F_1(L) = \int_X q \, dm(x) & \text{Mass integral} \\
F_2(L) = \int_X q^2 \, dm(x) & \text{Momentum integral} \\
F_3(L) = \int_X q^3 + \frac{1}{2} u'^2 \, dm(x) & \text{Energy integral}
\end{array}
$$

will serve as Hamiltonians of random KdV flows which can be defined by a Lax pair: If we take a suitable operators

$$B(q) = iK$$

with $K \in \mathcal{X}$ such that the differential equations

$$\dot{L} = [B, L]$$

make sense in the space $\mathcal{L}$ of Schrödinger operators, we have isospectral deformations of $q \in C^\infty(X)$. If the flow exists locally, it exists for all times. There is a whole hierarchy of such flows and the corresponding skew-symmetric B can be determined by making an ansatz and solving a linear system of equations.

The momentum functional F_2 generates translation flow $q \mapsto U_t q$ and the energy functional F_3 generates the usual KdV flow

$$q_t = 6qq_x - q_{xxx} .$$

There is a whole hierarchy of functionals leading to isospectral deformations.

Examples of dynamical systems.

• *Periodic KdV flow.* The dynamical system is $(X = \mathsf{T}^1, T_t(x) = x + t, dx)$. $\Phi_E = \phi_E(1)$ time-one map of the fundametal solution.

$$\Delta(E) = \frac{1}{2}\mathrm{tr}\Phi_E = \mu + \mu^{-1}$$

is called the discriminant. The spectrum of L is

$$\sigma(L) = \bigcup_{i=1}^{N} F_i ,$$

where F_i are closed intervals and $N \leq \infty$. The Floquet exponent is $w = \log(\mu)$.

• *Almost periodic KdV flow.* Take a Bohr-almost periodic function q and take $X = <q>$, the hull of $q(t)$. This is a compact topological group and translation $q(t) \mapsto a(t + s)$ gives a flow on X leaving invariant the Haar measure.

Jacobi operators over such dynamical systems have been considered in [Joh 82]. An integration has not yet been performed.

References

[Bel 85] J.Bellisard. *K theory of C^* algebras in solid state physics.*
Lecture Notes Phys., 257: 99-156, 1986

[Ber 86] Yu.M.Berezansky. *The integration of semi-infinite Toda Chain by means of inverse spectral problem.*
Reports on mathematical physics, 24:21-47, 1986

[Ber 86b] Yu.M.Berezansky, M.I.Gekhtman,M.E.Shmoish. *Integration of some chains of nonlinear difference equations by the method of the inverse spectral problem.*
Ukrainian. Math. J., 38:74-78, 1986

[Bog 88] O.I.Bogoyavlensky. *Five Constructions of Integrable Dynamical Systems.*
Acta Appl. Math., 13:227-266, 1988

[Bog 90] O.I.Bogoyavlensky. *On Integrable Generalizations of Volterra Systems.*
in Analysis, et cetera, ed. by P.Rabinowitz, E.Zehnder, Academic Press, Boston 1990

[Bog 76] O.I.Bogoyavlensky. *On Perturbations of the Periodic Toda lattice.*
Comm. Math. Phys., 51:201-209, 1976

[Chu 84] M.T.Chu. *On the global convergence of the Toda lattice for real normal matrices and ists applications to the eigenvalue problem.*
SIAM J. Math. Anal., 15 (1), (1984)

[Chu 84a] M.T.Chu. *The generalized Toda flow, the QR algorithm and the center manifold theory.*
SIAM J. Discrete Math., 5:187-201, 1984

[Chu 86] M.T.Chu. *A differential equation approach to the singular value decomposition of bidiagonal matrices.*
Lin. algebra and its applications, 80:71-79, 1986

[Car 87] R.Carmona, S.Kotani. *Inverse Spectral Theory for Random Jacobi Matrices.*
J. Statist. Phys., 46:1091-1114, 1987

[Car 90] R.Carmona, J.Lacroix. *Spectral Theory of Random Schrödinger Operators.*
Birkhäuser, Boston, 1990

[Con 90] A.Connes. *Geometrie non commutative.*
InterEditions, Paris 1990

[Cor 82] I.P.Cornfeld,S.V.Fomin,Ya.G.Sinai. *Ergodic Theory.*
Springer, New York, 1982

[Cyc 87] H.L.Cycon,R.G.Froese,W.Kirsch,B.Simon. *Schroedinger Operators.*
Texts and Monographs in Physics, Springer, 1987

[Dei 91] P.Deift, J.Demmel, L.Li,C.Tomei. *The bidiagonal singular value decomposition and Hamiltonian mechanics.*
SIAM J. Numer. Anal., 28:1463, 1991

[Dei 80] P.Deift,F.Lund,E.Trubowitz. *Nonlinear Wave Equations and Constrained Harmonic Motion.*
Comm. Math. Phys., 74:141-188, 1980

[Dei 85] P.Deift,L.C.Li,C.Tomei. *Toda flows with infinitely many variables.*
J. of Funct. Anal., 64:358-402, 1985

[Dei 89] P.Deift,L.C.Li,C.Tomei. *Matrix Factorizations and Integrable Systems.*
Comm. Pure Appl. Math., XLII: 443-521, 1989

[Dei 83] P.Deift,T.Nanda,C.Tomei. *Ordinary differential equations and the symmetric eigenvalue problem.*
SIAM J. Numer. Anal., 20:1-21, 1983

[Dei 89] P.Deift,L.C.Li. *Generalized Affine Lie Algebras and the Solution of a Class of Flows Associated with the QR Eigenvalue Algorithm.*
Comm. Pure Appl. Math., XLII: 963-991, 1989

[Del 83] F.Delyon, B.Souillard. *The Rotation Number for Finite Difference Operators and its Properties.*
Comm. Math. Phys., 89:415-426, 1983

[Die 68] J.Dieudonné. *Foundations of Modern Analysis, Tome I.*
Academic Press,New York and London 1968

[Dix 81] J.Dixmier. *Von Neumann Algebras.*
North-Holland mathematical Library, Vol 27, Amsterdam, 1981

[Fla 74] H.Flaschka. *On the Toda Lattice II.*
Prog. Theoret. Phys., 51:703-716, 1974

[Fad 86] L.D.Faddeev,L.A.Takhtajan. *Hamiltonian methods in the theory of Solitons.*
Springer, 1986

[Gor 85] D.Goroff. *Hyperbolic sets for twist maps.*
Ergod. Th. & Dynam. Sys., 5:337-339, 1985

[Joh 82] R.Johnson,J.Moser. *The Rotation Number for Almost Periodic Potentials.*
Comm. Math. Phys., 84:403-438, 1982

[Kac 75] M.Kac,P. van Moerbeke. *A complete solution of the periodic Toda problem.*
Proc. Nat. Acad. Sci. USA, 72, No.8, pp.2879-2880, 1975

[Kam 93] S.Kamvissis. *On the long time behavior of the doubly infinite Toda lattice under initial data decaying at infinity.*
Commun. Math. Phys., 153:479-519, 1993

[Koo 86] H.Kook,J.D.Meiss. *Periodic orbits for reversible, symplectic mappings.*
Physica D, 35:65-86, 1986

[Kos 79] B.Kostant. *The solution to a generalized Toda lattice and representation theory.*
Adv. in Math., 34:195-338, 1979

[Kot 88] S.Kotani, B.Simon. *Stochastic Schroedinger Operators and Jacobi Matrices on the Strip.*
Comm. Math. Phys., 119:403-429, 1988

[Li 87] L.C.Li. *Long time behavior of an infinite particle system.*
Comm. Math.Phys., 110:617-623, 1987

[Mat 68] J.Mather. *Characterization of Anosov Diffeomorphisms.*
Indag. Math., 30:479-483, 1968

[Mat 82] J.Mather. *Existence of quasi-periodic orbits for twist homeomorphism of the annulus.*
Topology, 21:457-467, 1982

[Mat 84] J.Mather. *Amount of Rotation About a Point and the Morse Index.*
Comm. Math. Phys., 94:141-153, 1984

[McK 79] H.P.McKean. *Integrable systems and algebraic curves.*
Lecture Notes in Math., No. 755, Springer 1979

[vMo 76] P.v.Moerbeke. *The spectrum of Jacobi Matrices.*
Invent. Math.,37:45-81, 1976

[vMo 78] P.v.Moerbeke. *About isospectral deformations of discrete Laplacians.*
Lecture Notes Math., No. 755, Springer 1978

[Mos 75] J.Moser. *Three Integrable Hamiltonian Systems Connected with Isospectral Deformations.*
Adv. in Math., 16:197-220, 1975

[Mos 75] J.Moser. *Finitly many mass points on the Line under the influence of an exponential potentials, an integrable system.*
Lecture Notes Phys., 38, 1975

[Per 90] A.M.Perelomov. *Integrable Systems of classical Mechanics and Lie Algebras I.*
Birkhäuser, Boston 1990

[San 86] V.S.Sanders. *An invitation to von Neumann algebras.*
Springer, New York, 1986

[Sch 87] R.Schmid. *Infinite dimensional Hamiltonian systems.*
Bibliopolis, edizioni di filosofia e scienze, Napoli, 1987

[Sim 83] B.Simon. *Kotani Theory for One Dimensional Stochastic Jacobi Matrices.*
Comm. Math. Phys., 89:227-234, 1983

[Sym 82] W.Symes. *The QR algorithm and scattering for the finite nonperiodic Toda lattice.*
Phys. D, 4:275-280, 1982

[Sym 80] W.Symes. *Systems of Toda Type, Inverse Spectral Problems, and Representation Theory.*
Invent. Math., 59, 13-51, 1980

[Tod 81] M.Toda. *Theory of nonlinear lattices.*
Springer, 1981

[Woj 85] M.Wojtkowski. *Invariant families of cones and Lyapunov exponents.*
Ergodic Theory & Dynamical Systems, 5:145-161, 1985.

Factorization of random Jacobi operators and Bäcklund transformations

Contents

Abstract

We show that a positive definite random Jacobi operator L over an abstract dynamical system $T : X \to X$ can be factorized as $L = D^2$, where D is again a random Jacobi operator but defined over a new dynamical system $S : Y \to Y$ which is an integral extension of T.

An isospectral random Toda deformation of L corresponds to an isospectral random Volterra deformation of D. The factorization leads to commuting Bäcklund transformations which can be written explicitly in terms of Titchmarsh-Weyl functions. In the periodic case, the Bäcklund transformations are time 1 maps of a Toda flow with a time dependent Hamiltonian.

1 Introduction

Bäcklund transformations for Toda lattices have been given in a non-explicit form by Toda and Wedati [Wed 75],[Tod 81]. Adler [Adl 81] found that Bäcklund transformations have their origin in a factorization $L = AA^*$ in analogy to the *Miura map* for the KdV equation. It has been mentioned already by Moser [Mos 75a] that the relation between the *Kac van Moerbeke system* and the Toda lattice has its algebraic origin in a factorization $L = D^2$, where D is a matrix on a vector space with twice the dimension of the vector space on which L acts. In those papers D or A are given first and L is obtained by forming $L = AA^* = D^2$. The Bäcklund transformed operator $\tilde{L}$ is obtained by commutation $\tilde{L} = A^*A$. Recently, the Poisson structure of the Bäcklund transformations was studied in [Dei 91] for the periodic Toda lattice and also in the more general context of Toda equations on Lie groups.

In [Kni 92] we studied Toda lattices with random boundary conditions. They were obtained by making isospectral deformations of random Jacobi operators. The random Toda lattice is a generalization of both the periodic and the tied Toda lattice. It is defined over an arbitrary abstract dynamical system. We will show here that Bäcklund transformations can also be done in this case. They generalize the Bäcklund transformations known for periodic and aperiodic Toda lattices investigated in [Tod 81],[Adl 81],[Dei 91]. What is new here, (beside the fact that we are working with random Jacobi operators and not with finite dimensional matrices), is that we have explicit formulas for the transformations in terms of Titchmarsh-Weyl functions. These functions are Green functions and play an important role for the study of spectral problems [Sim 83] and inverse spectral problems [Car 87] of stochastic Jacobi matrices.

We will prove that all of the Bäcklund transformations commute. This follows from the fact that there is an interpolation of the transformations by time dependent Hamiltonian flows in the periodic case. The Toda flow deformation of the operator L gives a *random Volterra* flow for D.

The relation of the Toda and the Volterra flows is discovered first by Henon in 1973 (in a letter to Flaschka). This bibliographic remark and the publication of the relation between these two systems can be found in [Mos 75a]. It is possible that different rediscoveries of this appeared later. Recent work [Dam 91] is dealing with this question. A very recent independent approach appeared in [Ges 93], where also explicit formulas for Bäcklund transformations are given.

There are known also other relations between Volterra and Toda systems. The fact that the Volterra system sits in the second Toda flow as a subsystem appeared in the paper [Mos 75b]. In [McK 78] the same relations have been shown for the periodic boundary conditions. The birational equivalence of the two systems goes back to a transformation of Stielties (1918).

In the last part of this paper we deal with symmetries of the Toda systems. We remark that the factorization $L = D^2$ leads to a kind of *super-symmetry* for random Jacobi matrices. This super-symmetry was invented by Witten and is also called zero-dimensional super-symmetry. The operator D plays the role of a charge- or Dirac operator. Under super-symmetry the Hilbert space splits into a direct sum of two Hilbert spaces, a Fermionic and a Bosonic part. The Bäcklund transformations interchange the Fermionic and the Bosonic parts.

An other symmetry of the Toda flows is CPT. The involution C (change of charge) flips the Fermionic and the Bosonic parts by interchanging the two copies of the probability spaces. The involution P (change of parity) is reversing the dynamical system. The transformation in the dynamical system S is replaced by its inverse S^{-1}. The involution T is changing the time parameter t of the Toda flow. Applying CPT together maps the system into itself.

2   Random Toda flows

We redefine shortly the definitions in [Kni 92] needed here: An ergodic *dynamical system* (X, T, μ) is a probability space (X, μ) together with a measurable ergodic invertible map T on X that preserves the measure μ. The *crossed product* $\mathcal{X}$ of $L^\infty(X)$ with the dynamical system (X, T, μ) is a C^* algebra and consists of sequences $K_n \in L^\infty(X)$ with convolution multiplication

$$(KM)_n(x) = \sum_{k+m=n} K_k(x) M_m(T^k x)$$

and involution

$$(K^*)_n(x) = \overline{K_{-n}(T^n x)} \ .$$

An element $K \in \mathcal{X}$ is written in the form

$$K = \sum_{n \in \mathbb{Z}} K_n \tau^n \ ,$$

where τ is a symbol. The multiplication in $\mathcal{X}$ is the multiplication of power series with the additional rule $\tau^k K_n = K_n(T^k)\tau^k$ for shifting the τ's to the right and the requirement $\tau^* = \tau^{-1}$. The norm on $\mathcal{X}$ is given by

$$|||K||| = |\ ||K(x)||\ |_\infty\ ,$$

where $K(x)$ is the infinite matrix

$$[K(x)]_{mn} = K_{n-m}(T^m x)\ .$$

The multiplication and involution in $\mathcal{X}$ is defined such that

$$K \in \mathcal{X} \mapsto K(x) \in \mathcal{B}(l^2(\mathbb{Z}))$$

is an algebra homomorphism:

$$KL(x) = K(x)L(x),\ K^*(x) = K(x)^*\ .$$

According to Pastur's theorem (adapted to the present situation), the spectrum of $K(x)$ is the same for almost all $x \in X$. The algebra $\mathcal{X}$ has the trace $\mathrm{tr}(K) = \int_X K_0\ d\mu$. An element K has the decomposition $K = K^- + K_0 + K^+$ defined by requiring $K^\pm = \sum_{\pm n > 0} K_n \tau^n$. With $\mathcal{L} \subset \mathcal{X}$ is denoted the real Banach space consisting of *random Jacobi operators*

$$L = a\tau + (a\tau)^* + b$$

if $a, b \in L^\infty(X, \mathbb{R})$. The number

$$M(L) = \exp\left(\int_X \log(a)\ d\mu\right)$$

is the *mass* of L. We say it has *positive definite mass* if there exists $\delta > 0$ such that $a(x) \geq \delta$ for almost all $x \in X$. For a Hamiltonian

$$H \in C^\omega(\mathcal{L}) = \{H(L) = \mathrm{tr}(h(L)) \mid h\ \text{entire},\ h(\mathbb{R}) \subset \mathbb{R}\}$$

the *random Toda lattice*

$$\dot{L} = [B_H(L), L],$$

with $B_H(L) = h'(L)^+ - h'(L)^-$ is an isospectral flow in $\mathcal{L}$. It reduces to the periodic Toda lattice in the case when $|X|$ is finite. The flows all commute and exist globally.

Remark. The assumption $a(x) \geq \delta > 0$ could be replaced by $|a(x)| \geq \delta > 0$ because every infinite Jacobi matrix $L(x)$ can be conjugated with a diagonal matrix $D(x)$ to a Jacobi matrix $D(x)L(x)D(x)^{-1}$ with non-negative side-diagonal entries. The map $x \mapsto D(x)$ is however not measurable in general. Only if $Y = \{x \in X \mid a(x) < 0\}$ is a *coboundary*, (which means that there exists a measurable set Z such that $Y = Z \triangle Z(T)$), the map $x \mapsto D(x)$ is measurable. The random operator $D = D_0$ is then defined by $D_0(x) = e^{i \cdot \pi 1_Z}$.

3   Factorization of random Jacobi operators

3.1   Definition of the Titchmarsh-Weyl functions.

Given a random Jacobi operator $L \in \mathcal{L}$ with positive definite mass. For almost all $x \in X$, $L(x)$ is a bounded operator on $l^2(\mathbb{Z})$. We consider it also as a matrix acting algebraically on $\mathbb{R}^{\mathbb{Z}}$. Fix an energy E outside the spectrum of L. The time-independent Schrödinger equation

$$L(x)u = Eu$$

admits a two dimensional family of solutions $\{u_n(x)\} \in \mathbb{R}^{\mathbb{Z}}$. If we fix for example u_0, u_1, all the other values u_n can be calculated recursively by

$$a_n u_{n+1} + a_{n-1} u_{n-1} + b_n u_n = E u_n ,$$

where $a_n = a(T^n x)$ and $b_n = b(T^n x)$. With the vector

$$w_n(x) = (a_n(x) u_{n+1}(x), u_n(x)) ,$$

the Schrödinger equation can be written as the first order system

$$A_E(x) w_{-1}(x) = w_0(x) ,$$

where A_E is the *transfer cocycle*

$$A_E(x) = a^{-1}(T^{-1}x) \begin{pmatrix} E - b(x) & -a^2(T^{-1}x) \\ 1 & 0 \end{pmatrix} .$$

The name "cocycle" is usually used for the function $\mathbb{Z} \times X \to SL(2, \mathbb{C})$,

$$(x, n) \mapsto A_E^n(x) = A_E(T^{n-1}(x)) \cdots A_E(Tx) A_E(x) .$$

Claim. For E outside the spectrum of L, the cocycle A_E has a positive *Lyapunov exponent*

$$\lambda(A_E) = \lim_{n \to \infty} n^{-1} \int_X \log(\|A_E^n(x)\|) \; d\mu(x) .$$

Proof. The *Thouless formula*

$$\mathrm{Re}(\mathrm{tr}(\log(L - E))) = \log(M) + \lambda(A_E)$$

shows that the Lyapunov exponent $E \to \lambda(A_E)$ is harmonic outside the spectrum. Because $\det(A_E(x)) = 1$, the Lyapunov exponent takes values ≥ 0. According to the maximum principle for harmonic functions, the minimum 0 can not occur in the resolvent set. $\qquad\square$

136

It follows from the *multiplicative ergodic theorem* (see [Rue 79]) that for E in the resolvent set, there exist one dimensional co-invariant stable and unstable vector spaces $W^\pm(x)$ such that

$$A_E(x)W^\pm(x) = W^\pm(Tx) .$$

For almost all $x \in X$ we can take a unit vector $w^\pm(x) \in W^\pm(x)$ and define $u^\pm(x)$ as the second coordinate of $w^\pm(x)$. Like this, there exist solutions $u_n^+, u_n^- \in \mathsf{R}^\mathsf{Z}$ of $L(x)u = Eu$ satisfying $\{u_n^+(x)\} \in l^2(\mathsf{N})$ and $\{u_n^-(x)\} \in l^2(-\mathsf{N})$. (The sequence u_n is determined by defining u_0 as the second coordinate of $w^\pm(x)$ and $a(x)u_1$ as the first coordinate of $w^\pm(x)$. The other entries u_n are then defined by $L(x)u = Eu$.) The *Titchmarsh-Weyl functions* are

$$m^+(x) = a(x)\frac{u^+(Tx)}{u^+(x)} \quad , \quad m^-(x) = a(x)\frac{u^-(Tx)}{u^-(x)} ,$$

$$n^+(x) = a(T^{-1}x)\frac{u^+(T^{-1}x)}{u^+(x)} \quad , \quad n^-(x) = a(T^{-1}x)\frac{u^-(T^{-1}x)}{u^-(x)} .$$

They are measurable according to the multiplicative ergodic theorem and are allowed to take the value ∞ or $-\infty$.

Remark. Contrary to $u_n^+(x), u_n^-(x)$, which were defined pointwise for $x \in X$ and only up to a multiplication with a nonzero constant, $m^+(x)$ and $m^-(x)$ are uniquely defined measurable functions.

Remark. We use slightly different Titchmarsh-Weyl functions than in the literature. In [Car 90] for example

$$m^+(x) = -\frac{u^+(Tx)}{a(x)u^+(x)}$$

is used. Often, (for example in [Sim 83],[Cyc 87],) stochastic Jacobi matrices are discussed with $a(x) = 1$.

3.2 The Titchmarsh-Weyl functions as Green functions.

Related to the operator $L(x) \in \mathcal{B}(l^2(\mathsf{Z}))$ are the operators $L^\mathsf{N}(x) \in \mathcal{B}(l^2(\mathsf{N}))$ defined by

$$[L^\mathsf{N}(x)]_{ij} = [L(x)]_{ij}, \quad i, j > 0$$

and $L^{-\mathsf{N}}(x) \in \mathcal{B}(l^2(-\mathsf{N}))$ by

$$[L^{-\mathsf{N}}(x)]_{ij} = [L(x)]_{ij}, \quad i, j < 0 .$$

By the spectral theorem there are two probability measures $d\sigma^+, d\sigma^-$ on the real axes such that

$$[(L^\mathsf{N}(x) - E)^{-1}]_{11} = \int_\mathsf{R} \frac{d\sigma^+(x)(E')}{E' - E} ,$$

$$[(L^{-\mathsf{N}}(x) - E)^{-1}]_{-1,-1} = \int_\mathsf{R} \frac{d\sigma^-(x)(E')}{E' - E} .$$

Lemma 3.1

$$m^+(x) = -a^2 \int_{\mathbf{R}} \frac{d\sigma^+(x)(E')}{E' - E}, \quad m^-(x) = -\left(\int_{\mathbf{R}} \frac{d\sigma^-(Tx)(E')}{E' - E}\right)^{-1},$$

$$n^+(x) = -\left(\int_{\mathbf{R}} \frac{d\sigma^+(T^{-1}x)(E')}{E' - E}\right)^{-1}, \quad n^-(x) = -a^2(T^{-1}x)\int_{\mathbf{R}} \frac{d\sigma^-(x)(E')}{E' - E}.$$

Proof.

$u_0^+(x)$ or $u_0^-(x)$ can't be zero, or else $u_n^\pm = u_0^\pm(T^n x)$ is an eigenfunction for the matrix $L^{\pm N}$. This is not possible since the spectra of $L^N(x)$ and $L^{-N}(x)$ are lying in an interval containing the whole spectrum of $L(x)$.

Define the solutions $\{v_n^+(x)\}$ and $\{v_n^-(x)\}$ of $L(x)u = Eu$ by imposing the boundary conditions $v_0^+(x) = v_0^-(x) = 0$ and $v_1^+(x) = v_{-1}^-(x) = 1$. Because both $u_0^+(x)$ and $u_0^-(x)$ never can get zero, v^+, u^+ and v^-, u^- are two pairs of linearly independent solutions of $L(x)u = Eu$. This implies that the two *Wronskians*

$$[v^\pm(x), u^\pm(x)]_n = \det\begin{pmatrix} a_n(x)v_{n+1}^\pm(x) & a_n(x)u_{n+1}^\pm(x) \\ v_n^\pm(x) & u_n^\pm(x) \end{pmatrix} = \det(W_n^\pm(x))$$

are both different from zero. Because $\det(A_E(x)) = 1$ and

$$A_E^n(x)W_{-1}^\pm(x) = W_{n-1}^\pm(x),$$

the Wronskians are independent of n. Define symmetric matrices $G^+(x), G^-(x)$ by requiring that for $m \le n$,

$$[G^+(x)]_{mn} = -\frac{v_m^+(x)u_n^+(x)}{[v^+(x), u^+(x)]},$$

$$[G^-(x)]_{-m,-n} = -\frac{v_{-m}^-(x)u_{-n}^-(x)}{[v^-(x), u^-(x)]}$$

and $[G^\pm(x)]_{nm} = [G^\pm(x)]_{mn}$. For all $n, m \in \mathbf{N}$ one has

$$[G^+(x)]_{mn}(x) = [(L^N(x) - E)^{-1}]_{mn},$$

$$[G^-(x)]_{-m,-n}(x) = [(L^{-N}(x) - E)^{-1}]_{-m,-n}.$$

To verify this, calculate in the case $k \le n - 1$ using $Lv = Ev$

$$\begin{aligned} R_{kn} &= \sum_{m \ge 1}((L^{\pm N} - E)_{km}[G^\pm]_{mn} \\ &= \frac{\sum_m (L^{\pm N} - E)_{km}v_m^\pm u_n^\pm}{[v^\pm, u^\pm]} \\ &= \frac{(a_k v_{k+1}^\pm + b_k v_k^\pm + a_{k-1}v_{k-1}^\pm - E v_k^\pm)u_n^\pm}{[v^\pm, u^\pm]} = 0. \end{aligned}$$

Using the symmetry of $G^{\pm}$ we get also $R_{kn} = 0$ for $k \geq n + 1$. For $k = n$ there is a sign change due to the symmetry of $G^{\pm}$ and we have

$$R_{kk} = \frac{a_k v_{k+1}^{\pm} u_k^{\pm} - a_k v_k^{\pm} u_{k+1}^{\pm}}{[v^{\pm}, u^{\pm}]} = 1 .$$

In particular

$$\begin{aligned}
[(L^{N}(x) - E)^{-1}]_{11} &= [G^{+}(x)]_{11} = -\frac{u_1^{+}(x)}{a(x)u_0^{+}(x)} \\
&= -\frac{m^{+}(x)}{a^2(x)} = -\frac{1}{n^{+}(Tx)}
\end{aligned}$$

and

$$\begin{aligned}
[(L^{-N}(x) - E)^{-1}]_{-1,-1} &= [G^{-}(x)]_{-1,-1} = -\frac{u_{-1}^{-}(x)}{a(T^{-1}x)u_0^{-}(x)} \\
&= -\frac{1}{m^{-}(T^{-1}x)} = -\frac{n^{-}(x)}{a^2(T^{-1}x)} .
\end{aligned}$$

$\square$

Remark. The lemma implies that $-m^{+}(x), -n^{-}(x), m^{-}, n^{+}$ are *Herglotz functions*: they are mapping the upper complex half plane into itself.

3.3 Factorization of a random Jacobi operator.

We will simplify the notation, by often leaving off the x in the variables $m^{\pm}, n^{\pm}, a, b$. If we avoid the superscripts $+, -$ in $m, n, etc.$, we mean that both equations (one with superscript $+$ and one with superscript $-$) are true.
One can recover the functions a, b from m, n by

$$\begin{aligned}
m + n &= E - b , \\
m \cdot n(T) &= a^2 .
\end{aligned}$$

We want to show now, how a random Jacobi operator L can be factorized as $L = D^2 + E$.

We take a special *integral extension* (Y, S, ν) of the dynamical system (X, T, μ) (see [Cor 82] for the general notion of an integral extension). It is defined like this: Y consists of two copies X_1, X_2 of the probability space (X, m). S is the identity map from X_1 to X_2 and the mapping T from X_2 to X_1. The S invariant measure ν is determined by $\nu(Z) = \mu(Z)/2$ for $Z \subset X_i$. Define on Y a new function c by requiring that for $x \in X = X_1$,

$$c(x) = -m(x), \ c(S^{-1}x) = -n(x) .$$

We have then for all $x \in X_1$,

$$c(x) + c(S^{-1}x) = -E + b(x) ,$$
$$c(x) \cdot c(Sx) = a^2(x) .$$

Because c is defined on Y, these formulas extend a, b to functions on Y. Define the C^* algebra $\mathcal{Y}$ analogously to $\mathcal{X}$ as the crossed product of $L^\infty(Y)$ through the dynamical system (Y, S, ν). The elements of $\mathcal{Y}$ can be represented as

$$K = \sum_n K_n \sigma^n ,$$

where $\sigma^2 = \tau$. Call ψ the map $\mathcal{Y} \to \mathcal{X}$

$$K = \sum_n K_n \sigma^n \mapsto \sum_n \tilde{K}_n \tau^n ,$$

where $\tilde{K}_n(x) = K_{2n}(x)$ for $x \in X_1 = X$. The mapping ψ gives for $x \in X_1$,

$$[\psi(K)(x)]_{nm} = [K(x)]_{2n,2m} .$$

Theorem 3.2 *a) The random Jacobi operators*

$$D = (\sqrt{c}\sigma + \sigma^* \sqrt{c}) \in \mathcal{Y}$$

are bounded for E outside an interval containing the spectrum $\Sigma(L)$ and

$$\psi(D^2) = L - E .$$

D is selfadjoint if E is real and below $\Sigma(L)$.
b) The operators

$$BT_E^\pm(L) := \psi((D^\pm)^2(S) + E)$$

have the same spectrum as L.

Proof.
a) If E is real and below the spectrum of L, we have from Lemma 3.1,

$$-m^\pm(x) > 0, \quad -n^\pm > 0 .$$

If E is outside an interval containing the spectrum of L, $m^\pm$ take complex values in general. But they are bounded in modulus by the inverse of the distance from E to the interval containing the spectrum of L. The relation $\psi(D^2) = L - E$ follows from the definition:

$$\psi(D^2) = \psi(\sqrt{c \cdot c(S)}\sigma^2 + (c + c(S^{-1})) + \sqrt{c(S^{-2}) \cdot c(S^{-1})}\sigma^{-2}$$
$$= a\tau + b - E + a(T^{-1})\tau^* = L - E .$$

b) S is ergodic as an integral extension of T [Cor 82] and D is an ergodic random Jacobi operator over the ergodic dynamical system (Y, S, ν). The spectrum $\Sigma(D(x))$ is constant almost everywhere. Especially it is translational invariant: the spectrum of D is the same as the spectrum of $D(S)$. Thus, also the spectrum of

$$D^2 + E$$

is the same as the spectrum of

$$D^2(S) + E .$$

But this is not yet enough to prove that L and $L(S)$ have the same spectrum.

Take first the periodic ergodic case, where $N = |X|$ is finite and where we can build for each periodic $N \times N$ Jacobi matrix L of positive mass a periodic $2N \times 2N$ Jacobi matrix D such that $D^2 + E$ is the direct sum of two $N \times N$ matrices L and $BT_E(L)$. The spectrum of periodic Jacobi operators is generically simple and the multiplicity of an eigenvalue is maximally two. (See the appendix).

(i) Assume therefore first that L has N simple eigenvalues. We want to show that $BT^{\pm}(L)$ has in this case the same spectrum as L. The Jacobi matrix D has a spectrum $\pm\lambda_1, \ldots, \pm\lambda_N$ symmetric with respect to the imaginary axis because if λ is an eigenvalue with eigenvector $(u_1, \ldots, u_{2N})$ then $-\lambda$ is an eigenvalue with eigenvector $(u_1, -u_2, \ldots, -u_{2N})$. If the eigenvalue 0 of D occurs then it must have multiplicity 2 because the multiplicities of all eigenvalues must add up to an even sum. The matrix $D^2 + E$ is the direct sum of two Jacobi matrices $L, BT_E(L)$ and has the eigenvalues $\lambda_i^2 + E$, where each multiplicity is exactly 2. Because L has by assumption simple spectrum we get that $\sigma(L) = \{\lambda_i^2 + E \mid i = 1, \ldots, N\}$ and the operator $BT_E(L)$ must have the same spectrum as L because each eigenvalue of $D^2 + E$ has multiplicity 2.

(ii) In the case, when L has not simple spectrum, the claim follows because the spectrum depends continuously on the matrix in the weak operator topology and because we can approximate a general Jacobi operator in the weak operator topology with matrices having simple spectrum.

(iii) In the general infinite dimensional case, we can approximate a Jacobi matrix $L(x)$ in the weak operator topology by periodic Jacobi matrices $L^{(N)}(x)$ and the spectra of these approximations converge for $N \to \infty$ to the spectrum of $L(x)$ by the lemma of Avron-Simon. The Bäcklund transformed matrices $BT_E(L^{(N)})$ converge for $N \to \infty$ in the weak operator topology to $BT_E(L)$. So, the spectrum of $BT_E(L)$ is the same as the spectrum of L. □

For simplicity, we will omit in the future the restriction map ψ and write just $L = D^2 + E$ instead of $L = \psi(D^2) + E$, where it does not lead to confusion.

Remarks. The requirement that L has positive definite mass could be weakened. If L has positive mass then

$$\int \log^+ \left(\|A_E^{\pm}\| \right) \, d\mu$$

141

is finite and Oseledec's theorem is still applicable to define the Titchmarsh Weyl functions. If the mass is zero and $a(x) > 0$ for almost all $x \in X$ then one can consider the cocycle $a(T^{-1}) \cdot A_E$ to find the Titchmarsh Weyl functions. If $a(x) = 0$ on a set of positive measure, the Jacobi matrix $L(x)$ is block diagonal for almost all $x \in X$ and a decomposition $L = D^2 + E$ is then in general no longer possible.

The factorization $L = D^2 + E$ can also be written as

$$L - E = D^2 = AA^* ,$$

with $A = D\sigma$ and $A^* = \sigma^* D$. We will consider in the next section the Bäcklund transformation

$$AA^* \mapsto A^*A .$$

4 Bäcklund transformations

4.1 Bäcklund transformations as isospectral transformations

Theorem 4.1 *For $H \in C^\omega(\mathcal{L})$, the Toda flow $\dot{L} = [B_H(L), L]$ is with $D^2 = L - E$ equivalent to the Volterra flow*

$$\dot{D} = [B_H(D^2 + E), D] .$$

The mapping

$$BT_E : L \mapsto L(S)$$

is a Bäcklund transformation: It is isospectral and commutes with each Toda flow.

In order to prove Theorem 4.1, we need a lemma

Lemma 4.2 *Given two random operators $D = d\sigma + \sigma^* d$, $R = r\sigma + \sigma^* r$ over the ergodic dynamical system (Y, S, ν). If d^2 is not constant on Y and $DR + RD = 0$ then $R = 0$.*

Proof. The equation

$$
\begin{aligned}
RD + DR &= (rd(S) + dr(S))\sigma^2 + 2(dr + (dr)(S^{-1})) \\
&+ \left((rd(S) + dr(S))\sigma^2 \right)^* = 0
\end{aligned}
$$

is equivalent to

$$dr + (dr)(S^{-1}) = 0 , \tag{1}$$

$$r \cdot d(S) + d \cdot r(S) = 0. \tag{2}$$

From (1), we get $dr = -dr(S^{-1})$ or $dr = dr(S^{-2})$. If S^2 is ergodic, then $dr = C_0 = const$ and from (1) follows $C_0 = 0$ and so $r = 0$. If S^2 is not ergodic, then $Y = X_1 \cup X_2$ and S^2 is ergodic on X_i. This implies that

$$dr = -(dr)(S) = C_0 = const$$

when restricted to X_1. Equation (2) gives

$$C_0 \cdot \left(\frac{d(S)}{d} - \frac{d}{d(S)} \right) = C_0 \cdot \frac{(d(S)^2 - d^2)}{dd(S)} = 0$$

which implies that $C_0 = 0$ unless $d^2(S) = d^2$ almost everywhere. By ergodicity of S, the equation $d^2(S) = d^2$ is equivalent to $d^2 = const$ which was excluded by assumption. Therefore $r = 0$ and so $R = 0$. $\qquad \square$

We prove Theorem 4.1: Proof. If D fulfills the equation $\dot{D} = [B_H(D^2 + E), D]$, then $L(t) = D^2(t) + E$ satisfies the differential equation $\dot{L} = [B_H(L), L]$:

$$\begin{aligned}
\frac{d}{dt}L(t) &= \frac{d}{dt}(D^2(t) + E) = \dot{D}D + D\dot{D} \\
&= [B_H(D^2 + E), D]D + D[B_H(D^2 + E), D] \\
&= [B_H(D^2 + E), D^2] = [B_H(L), L] \,.
\end{aligned}$$

If on the other hand $L(t)$ satisfies $\dot{L} = [B_H(L), L]$, then

$$\dot{D}D + D\dot{D} = [B_H(D^2 + E), D^2] = [B_H(D^2 + E), D]D + D[B_H(D^2 + E), D] \,,$$

where $D(t)$ is defined by $L(t) = D(t)^2 + E$. With

$$R = \dot{D} - [B_H(D^2 + E), D]$$

we can write this as

$$RD + DR = 0 \,.$$

From Lemma 4.2 we have $R = 0$, unless D is constant. But in the later case $\dot{D} = \dot{L} = 0$ anyway.

Each Toda flow commutes with $L \mapsto L(T)$ and in the same way, each Volterra flow commutes with $D \mapsto D(S)$. The just proved relation between the Toda and the Volterra flow shows that the Toda flows are commuting with $L \mapsto L(S)$ and the operators $BT_E^{\pm}L$ satisfies the same differential equation as L. A transformation with this property is called a *Bäcklund transformation*. $\qquad \square$

Example. In the case $h(L) = L^2/2$, the motion of D is given by the differential equation

$$\dot{c} = 2c(c(S) - c(S^{-1}))$$

143

which is called the *Volterra, Kac Moerbeke or Langmuir lattice*. It is a conservation law for the integral

$$\int_Y \log(c) \, d\nu$$

and in terms of the Titchmarsh-Weyl functions m, n it can be rewritten as

$$\begin{aligned}
\dot{m} &= 2m(n - n(T)) \,, \\
\dot{n} &= 2n(m(T^{-1}) - m) \,.
\end{aligned}$$

Also these differential equations are parametrized by a parameter E.

Remark. With $d = \sqrt{c}$ we can build the *weighted composition operator* $A = d\sigma^*$ and the Volterra lattice can also be written as

$$\dot{A} = [A^*A + AA^*, A] \,.$$

If d is more generally a matrix-valued cocycle in $L^\infty(X, M(N, \mathbf{R}))$, where $M(N, \mathbf{R})$ is the algebra of real $N \times N$ matrices, we get isospectral deformations

$$\dot{A} = A^2(T)A - AA^2(T^{-1})$$

of matrix cocycles. Since all the Volterra equations make sense also in the non-abelian case and the flows exists for all times, the isospectral deformation corresponding to the first Volterra flow

$$\dot{A} = A^2(T)A - AA^2(T^{-1})$$

exists for all times also in the non-abelian case.

A historical remark. The Volterra system appeared first in 1931 in Volterra's work. He studied the evolution of a hierarchical system of competing individuals. Henon mentions in a letter (1973) to Flaschka the relation of the Toda lattice with the Volterra system. In 1975 the version with aperiodic boundary conditions was solved by Moser [Mos 75a]. In the same paper, the relation with the Toda lattice is published. There are other relations: The fact that the Volterra system sits in the second Toda flow as a subsystem appeared in the paper [Mos 75a]. In [McK 78] the same relations have been established for the periodic boundary conditions.

Remark. Bäcklund transformations are also defined for complex values E outside the convex hull of the spectrum. They still preserve the spectrum, but the images are no longer selfadjoint operators. The norm can blow up in an isospectral way. Indeed, if E approaches a pole of $m^+(x)$ in a gap of the spectrum, then $\|BT^+(E)L(x)\| \to \infty$.

144

4.2 Bäcklund transformations in the coordinates of Flaschka

A Bäcklund transformation can also be described in the canonical coordinates $q, p \in L^\infty(X)$ if they exist. If $\log(a)$ is an *additive coboundary*: $\exists f \in L^\infty(X)$

$$\log(a) = f(T) - f ,$$

we can define q, p by

$$4a^2 = e^{q(T)-q}, \ 2b = p .$$

We will see that there is an additional free parameter for doing the Bäcklund transformations in the coordinates q, p. The generating function and the implicit canonical transformations in the following proposition have been given by Toda and Wedati [Wed 75] in the case of aperiodic Toda lattices, where the Bäcklund transformations are not isospectral.

Theorem 4.3 *For E outside an interval containing the spectrum $\Sigma(L)$, the Bäcklund transformations $BT_E^\pm$*

$$b' = b + n - n(T), \ a'^2 = a^2 \frac{m(T)}{m}$$

can be written in the canonical variables q, p as canonical transformations $BT_E^\pm$:
$(q, p) \rightarrow (q', p')$

$$p = \frac{\partial W}{\partial q} = -e^{q'-q-C} - e^{q-q'(T^{-1})+C} + 2E ,$$

$$p' = -\frac{\partial W}{\partial q'} = -e^{q'-q-C} - e^{q(T)-q'+C} + 2E$$

with a generating function

$$W(q, q') = \int_X e^{q'-q-C} - e^{q(T)-q'+C} - 2E \cdot (q' - q) \, d\mu ,$$

where C is a parameter. Explicitly

$$q' = q + \log(2m) + C ,$$
$$p' = p + 2n - 2n(T) .$$

Proof. From $b' = b + n - n(T)$ we get $p' = p + 2n - 2n(T)$ which gives together with $b = -m - n + E$

$$p = -2m - 2n + 2E ,$$
$$p' = -2m - 2n(T) + 2E .$$

Taking the difference of these two equations gives

$$p' = p + 2n - 2n(T) .$$

From $a'^2 = a^2 \frac{m(T)}{m}$ we obtain

$$\log(4a'^2) = \log(4a^2) + \log(m(T)) - \log(m)$$

and

$$q'(T) - q(T) - \log(m(T)) = q' - q - \log(m) .$$

The ergodicity of T implies that

$$q' - q = \log(2m) + C ,$$

where C is a constant. Together with

$$\log(2m) + \log(2n(T)) = \log(4a^2) = q(T) - q$$

this gives

$$q' - q(T) = C - \log(2n(T))$$

and so

$$
\begin{aligned}
p &= -2m - 2n + 2E = -e^{q'-q-C} - e^{q-q'(T^{-1})+C} + 2E , \\
p' &= -2m - 2n(T) + 2E = -e^{q'-q-C} - e^{q(T)-q'+C} + 2E .
\end{aligned}
$$

We verify

$$p = \frac{\partial W}{\partial q}, \; p' = -\frac{\partial W}{\partial q'} .$$

$\square$

4.3 Asymptotic behavior for $E \to -\infty$

Proposition 4.4 *a)*

$$
\begin{aligned}
\lim_{E \to -\infty} BT_E^+(L) &= L(T) , \\
\lim_{E \to -\infty} BT_E^-(L) &= L .
\end{aligned}
$$

b) For all E outside an interval containing $\Sigma(L)$,

$$BT_E^+ \circ BT_E^-(L) = L(T) .$$

Proof.
a) From Lemma **??**, we see that the functions

$$\log\left(m^+(x)\cdot E\right) = \log(a^2(x)) + \log\left([(1 - \frac{L^N(x)}{E})^{-1}]_{11}\right)$$
$$= \log\left(a^2(x)\right) + \log\left(\sum_{n=0}^{\infty} \frac{s_n^+(x)}{E^n}\right),$$
$$\log\left(\frac{m^-(x)}{E}\right) = -\log\left(E[(E - L^N(Tx))^{-1}]_{-1,-1}\right)$$
$$= -\log\left(\sum_{n=0}^{\infty} \frac{s_n^-(Tx)}{E^n}\right)$$

with $s_n^+(x) = [(L^N(x))^n]_{11}$ and $s_n^-(x) = [(L^{-N}(x))^n]_{-1,-1}$ are analytic in a disc around ∞. We have thus the Taylor expansion in the variable $1/E$ (compare [Car 87])

$$\log\left(m^+(x)\cdot E\right) = \log(a^2(x)) + s_1^+(x)\frac{1}{E} + \left(s_2^+(x) - \frac{s_1^+(x)^2}{2}\right)\frac{1}{E^2} + \cdots,$$
$$\log\left(\frac{m^-(x)}{E}\right) = s_1^-(Tx)\frac{1}{E} + \left(s_2^-(Tx) - \frac{s_1^-(Tx)^2}{2}\right)\frac{1}{E^2} + \cdots,$$
$$\log\left(\frac{n^+(x)}{E}\right) = s_1^+(T^{-1}x)\frac{1}{E} + \left(s_2^+(T^{-1}x) - \frac{s_1^+(T^{-1}x)^2}{2}\right)\frac{1}{E^2} + \cdots,$$
$$\log\left(n^-(x)\cdot E\right) = \log\left(a^2(T^{-1}x)\right) + s_1^-(x)\frac{1}{E} + \left(s_2^-(x) - \frac{s_1^-(x)^2}{2}\right)\frac{1}{E^2} + \cdots$$

leading to

$$\log\left(m^+(T)\right) - \log(m^+) = \log\left(a^2(T)\right) - \log(a^2) + (s_1^+(T) - s_1^+)\frac{1}{E} + \cdots$$

and

$$\lim_{E\to-\infty} \frac{m^+(T)}{m^+} = \frac{a^2(T)}{a^2}.$$

Because $n^+ = E - b - m^+$, we get also

$$\lim_{E\to-\infty} n^+(T) - n^+ = b - b(T).$$

From these two formulas $\lim_{E\to-\infty} BT_E^+(L) = L(T)$ follows. Similar, we deduce from the Taylor expansion that

$$\lim_{E\to-\infty} \frac{m^-(T)}{m^-} = 1, \quad \lim_{E\to-\infty} n^- = 0$$

147

and $\lim_{E \to -\infty} BT_E^-(L) = L$.

b) With $L = a\sigma + (a\sigma) + b$ and $L'' = BT_E^+ \circ BT_E^- L = a''\sigma + (a''\sigma)^* + b''$, we obtain

$$\log(a'') = \log(a) + \frac{1}{2}\log\left(m^+(T)\right) - \frac{1}{2}\log(m^+) + \frac{1}{2}\log\left(n^-(T)\right) - \frac{1}{2}\log(n^-)$$

$$= \log(a) + \frac{1}{2}\log\left(a^2(T)\right) - \frac{1}{2}\log(a^2) = \log(a(T)) ,$$

$$b'' = b + n^+ - n^+(T) + m^- - m^-(T) = b + (E - b) - (E - b(T)) = b(T) .$$

$\square$

Each random Toda flow $L(t)$ is now embedded in a one parameter family of flows

$$t \mapsto BT_E L(t)$$

where E is a parameter. The random flow itself is obtained for $E \to -\infty$.

Remark. In the case $|X| < \infty$, the boundaries of the curves $E \mapsto BT_E^-(L)$ and $E \mapsto BT_E^+(L(T^{-1}))$ have nonempty intersection. This gives the possibility to deform L into $L(T)$ inside the isospectral set. For aperiodic dynamical systems, one can't expect that a deformation of L into $L(T)$ can always be done because in general, m^+ and m^- are different everywhere.

4.4 Commutation of Bäcklund transformations

Assume, the Hamiltonian $H_E(L) = \text{tr}(h_E(L))$ is dependent on a parameter $E \in U \subset \mathbf{R}$, where U is an open interval in $\mathbf{R}$. Together with a smooth curve $t \to E(t)$ in U, we can define an isospectral deformation

$$\frac{d}{dt}L(t) = [B_{H(E(t))}(L), L] .$$

Theorem 4.5 *a) Assume $|X|$ is finite. For all $L \in \mathcal{L}$, there exist time dependent Hamiltonians $H_E^{\pm}(L) = \text{tr}(h_E^{\pm}(L))$, such that $E \mapsto L(E) = BT^{\pm}(E)$ is the Toda orbit of*

$$\frac{d}{dt}L(E(t)) = [B_{H^{\pm}(E(t))}(L), L] .$$

b) In general, for all real $E', E'' < \inf(\Sigma(L))$ and for all $\sigma, \mu \in \{+, -\}$

$$BT^{\sigma}(E')BT^{\mu}(E'')L = BT^{\mu}(E'')BT^{\sigma}(E')L .$$

For the proof we need the following little lemma:

148

Lemma 4.6 *Given d linear independent constant real vector fields $f^{(i)}$ on the d dimensional torus T^d.*
a) If the smooth vector field $\frac{d}{dE}\chi = f(E, \chi)$ commutes with the vector fields $f^{(i)}$, then there exist $a_i(E, \chi) : \mathsf{R} \times \mathsf{T}^d \to \mathsf{R}$ independent of χ such that

$$f(E, \chi) = \sum_{i=1}^{d} a_i(E) f^{(i)} .$$

b) For any differentiable functions $a_i(E), b_i(E)$, the time-dependent vector fields

$$F(E) = \sum_{i=1}^{d} a_i(E) f^{(i)}, \; G(E) = \sum_{i=1}^{d} b_i(E) f^{(i)}$$

are commuting.

Proof.
a)

$$0 = [f, f^{(i)}]_j = (\nabla f_j) \cdot f^{(i)} - f \cdot (\nabla f_j^{(i)}) = (\nabla f_j) \cdot f^{(i)}$$

implies

$$\nabla f_j = 0 ,$$

for all $j = 1, \ldots, d$.
b)

$$\begin{aligned}[F(E), G(E)] &= [\sum_i a_i(E) f^{(i)}, \sum_j b_j(E) f^{(j)}] \\ &= \sum_{i,j} a_i(E) b_j(E) \left[f^{(i)}, f^{(j)} \right] = 0.\end{aligned}$$

$\square$

Now to the proof of Theorem 4.5
Proof. a) The set $\text{Iso}(L) \subset \mathcal{L}$ of Jacobi operators with the same spectrum and mass forms a d dimensional real torus T^d, where $d \leq |X| - 1$ (see [vMor 76]). There are d linearly independent real vector fields f_i on T^d which correspond to d different Toda flows [vMor 78]. The real analytic curves

$$E \in [-\infty, \inf(\Sigma(L))] \mapsto L(E) = BT_E^{\pm}(L)$$

on the isospectral set $\text{Iso}(L)$ correspond to real analytic curves $E \mapsto \chi^{\pm}(E)$ on the torus T^d. Because these curves are smooth and passing through every point $\chi \in \mathsf{T}^d$, they are integral curves of time dependent vector fields

$$f^{\pm}(E, \chi) = \frac{d}{dE} \chi^{\pm}(E) .$$

We have seen that a Bäcklund transformation commutes with each constant Toda flow. Therefore, the vector fields $f^{\pm}(E, \chi)$ and f_i are commuting. Application of

149

Lemma 4.6 a) implies that $f^\pm(E, \chi)$ are independent of χ. In the original operator coordinates, this means that the time dependent Hamiltonian fields

$$\frac{d}{dE}L = [B_{H(E)}(L), L]$$

with Hamiltonian

$$H_E(L) = \text{tr}(h_E(L)) = \sum_n h_{E,n}\text{tr}(L^n)$$

have coefficients $h_{E,n}$, which are independent of L.

b) Assume first $|X|$ is finite. Assume $\sigma = +$ and $\mu = -$. The other cases go in the same way. Take from a) the time dependent vector fields $f^\pm(E)$ on T^d which are independent of the coordinate $\chi \in \mathsf{T}^d$. We know from Lemma 4.6 b) that for each E', E'', the flows of the vector fields

$$\begin{aligned} t &\mapsto F^+(t) = f^+(E'/t)\,, \\ t &\mapsto F^-(t) = f^-(E''/t) \end{aligned}$$

are commuting. As $BT^-_{-\infty}L = L$ (see Proposition 4.4), the transformation $BT^-_{E''}$ is obtained by integrating up the time dependent vector field f^-_E from $E = -\infty$ to $E = E''$ which is just the time 1 map of the flow given by the vector field $F^-(t)$. Because $BT^+_{-\infty}L = L(T)$ (again Proposition 4.4), the transformation $BT^+_{E'}$ is obtained by shifting $L \mapsto L(T)$ and then integrating up the vector field f^+_E from $E = -\infty$ to $E = E'$. This is a shift T followed with a time 1 map of the vector field $F^+(t)$. We have thus interpolated the Bäcklund transformations by Toda flows with time dependent Hamiltonians. From the commutation of the vector fields and the commutation of the Bäcklund transformations with the shift $T : L \mapsto L(T)$, the claim follows.

In general, let $L^{(N)}(x)$ be a periodic approximation of period N such that for $-N/2 \leq i, j < N/2$,

$$[L^{(N)}(x)]_{i+N,j+N} = [L^{(N)}(x)]_{ij} = [L(x)]_{ij}\,.$$

Then

$$(L^{(N)}(x))^\mathsf{N} \to L^\mathsf{N}(x)$$

in the strong operator topology (the strong and weak operator topologies coincide on the space of tridiagonal operators) and so in the strong resolvent sense (see [Ree 80] p. 292). This implies, that the Green functions of $(L^{(N)})^\mathsf{N}$ converge to the Green functions of $L^\mathsf{N}(x)$. Therefore, the Titchmarsh-Weyl functions of $L^{(N)}$ converge pointwise to the Titchmarsh-Weyl functions of $L(x)$ and so

$$BT_E L^{(N)}(x) \to BT_E L(x)$$

in the weak operator topology. This gives

$$BT^\sigma_{E'} BT^\mu_{E''} L^{(N)}(x) \to BT^\sigma_{E'} BT^\mu_{E''} L(x)$$

in the weak operator topology and finally

$$BT_{E'}^\sigma BT_{E''}^\mu L(x) = \lim_{N\to\infty} BT_{E'}^\sigma BT_{E''}^\mu L^{(N)}(x)$$
$$= \lim_{N\to\infty} BT_{E''}^\mu BT_{E'}^\sigma L^{(N)}(x) = BT_{E''}^\mu BT_{E'}^\sigma L(x) .$$

$\square$

Remark. For the tied Toda lattice, the Bäcklund transformation is not an isospectral transformation and it can be used to produce soliton solutions out of the trivial solution $q = \dot{q} = 0$. (see [Tod 81] and the appendix). This doesn't work in the periodic case: if we start with $q = \dot{q} = 0$, we get after the transform the same solution.

Illustration. As we visualized in the last chapter the motion of the Toda lattice as "particles" with coordinates

$$z_n = (\log(a_n), b_n) ,$$

we want to see how a Bäcklund transformation acts on a curve

$$x \in \mathsf{T}^1 \mapsto (\log(a) + ib)(x) \in \mathsf{C}$$

in the complex plane. Also Bäcklund transformations preserve the integrals

$$\int_X \log(a)\, dm, \ \int_X b\, dm$$

and the center of mass of the curve will keep constant also. We calculated numerically some Bäcklund transformations with the following Mathematical program

```
m[i_Integer,n_Integer]:=Mod[i-1,n]+1;
a[n_Integer]:=Table[N[2+Sin[k 2 Pi/n]],{k,n}];
b[n_Integer]:=Table[N[Cos[k 2 Pi/n]],{k,n}];
A[a_List,b_List,EE_]:=Table[1/a[[m[i-1,Length[a]]]]*
  {{EE-b[[i]],-(a[[m[i-1,Length[a]]]])^2},{1,0}},{i,1,Length[a]}];
Monodro[a_List,b_List,EE_]:=Block[{t=0,A=A[a,b,EE],B=IdentityMatrix[2]},
  Do[B=A[[i]].B;t=t+Re[Log[B[[1,1]]]];B=B/B[[1,1]],{i,Length[a]}];{B,t}];
mplus[a_List,b_List,EE_]:=Module[
  {M=Monodro[a,b,EE][[1]],s,m0,ad,n=Length[a]},ad=M[[1,1]]-M[[2,2]];
  m0=(ad-Sqrt[ad^2+4*M[[1,2]]*M[[2,1]]])/(2*M[[2,1]]);s={m0};
  Do[s=Prepend[s,a[[m[n-i,n]]]^2/(EE-First[s]-b[[m[n-(i-1),n]]])],{i,n-1}]; s];
BTplus[{a_List,b_List},EE_]:=Module[{mpl=mplus[a,b,EE],npl},
  npl=EE-b-mpl;{a Sqrt[RotateLeft[mpl]/mpl],b+npl-RotateLeft[npl]}];
Init[n_]:=Block[{s=10.1231*Pi/n},
  {Table[N[Exp[Cos[j*s]]],{j,n}],Table[N[Sin[j*s]],{j,n}]}];
Pict[{a_,b_}]:=ListPlot[Table[{Log[a[[i]]],b[[i]]},{i,Length[a]}],
      PlotRange->{{-3,3},{-3,3}},DisplayFunction->Identity,Axes->False];
Film[NPart_,NPict_,TimeInt_]:=Block[{c=Init[NPart],Movie={},
```

```
Do[Movie=Append[Movie,Pict[c]];c=BTplus[c,-5-(N[1/(i*TimeInt)])],{i,NPict}];
Movie];
Display["!psfix -land -stretch > backlundfilm.ps",
  Show[GraphicsArray[Partition[Film[101,16,1/16],4]],
  DisplayFunction->$DisplayFunction,Frame->True,
  PlotLabel->FontForm["Backlund transformations",{"Helvetica",12}]]]]
```


Backlund transformations
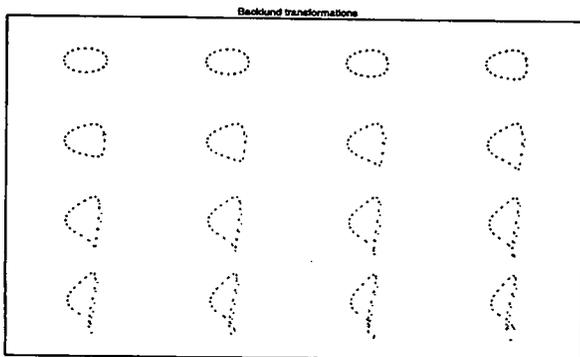
5 Symmetries

5.1 A simple version of super symmetry

The factorization $L - E = D^2$ leads to the simplest version of *super symmetry*. Define the elements

$$H = \begin{pmatrix} L - E & 0 \\ 0 & L(S) - E \end{pmatrix}, Q = \begin{pmatrix} 0 & D\sigma \\ (D\sigma)^* & 0 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in $M(2, \mathcal{L})$. The property

$$Q^2 = H, P^2 = 1, \{Q, P\} = QP + PQ = 0$$

is called *super symmetry* ([Cyc 87] p.121). It is believed to be important for the understanding of super-symmetry breakdown in realistic field theories. One calls the operators

$$Q^+ = \begin{pmatrix} 0 & D\sigma \\ 0 & 0 \end{pmatrix}, Q^- = \begin{pmatrix} 0 & 0 \\ (D\sigma)^* & 0 \end{pmatrix}$$

152

super charge operators. They satisfy

$$(Q^+)^2 = (Q^-)^2 = 0, \ Q^+ + Q^- = Q .$$

The eigenspace of the eigenvalue 1 of P is the space of *Bosonic states* and the eigenspace to the eigenvalue -1 is the space of *Fermionic states*. The operator $L - E$ is the restriction of H on the Bosonic states and the Bäcklund transformation $L(S) - E$ is the restriction of H on the Fermionic states. D is also called *charge operator*. This suggests to denote the invariants

$$Q^-(E) = \exp\left(\int_X \log\sqrt{m^-}\, d\mu\right) = \exp\left(\frac{-w(E)}{2}\right)$$

the *charge functions* of the operator D^- and

$$Q^+(E) = \exp\left(\int_X \log\sqrt{m^+}\, d\mu\right) = M \cdot \exp\left(\frac{w(E)}{2}\right)$$

the charge function of the "anti-operator" D^+.

5.2   CPT symmetry

We have seen that we can write the first Toda lattice as the differential equation

$$\begin{aligned}
\dot{m} &= 2m(n - n(T)) \\
\dot{n} &= 2n(m(T^{-1}) - m)
\end{aligned}$$

in $L^\infty(X) \times L^\infty(X)$. Define the transpositions

$$\begin{aligned}
C &: \ m^\pm \leftrightarrow n^\pm , \\
P &: \ T \leftrightarrow T^{-1} , \\
T &: \ t \leftrightarrow -t.
\end{aligned}$$

One can see that the above equations for the motion of the Titchmarsh-Weyl functions m, n satisfy the symmetry CPT in that applying the transformation $C \circ P \circ T$ leaves the equations invariant. We could call the transformations a change of *Charge, Parity and Time.* The name charge function is consistent with calling a Dirac operator like $D = \sqrt{c}\sigma + (\sqrt{c}\sigma)^*$ the *charge* operator. Notice also that the linearization of the first Toda equation, the *random wave equation*

$$\begin{aligned}
\dot{m} &= n - n(T) , \\
\dot{n} &= m(T^{-1}) - m
\end{aligned}$$

has this CPT symmetry while the continuous analogue on $\mathbf{R}$

$$\begin{aligned}
\dot{m} &= n_x , \\
\dot{n} &= m_x
\end{aligned}$$

has more symmetry, namely C and TP where $P : x \mapsto -x$. The discretization changes the symmetry. The doubling of the lattice simplifies the Toda equation to

$$\dot{c} = 2c(c(S) - c(S^{-1}))$$

and the C transformation which was an involution before becomes now the shift $C : c \mapsto c(S)$. We have still TP symmetry where $P : S \mapsto S^{-1}$. The doubling of the lattice changed also the symmetry.

5.3 Remarks

Taking terms from physics to describe mathematical structures has some danger in that a relation to real physics is pretended even if there is no physical experiment for which such a relation could be helpful.

On the other hand, there is hardly any mathematical structure which is not realized in some sense in physics. Today physics suffers from the lack of experimental facilities to eliminate mathematical theories as candidates for explaining the world. The link between the models and reality is lost. This leads to the nowadays observed inflation of mathematical theories having physical content. On the other hand, using physical terms in mathematics simplifies heuristic thinking and makes things more exciting because there could be a relevant relation between any given mathematical structure and the physical world. This remark should show that the following statements should be taken "cum grano salis".

CPT symmetry, internal symmetry:
It is believed today that CPT is a symmetry in modern particle physics. Our above simple mathematical model shows that this finite symmetry appears in a natural way after a discretisation of space. This *could* be an indication that a space is discrete at the Planck length, an opinion shared by a large community of physicists. On the other hand, we don't believe that the "time" in the Toda systems have anything to do with the usual time in physics even though also in classical quantum mechanics the time evolution is an isospectral deformation. The Toda (and Volterra) systems should be considered as *internal symmetries* of a one-dimensional classical quantum mechanical model with Hamiltonian L.

Abstract conductivity:
In the periodic finite dimensional case, also space translation $L \mapsto L(T)$ can be realised as a Toda deformation. It is an interesting question if this is possible for general random operators because one could interpret the ability to move in the internal symmetry space from L to $L(T)$ as an abstract concept of "conductivity".

In the finite dimensional case, there is a path $BT^+(E)L$ connecting L with $L(T)$: The Jacobi operator L defines a hyperelliptic curve

$$\mathcal{R} = \{(E, y) \in \mathbb{C}^2 \mid y^2 = \det(L - E) = e^{-w(E)}\}$$

154

which has a universal covering serving as a complex time axis: Given L_0, there exists a parametrized Hamiltonian $H_E = \operatorname{tr}(h(\cdot))$, where the "time" E is lying in the universal covering of $\mathcal{R}$. There is a real path $t \mapsto E(t)$ which is the union of two paths $E'(t), E''(t)$. The first path E' is passing from $E(0) = \infty' = (-\infty, -\infty)$ to the lowest branch point $E'(1/2) = \inf\{\lambda \in \sigma(L)\}$ and can be given as

$$E'(t) = \frac{2E_0}{t} \ .$$

The second path is starting at $E''(0) = E'(1/2)$ and is going back on the other sheet to the point $E''(1) = \infty = (-\infty, \infty)$. It can be parametrized as

$$E''(t) = \frac{2E_0}{1/2 - t} \ .$$

If we do the Toda flow

$$\frac{d}{dt}L(E(t)) = [B_{H(E(t))}(L), L]$$

starting with $L_0 = L(0)$ taking this Hamiltonian along the curve $E(t)$ we get

$$
\begin{aligned}
L(t) &= BT^-(E(t)), \quad t \in [0, 1/2] \ , \\
L(t) &= BT^+(E(t)), \quad t \in [1/2, 1]
\end{aligned}
$$

and L is connected with $L(T)$. Because in general, m^+ and m^- are no more coinciding at the infimum of the spectrum, such a connection does not exist in general. This does not exclude other connections in the isospectral set but it indicates that properties of the bottom the spectrum might have a relation with the mobility in space.

An other measure for conductivity is the *rate of ballistic motion* which measures the decay of the wave function $q(t)u$ for the position operator $(qu)_n = n \cdot u_n$. Given a wave function u in a domain

$$D(q) = \{u \mid \sum_n |n| \cdot |u_n|^2 < \infty\}$$

which makes q self adjoint. The value

$$r(u) = \lim_{t \to \infty} \frac{1}{t^2}||q(t)u|| = t^{-2} \cdot ||xe^{-itL}u||$$

is positive for a free particle. Simon [Sim 90] has shown that point spectrum of $L = \tau + \tau^* + V$ implies that $r(u) = 0$ for $u \in D(u)$ ("absence of ballistic motion"). In solid state physics $r(u)$ is proportional to the conductivity.

Mass and charge:
The name *mass* for the integral $M = \exp(\int_X \log(a) \ dm)$ was chosen because we

needed a name for this conserved quantity. A justification is that we can view the discrete Schrödinger operator as a sum

$$L = K + P = (a\tau + a\tau^*) + b$$

of a *kinetic energy* and a *potential energy* and a as a local mass function which gives an averaged "mass" $\exp(\int \log(a)\ dm$. (The average $\int a\ dm$ has the "disadvantage" that it is not an integral of the Toda flow.)

The name "charge function" for the functions $Q^\pm$ was chosen because $D^\pm$ can also be called *charge operator*. There is another more physical justification (which however mixes different unrelated physical topics). We have defined the charge function

$$Q^-(E) = \exp\left(\frac{-w(E)}{2}\right)\ .$$

We use the density of states dk to average these functions to the *charge*:

$$Q^- := \exp\left(\int_C \frac{-w(E)}{2}\ dk(E)\right)\ ,$$

Using the Thouless formula we can write

$$
\begin{aligned}
C^- := (Q^-)^2 &= \exp\left(\int_C -w(E)\ dk(E)\right) \\
&= \exp\left(\int_C \operatorname{tr}(\log|L - E|)\ dk(E)\right) \\
&= \exp\left(\int_C \int_C \log|E - E|)\ dk(E')\ dk(E)\right) \\
&=: \exp(I(L))\,,
\end{aligned}
$$

which is called the *capacity* of the set $\sigma(L)$ if the density of states measure dk is maximizing $\exp(I(\mu))$ under all probability measures μ on the spectrum $\sigma(L)$. $I(L)$ is the *energy* of the measure dk. The relation between capacity C and charge Q is (at least in a condenser with potential energy 1) given by the formula $C = Q^2$. This acrobatic playing with analogies has related the charge motivated by the Dirac operator of L with the *potential theoretic charge* of its spectrum $\sigma(L)$. The definition of the charge Q can be done in the same way for higher dimensional Laplacians.

6   Some questions

• What is the set of potentials we can get from one a, b by making Bäcklund transformations? Can we reach like this a dense set of the isospectral set of random operators.

• Under which conditions is $L \mapsto L(T)$ the time one map of a Hamiltonian isospectral flow in $\mathcal{L}$? For which ergodic automorphisms $\tilde{T}$ commuting with T can one connect L with $L(\tilde{T})$ inside the isospectral set?

• Can one find analogous factorizations and Bäcklund transformations of higher dimensional random Jacobi operators in the crossed product $\mathcal{X}$ of $L^\infty(X)$ with a Z^d dynamical system? Jacobi operators are then of the form

$$\sum_{i=1}^{d} \left(a_i \tau^i + (a_i \tau^i)^* \right) + b \, .$$

7 Appendix: The spectrum of periodic Jacobi matrices

In this appendix we prove an easy (and probably well known) fact about the spectrum of periodic Jacobi matrices which we give here because we couldn't find in the literature. Our sources are [vMor 76] and [Mos 84]. In contrast to aperiodic Jacobi matrices the spectrum of periodic Jacobi matrices

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdot & & 0 & a_N \\ a_1 & b_2 & a_2 & \cdot & & \cdot & 0 \\ 0 & a_2 & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & a_{N-2} & 0 \\ 0 & \cdot & \cdot & a_{N-2} & b_{N-1} & a_{N-1} \\ a_N & 0 & \cdot & 0 & a_{N-1} & b_N \end{pmatrix} \, .$$

needs not to be simple as the example

$$L = \begin{pmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{pmatrix}$$

with spectrum $-a, -a, 2a$ shows. We want to prove now that for a generic set of coefficients a_n, b_n, the spectrum is simple.

Lemma 7.1 *There exists an open dense set of coefficients $a, b \in L^\infty(X) = R^{2N}$ such that the Jacobi operator $L = a\tau + a\tau^* + b$ has simple spectrum.*

Proof. Let $L = L_t$ be a one-parameter family of Jacobi operators and denote with $u = u_t$ an eigenvector with eigenvalue E_t. Differentiation of

$$(L - E)u = 0$$

gives $(L' - E')u + (L - E)u' = 0$. Using the symmetry of $L - E$ we get

$$0 = (u, (L' - E')u) + (u, (L - E)u') = (u, (L' - E')u)$$

which leads to the *Rayleigh quotient*

$$E' = \frac{(L'u, u)}{(u, u)} \; .$$

If u has norm 1, this formula gives

$$\frac{\delta}{\delta b_k} L \;=\; u_k^2 \, ,$$

$$\frac{\delta}{\delta a_k} L \;=\; 2u_k u_{k+1} \, .$$

If E is an eigenvalue with multiplicity 2, it has two different eigenfunctions u, v. Let k be an index such that $u_k \neq v_k$. We can also assume $u_k \neq -v_k$ because we can replace else v by $-v$ and find another entry $u_l \neq v_l$. From the above formula we deduce that changing b_k changes the two eigenvalues differently. For small ϵ a change of b_k to $b_k + \epsilon$ splits the doubled eigenvalue E and keeps the already simple eigenvalues simple. Proceeding inductively we can perturb the matrix in order to get only simple eigenvalues. The set of Jacobi matrices with simple eigenvalues is open because the eigenvalues depend in finite dimensions continuously on parameters. □

Remark. A Jacobi matrix L with period N can be viewed as an infinite periodic matrix acting on the finite-dimensional space of N-periodic sequences in $l^2(\mathbf{Z})$. If the periodic matrix acts on the whole Hilbert space $l^2(\mathbf{Z})$, the spectrum of L becomes a *band spectrum*. Let

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2N}$$

be the eigenvalues of the infinite matrix L when restricted to the $2N$ dimensional space of $2N$ periodic sequences in $l^2(\mathbf{Z})$.

The proof of the following Lemma can be found in [vMor 76]

Lemma 7.2 *The spectrum of L acting on $l^2(\mathbf{Z})$ consists of the intervals (bands)*

$$[\lambda_1, \lambda_2], [\lambda_3, \lambda_4], \ldots, [\lambda_{2N-1}, \lambda_{2N}] \; .$$

The spectrum of L acting on N periodic sequences is

$$\{\lambda_1, \lambda_4, \lambda_5, \ldots, \lambda_{2N-3}, \lambda_{2N}\} \; .$$

The intervals between the bands are called *gaps* .

8 Appendix: Bäcklund transformations for random Schrödinger operators

There is a complete parallel theory of isospectral deformations and Bäcklund transformations for random Schrödinger operators over a flow. The set up is the following:

Let X be a compact Hausdorff space and T_t a flow on X with an ergodic invariant measure μ. For $q \in C(X, \mathbb{R})$ one looks at the operator

$$L = -\frac{d^2}{dx^2} + q \, ,$$

acting on $L^2(X)$. The simplest (and well investigated) case of a random Schrödinger operator is a periodic operator

$$L = -\frac{d^2}{dx^2} + q \, ,$$

where q is a continuous say 1 periodic function. If u^+, u^- are the Floquet solutions of the Schrödinger equation

$$Lu = Eu$$

satisfying

$$
\begin{aligned}
u^+(x+1) &= e^\lambda u^+ \\
u^+ - x + 1) &= e^{-\lambda} u^+
\end{aligned}
$$

define the *Weyl functions*

$$m^\pm = (\log u^\pm)' = \frac{\frac{d}{dx} u^\pm}{u^\pm} \, .$$

The Bäcklund transformation

$$BT^\pm(q) = q - 2(\log u^\pm)'' = q - 2\frac{d}{dx} m$$

is a Bäcklund transformation. It is isospectral. (see [Dei 78] Theorem 11). This transformation was found by Darboux in 1882 (see [Ehl 82]) and was worked out by Miura.

An other interesting case are almost periodic Schrödinger operators. Isospectral deformations of these almost periodic operators have been considered in [Joh 82]. An integration of such systems is still missing.

9 Appendix: Bäcklund transformations for aperiodic Jacobi matrices

We discuss shortly Bäcklund transformations for aperiodic Jacobi matrices.
Let L be an aperiodic $N \times N$ Jacobi matrix

$$
L = \begin{pmatrix}
b_1 & a_1 & 0 & \cdot & & \cdot & 0 \\
a_1 & b_2 & a_2 & \cdot & & \cdot & \cdot \\
0 & a_2 & \cdot & \cdot & & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & & a_{N-2} & 0 \\
\cdot & \cdot & \cdot & a_{N-2} & & b_{N-1} & a_{N-1} \\
0 & \cdot & \cdot & 0 & & a_{N-1} & b_N
\end{pmatrix}
$$

satisfying $a_n > 0$ for all $n = 1, \ldots N$. Assume L has the eigenvalues

$$\lambda^{(1)} < \lambda^{(2)} \ldots < \lambda^{(N)}$$

and corresponding eigenvectors

$$u^{(i)}, \ldots, u^{(N)} \ .$$

For E in the spectrum, we write also $u = u(E)$ for the eigenfunction of E. We see from

$$(Lu) = a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1} = E u_n$$

that $u_n = 0$ is not possible as long as $a_{n+1}, a_{n-1} > 0$.
Denote by

$$
\begin{aligned}
m_k(E) &= a_k \frac{u(E)_{k+1}}{u(E)_k} \ , \\
n_k(E) &= a_{k-1} \frac{u(E)_{k-1}}{u(E)_k}
\end{aligned}
$$

the *Titchmarsh-Weyl functions* belonging to the eigenvalue E. We understand in the above definitions $m(E)_N = n(E)_1 = 0$. We get

$$
\begin{aligned}
m(E)_k + n(E)_k &= E - b_k \ , \\
m(E)_k n(E)_{k+1} &= a_k^2 \ .
\end{aligned}
$$

On a doubled lattice we can define

$$c_{2k} = -m_k, \ c_{2k-1} = -n_k \ , k = 1, \ldots n \ .$$

The new Jacobi matrix

$$
D = \begin{pmatrix}
0 & 0 & 0 & \cdot & & \cdot & & 0 \\
0 & 0 & d_2 & \cdot & & \cdot & & \cdot \\
0 & d_2 & \cdot & \cdot & & \cdot & & \cdot \\
\cdot & \cdot & \cdot & \cdot & & d_{N-2} & & 0 \\
\cdot & \cdot & \cdot & d_{2N-2} & & 0 & & d_{2N-1} \\
0 & \cdot & \cdot & 0 & & d_{2N-1} & & 0
\end{pmatrix}
$$

with $d_i = \sqrt{c_i}$ satisfies like in the periodic case

$$[D^2 + E]_{2k,2l} = [L]_{k,l} \ .$$

The entries d_k however are complex in general because c_k is not necessarily positive. If we do the Bäcklund transformation with the *ground state*, the lowest eigenfunction, we get however a real Jacobi matrix D because the Titchmarsh Weyl functions m, n

are then negative. To proof this, we approximate the aperiodic Jacobi matrix by periodic Jacobi matrices

$$L_\epsilon = \begin{pmatrix} b_1 & a_1 & 0 & \cdot & 0 & \epsilon \\ a_1 & b_2 & a_2 & \cdot & \cdot & 0 \\ 0 & a_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_{N-2} & 0 \\ 0 & \cdot & \cdot & a_{N-2} & b_{N-1} & a_{N-1} \\ \epsilon & 0 & \cdot & 0 & a_{N-1} & b_N \end{pmatrix} .$$

The Titchmarsh-Weyl functions $m_\epsilon(\lambda^{(1)}), n_\epsilon(\lambda_\epsilon^{(1)})$ of L_ϵ at the lowest eigenvalue $\lambda_\epsilon^{(1)}$ of L_ϵ converge to the Titchmarsh-Weyl functions $m(\lambda^{(1)}), n(\lambda^{(1)})$ of L at the lowest eigenvalue $\lambda^{(1)}$ of L. Because $-m_\epsilon, -n_\epsilon$ are nonnegative, also $-m, -n$ are nonnegative. They can't be zero and we conclude that the numbers d_i are real and positive.

The map
$$L \mapsto BT_E(L)$$
with
$$[\tilde{L}]_{k,l} = [BT_E(L)]_{k,l} = [D^2 + E]_{2k+1,2l+1}$$
is the Bäcklund transformation. The proof that it is isospectral goes like in the periodic case. If $a_n > 0$ for all $n \in \mathbb{N}$, then the Bäcklund transformation with the ground state $u^{(1)}$ has the property that a $BT(L)$ is again real. The Bäcklund transformation can be written explicitly as

$$\tilde{a}_k = a_k \sqrt{\frac{m_{k+1}}{m_k}}, \ \tilde{b}_k = b + n_k - n_{k+1} \ ,$$

where we understand $n_{n_1} = m_{n+1} = 0$. The Jacobi matrix

$$\tilde{L} = \begin{pmatrix} \tilde{b}_1 & \tilde{a}_1 & 0 & \cdot & \cdot & 0 \\ \tilde{a}_1 & \tilde{b}_2 & \tilde{a}_2 & \cdot & \cdot & \cdot \\ 0 & \tilde{a}_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \tilde{a}_{N-2} & 0 \\ \cdot & \cdot & \cdot & \tilde{a}_{N-2} & \tilde{b}_{N-1} & 0 \\ 0 & \cdot & \cdot & 0 & 0 & \tilde{b}_N \end{pmatrix}$$

is isospectral to L but it satisfies $\tilde{a}_{N-1} = 0$. It is not possible to transform it again with the ground state because we have now zero'1s in the side diagonal. However, we can get a new $(N-1) \times (N-1)$ Jacobi matrix by projecting $\tilde{L}$ onto the eigenspace generated by the eigenvalues different from $\lambda^{(1)}$. This is then a new Jacobi matrix

$$\tilde{L}' = \begin{pmatrix} \tilde{b}_1 & \tilde{a}_1 & 0 & \cdot & \cdot \\ \tilde{a}_1 & \tilde{b}_2 & \tilde{a}_2 & \cdot & \cdot \\ 0 & \tilde{a}_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \tilde{a}_{N-2} \\ \cdot & \cdot & \cdot & \tilde{a}_{N-2} & \tilde{b}_{N-1} \end{pmatrix}$$

for which the Bäcklund transformation with its ground state makes again sense. We can repeat this procedure getting smaller and smaller matrices. The Bäcklund map from the isospectral set of Jacobi operators with spectrum

$$\lambda^{(1)} < \lambda^{(2)} \ldots < \lambda^{(n)}$$

in the isospectral set of Jacobi operators with spectrum

$$\lambda^{(2)} < \ldots < \lambda^{(n)}$$

is a continuous map from an n dimensional simplex into one of its faces.

The reversed process is also interesting because it allows to build up bigger and bigger Jacobi matrices with prescribed spectrum. In each step one gains an eigenvalue more. A possibility to construct the inverse process would be to approximate the Jacobi matrices by periodic matrices L_ϵ, where the Bäcklund transformation can be reversed and to do the limit $\epsilon \to 0$ in a suitable way.

References

[Adl 81] M.Adler. *On the Bäcklund Transformation for the Gelfand-Dickey Equations.*
Comm. Math. Phys., 80:517-527, 1981

[Car 87] R.Carmona, S.Kotani. *Inverse Spectral Theory for Random Jacobi Matrices.*
J. Statist. Phys., 46:1091-1114, 1987

[Car 90] R.Carmona, J.Lacroix. *Spectral Theory of Random Schrödinger Operators.*
Birkhäuser, Boston, 1990

[Cor 82] I.P.Cornfeld,S.V.Fomin,Ya.G.Sinai. *Ergodic Theory.*
Springer, New York, 1982

[Cyc 87] H.L.Cycon,R.G.Froese,W.Kirsch,B.Simon. *Schroedinger Operators.*
Texts and Monographs in Physics, Springer, 1987

[Dam 91] P.Damianou. *About the Toda and Volterra system.*
Phys. Lett. A, 1991

[Dei 78] P.Deift. *Applications of a commutation formula.*
Duke Math. J., 45: 267-310, 1978

[Dei 91] P.Deift,L.C.Li. *Poisson Geometry of the Analog of the Miura Maps and Bäcklund-Darboux Transformations for Equations of Toda Type and periodic Toda Flows.*
Comm. Math. Phys. 143:201-214, 1991

[Ehl 82] F.Ehlers, H.Knörrer. *An algebro-geometric interpretation of the Bäcklund transformation for the Korteweg-de Vries equation.*
Comment. Math. Helv., 57:1-10, 1982

[Ges 93] F.Gesztesy, Z.Zhao. *Critical and subcritical Jacobi operators defined as Friedrichs extensions.*
J.Diff. Equ., 103:68-93, 1993

[Joh 82] R.Johnson,J.Moser. *The Rotation Number for Almost Periodic Potentials.*
Comm. Math. Phys., 84:403-438, 1982

[Kni 92] O.Knill. *Isospectral deformations of random Jacobi operators.*
Commun. Math. Phys., 151:403-426, 1993

[McK 78] H.P.McKean. *Integrable systems and algebraic curves.*
Lecture Notes in Math., No. 755, Springer 1978

[vMor 76] P.v.Moerbeke. *The spectrum of Jacobi Matrices.*
Invent. Math.,37:45-81, 1976

[vMor 78] P.v.Moerbeke. *About isospectral deformations of discrete Laplacians.*
Lecture Notes in Math., No. 755, Springer 1978

[Mos 75a] J.Moser. *Three Integrable Hamiltonian Systems Connected with Isospectral Deformations.*
Adv. in Math., 16:197-220, 1975

[Mos 75b] J.Moser. *Finitely many mass points on the Line under the influence of an exponential potentials, an integrable system.*
Lecture Notes in Physics, 38, 1975

[Mos 84] J.Moser. *Drei Kapitel über integrable Hamiltonsche Systeme.*
ETH lecture , Summer 1984

[Ree 80] M.Reed, B.Simon. *Methods of modern mathematical physics I, Functional analysis.*
Academic Press, Orlando, 1980

[Rue 79] D.Ruelle. *Ergodic theory of differentiable dynamical systems.*
Publ. Math. IHES, 50:27-58, 1979

[Sim 83] B.Simon. *Kotani Theory for One Dimensional Stochastic Jacobi Matrices.*
Comm. Math. Phys, 89:227-234, 1983

[Sim 90] B.Simon. *Absence of Ballistic Motion.*
Comun. Math. Phys., 134:209-212, 1990

[Tod 81] H.Toda. *Theory of nonlinear lattices.*
Springer. 1981

[Wed 75] M.Wedati,M.Toda. *Bäcklund transformation for the exponential lattice.*
J. Phys. Soc. Japan, 39:1196, 1975

Cohomology of cocycles and zero curvature equations for random Toda flows

Contents

Abstract

We show that transfer cocycles of random Jacobi operators move according to zero curvature equations, when the Jacobi operators are deformed in an isospectral way. The zero curvature equations are isospectral deformations of $SL(2, \mathbb{R})$ cocycles.

We show that every $SL(2, \mathbb{R})$ cocycle is cohomologuous to a transfer cocycle of a random Jacobi operator if the dynamical system (X, T, m) is ergodic.

We consider discrete d−dimensional zero curvature $SL(2, \mathbb{R})$ gauge fields over a $\mathbb{Z}^d$ dynamical system. These gauge fields are Toda systems with discrete space and time. We begin to study the moduli space of zero curvature gauge fields in the case of an abelian structure group and $d = 2$.

In the case, when the structure group is $\mathbb{T}^1$, a result of Feldman-Moore has the following reinterpretation: for any field h, there is a gauge potential (f, g), such that the curvature of (f, g) is h. This leads to the existence of random Harper models with arbitrary space dependent magnetic flux.

1   Introduction

Most integrable systems allow a zero-curvature representation (see [Fad 86]). Also for the Toda lattice such a representation has been given (see [Fad 86] p. 471). We want to adapt those zero curvature representations for the random Toda lattice. The Toda system is a semi-discrete system in which time is continuous and the space is discrete. We want also to formulate higher dimensional zero curvature representation of discrete Toda equations in any dimension. In two dimensions, we compare the continuous (KdV equation), semi-discrete (usual Toda system), and discrete (lattice gauge field) set up. To get a symmetry between continuous and discrete case, we formulate everything in a dynamical setting. The idea is to replace the discrete d− dimensional lattice $\mathbb{Z}^d$ by the orbit of an aperiodic $\mathbb{Z}^d$ action. In the continuous case, the d− dimensional plane $\mathbb{R}^d$ is replaced by an orbit of an aperiodic $\mathbb{R}^n$ action $T_1, \ldots T_d$ on a compact metric space. The advantage to replace manifolds by orbits of dynamical systems is that the embedding in a compact metric or a probability space is useful for example for integration. The notations get simpler and there is more symmetry between discrete and and continuous systems. The idea is used in statistical mechanics, where the thermodynamic formalism has been generalized to lattices formed by orbits of dynamical systems. Advantages of such a set-up is a simplification of notation and a stimulation of both of the subjects dynamical systems and statistical mechanics. The general idea of doing mathematics on orbits of group actions seems to be due to Mackey [Mac 63].

The embedding of the lattice inside a probability space is always connected with cohomology constraints which manifest in an algebraic way the *random boundary*

conditions. . Not everything which is possible in the free lattice is also possible on the embedded lattice. Also the properties of the dynamical system play a role and the choice of the ergodic dynamical system corresponds to a choice of a boundary condition.

An important question is to know the moduli space of zero curvature fields or even to know the moduli space of all fields.

2 Zero curvature equations in two dimensions

We compare here the continuous, the semi-discrete and the discrete set up for two-dimensional zero curvature fields with structure group $SL(2, \mathbb{C})$. An example of a continuous field is the *KdV flow* an example for the semi-discrete field is the *Toda flow* and an example for the discrete field is the *discrete Toda lattice*.

Let X be a compact metric space which is also a probability space. Let $\mathcal{R}$ be a two-dimensional group acting on X by measure preserving transformations. We distinguish continuous, the semi-discrete and the discrete case (i) $\mathcal{R} = \mathbb{R} \times \mathbb{R}$, (ii) $\mathcal{R} = \mathbb{R} \times \mathbb{Z}$ and (iii) $\mathcal{R} = \mathbb{Z} \times \mathbb{Z}$ (the discrete case). We denote with T^t, S^s the actions of $\mathcal{R}$ and with δ_t, δ_x the Lie-derivatives which are in the discrete case $\tau_x f = f(T)$ and $\tau_t f = f(S)$, in the continuous case

$$\delta_x f = \frac{d}{dx} f(T_x)|_{x=0} \ , \delta_t f = \frac{d}{dt} f(T_t)|_{t=0} \ .$$

A matrix Lie group $G = SL(2, \mathbb{C})$ called the *structure group* and denote with $g = sl(2, \mathbb{C})$ its Lie algebra. The space of *Gauge fields* $\mathcal{G} \subset L^\infty(X, G)$ consists of functions $A \in L^\infty(X, G)$ having the property that on each orbit, the mapping

$$\mathcal{R} \to G, \ r \mapsto A(rx)$$

is smooth. A *connection* or *gauge potentials* is a smooth function in (i) $L^\infty(X, g \times g)$, (ii) $L^\infty(X, \mathcal{G} \times g)$ or (iii) $L^\infty(X, G \times G)$. To a connection is attached a *covariant derivative* which is

$$
\begin{aligned}
(i) &\quad \nabla_t = \delta_t - V \ , \nabla_x = \delta_x - U \ , \\
(ii) &\quad \nabla_t = \delta_t - V \ , \nabla_x = L\tau^* \ , \\
(iii) &\quad \nabla_t = K\tau_t^* \ , \nabla_x = L\tau_x^* \ .
\end{aligned}
$$

The commutator $[\nabla_t, \nabla_x]$ is the *additive curvature*. Let u be a vector field on X. The integrability condition for the two over-determined equations $\nabla_t u = 0$, $\nabla_x u = 0$ is $[\nabla_t, \nabla_x] = 0$ called *zero curvature condition*. It can be written in the three cases as

$$
\begin{aligned}
(i) &\quad U_t - V_x + [U, V] = 0 \ , \\
(ii) &\quad L_t \tau_x^* + [L\tau_x^*, V] = 0 \ , \\
(iii) &\quad [L\tau_x^*, K\tau_t^*] = 0 \ .
\end{aligned}
$$

Given a connection (U, V) and $A \in \mathcal{G}$, the transformations

$$
\begin{aligned}
(i) \quad & (U, V) \mapsto \left(AUA^{-1} + A_x A^{-1}, AVA^{-1} + A_t A^{-1} \right), \\
(ii) \quad & (L, V) \mapsto \left(ALA^{-1}(T^{-1}), AVA^{-1} + A_t A^{-1} \right), \\
(iii) \quad & (L, K) \mapsto \left(ALA^{-1}(T^{-1}), AKA^{-1}(S^{-1}) \right)
\end{aligned}
$$

are called *gauge transformation*. The zero curvature equation stays invariant under such transformations.

Given a connection (U, V) one defines a *parallel transport* of a vector. The definition is the simplest in the discrete case (iii), where a path is a sequence of points $\Gamma = \{x_1, \ldots, x_n\}$, where $x_n = Tx_{n-1}$ for $T \in \{T_1, T_2, T_1^{-1}, T_2^{-1}\}$. We represent Γ also as a sequence $(T_{i_1}, T_{i_2}, \ldots, T_{i_{n-1}})$.

$$
\int_\Gamma A(x) = A_i(T_{i_{n-1}} \cdots T_{i_1} x) \cdots A(T_{i_1} x) A(x)
$$

which defines a *parallel transport* along the path. The parallel transport in the continuous case can be reduced to the discrete case by approximating a curve in $\mathbb{R}^2$ through a polygon and taking $A_i = \exp L_i$ on the edges of the polygon.

Examples.
• Case (i): Take $X = T^4$ and $T_t(x, y) = (x + t\alpha, y), T_s(x, y) = (w, y + s\beta)$, where $x, y \in T^2$. An orbit can be a torus, if $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2)$ are both rational vectors, a cylinder, if one of the two vectors α, β is rational, or the two dimensional plane, if both α, β are irrational. Define for a spectral parameter E the connection (see [Fad 86] p. 307)

$$
\begin{aligned}
U &= -2iE \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \\
V &= -2iE^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + E^2 \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \\
&\quad -iE \begin{pmatrix} u & 0 \\ u_x & -u \end{pmatrix} + \begin{pmatrix} -u_x & 2u \\ 2u^2 - u_{xx} & u_x \end{pmatrix} .
\end{aligned}
$$

The zero curvature condition gives the *Korteweg de Vries equation*

$$
u_t - 6uu_x + u_{xxx} = 0 .
$$

If α is rational, we are looking for periodic solutions, if β is rational, we have a KdV with periodic boundary conditions.

• Case (ii): Take $X = X_1, \times X_2$, where $X_1 = T^2$ and X_2 is a probability space. Take $T_t(y) = (y + \alpha)$ and T_s is generated by an automorphism T of the probability space.

The zero curvature equation for the connection

$$L = \begin{pmatrix} p + E & e^q \\ -e^{-q} & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & -e^{q(T)} \\ e^{-q} & 0 \end{pmatrix}.$$

gives $\dot{L}\tau = [V, L\tau]$ or

$$\begin{aligned} \dot{q} &= p + E, \\ \dot{p} &= e^{q(T)-q} - e^{q-q(T^{-1})}, \end{aligned}$$

which is the Toda lattice. By the gauge transformation

$$C = \begin{pmatrix} e^{-q(T)/2} & 0 \\ 0 & e^{q/2} \end{pmatrix}$$

the Toda flow in the (a, b) coordinates can be recovered

$$\begin{aligned} L' &= CLC^{-1}(T^{-1}) = a^{-1} \begin{pmatrix} b + E & -a(T^{-1}) \\ a & 0 \end{pmatrix}, \\ V' &= CVC^{-1} + C_t C^{-1} = \begin{pmatrix} -E - b(T) & 2a \\ -2a & b \end{pmatrix}. \end{aligned}$$

With a further gauge transformation with

$$C' = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix},$$

we get

$$\begin{aligned} L'' &= C'L'C'^{-1}(T^{-1}) = a(T^{-1})^{-1} \begin{pmatrix} b + E & -a^2(T^{-1}) \\ 1 & 0 \end{pmatrix}, \\ V'' &= C'VC'^{-1} + C'_t C'^{-1} = \begin{pmatrix} -E - b & 2a^2 \\ -2 & b + E \end{pmatrix} \end{aligned}$$

with $L''(x) \in SL(2, \mathbf{R})$, $V''(x) \in sl(2, \mathbf{R})$.

Remark. Also the *random Volterra differential equation* $\dot{u} = u(u(S) - u(S^{-1}))$ has a zero curvature representation (see [Fad 86] p.296). With

$$L = \begin{pmatrix} E & u \\ -1 & 0 \end{pmatrix}, V = \begin{pmatrix} u(S) & -Eu(S) \\ -E & -E^2 + u \end{pmatrix}$$

one gets

$$\dot{A}\sigma^* = [V, A\sigma^*].$$

- Case (iii): Given two automorphisms S, T of the probability space. The problem consists of finding $K, L \in L^\infty(X, SL(2, \mathbf{R}))$ satisfying

$$LK(T) = KL(S) .$$

Solutions can be found by taking an element $M \in L^\infty(X, SL(2, \mathbf{R}))$ and define (L, M) to be the gradient of M

$$L = MM(T)^{-1} , K = MM(S)^{-1} .$$

One checks then that $LK(T) = MM(TS)$ and $KL(S) = MM(TS)$. Such solutions are gauge equivalent to the identity.

There are candidates for nontrivial examples of random discrete zero curvature equations [Sur 90],[Sur 91],[Qui 91],[Cap 91]. In these articles the finite case X is treated and the notation is different. We take an example of Suris in [Sur 91]: The equation for $q \in L^\infty(X)$

$$q(S) - 2q + q(S^{-1}) = \log\left(\frac{1 + h^2 \cdot e^{q(T) - q}}{1 + h^2 \cdot e^{q - q(T^{-1})}}\right)$$

is equivalent to

$$e^{q(S) - q} - e^{q - q(S^{-1})} = h^2 \cdot e^{q(T) - q(S^{-1})} - h^2 \cdot e^{q(S) - q(T^{-1})}$$

and is called *discrete time Toda equation*. It can be written as a zero curvature equation

$$L(S)M = M(T)L$$

with

$$L = \begin{pmatrix} Ee^{q - q(S^{-1})} & -E^{-1}he^q \\ -he^{-q(T^{-1})} & 0 \end{pmatrix} , M = \begin{pmatrix} E^{-1} & -he^q \\ Ehe^{-q(T^{-1})} & E \end{pmatrix} ,$$

where E is an additional spectral parameter. The problem is to find solutions q of these zero curvature equation. An other example is the *Hirota equation* [Bbgp 92] which is translated in the random language given by a solution $v \in L^\infty(X)$ of

$$vv(S^{-1})v(T)v(S^{-1}T) = 1 + k(vv(S^{-1}T) - v(S^{-1})v(T)) ,$$

where k is a parameter. This equation can be written as a zero curvature equation

$$ML(T) = LM(S)$$

for

$$L = \begin{pmatrix} v/v(S^{-1}) & \lambda \\ \lambda & v(S^{-1})/v \end{pmatrix} , M = \begin{pmatrix} k & (\lambda v(S^{-1})v(S^{-1}T))^{-1} \\ \lambda^{-1}v(S^{-1}T)v(S^{-1}) & k \end{pmatrix} ,$$

170

where λ is an additional parameter. Bobenko et al. also remark that the Hirota equation can be written as $v(TS^{-1})v = M(v(T)v(S^{-1}))$, where M is the Moebius transform

$$z \mapsto \frac{1+kz}{k+z} \ .$$

They study the map also over finite fields for which the Möbius transform on a quadratic extension of the field makes sense and maps $S = \{v\bar{v}\}$ into itself. Such a system can be viewed as a *cellular automata*.

3 Zero curvature equations for random Toda flows

We quickly redefine the definition of the random Toda lattice. which is an isospectral deformation in $\mathcal{X}$, the *crossed product* of $L^\infty(X)$ with the dynamical system (X, T, m). The von Neumann algebra $\mathcal{X}$ consists of elements $K = \sum_n K_n \tau^n$ with convolution multiplication, where the rule $\tau K_n = K_n(T)\tau$ is used to shift the τ to the right and $\tau^* = \tau^{-1}$ and the norm is given by

$$|||K||| = |\ ||K(x)||\ |_\infty \ ,$$

where $K(x)$ is the infinite matrix $[K(x)]_{mn} = K_{n-m}(T^m x)$. There is a trace $\text{tr}(K) = \int_X K_0\ dm$. Each element K has the decomposition $K = K^- + K_0 + K^+$ defined by $K^+ = \sum_{n=1}^\infty K_n$. With $\mathcal{L} \subset \mathcal{X}$ is denoted the Banach space of *random Jacobi operators* $L = a\tau + (a\tau)^* + b$ if $a, b \in L^\infty(X)$ are real. For a Hamiltonian

$$H \in C^\omega(\mathcal{L}) = \{H(L) = \text{tr}(h(L))|\ h \text{ entire}, \ h(\mathbb{R}) \subset \mathbb{R}\}$$

the *random Toda lattice*

$$\dot{L} = [B_H(L), L],$$

with $B_H(L) = h'(L)^+ - h'(L)^-$ and $H(L) = \text{tr}(h(L))$ is an isospectral flow in $\mathcal{L}$. It reduces to the periodic Toda lattice in the case when $|X|$ is finite.

We have seen in the last paragraph, how the first Toda lattice can be written as a zero curvature equation. We show now how each random Toda flow is a zero curvature condition for a certain gauge field over a semi-discrete space time.

We assume now $\mathcal{G} = L^\infty(X, SL(2, \mathbb{R}))$. If we take the connection $C = (A_E, V_E)$ where V_E is the $sl(2, \mathbb{R})$ cocycle

$$V_E = \begin{pmatrix} -E - b & 2a^2 \\ -2 & E + b \end{pmatrix} \ ,$$

the zero curvature equation

$$\frac{d}{dt} A\tau^* = V A\tau^* - AV(T^{-1})\tau^* = [V, A\tau^*]$$

171

describes the *first Toda flow*

$$\dot{a} = a(b(T) - b),$$
$$\dot{b} = 2 \cdot a^2 - 2 \cdot a^2(T^{-1}),$$

with Hamiltonian $H(L) = \operatorname{tr}(\frac{L^2}{2})$.

Parallel transport along the time axis gives the motion of the Toda lattice. If we make parallel transport along space, the average growth rate of the vector is $e^{\lambda(A)}$, a Gauge invariant number.

Two cocycles $A, B \in \mathcal{A} = L^{\infty}(X, SL(2, \mathbf{R}))$ are called *cohomologuous* or *Gauge equivalent*, if there exists $C \in \mathcal{A}$ such that $B = C(T)AC^{-1}$. A cocycle $A \in \mathcal{A}$ is also called a *weighted composition operator* $A\tau^*$ acting on $l^2(\mathbf{Z})^2$ as $A\tau^* w_n = A(x)w_{n-1}$.

Theorem 3.1 *For each Toda flow* $\dot{L} = [B_H(L), L]$ *with* $H \in C^{\omega}(\mathcal{L})$ *there is a zero curvature equation*

$$\dot{A}_E \tau^* = [V_{h,E}, A_E \tau^*].$$

Each of these flows defines an isospectral deformation of the cocycle $A\tau^*$. *The deformed cocycles are all cohomologuous: there exists a curve* $U(t) \subset \mathcal{A}$ *with*

$$A(t) = U(t)^{-1}A(0)U(t)(T^{-1}).$$

Proof. We show this first for a Hamiltonian

$$H_k(L) = \frac{\operatorname{tr}(L^k)}{k}$$

which gives the Toda flow $\dot{L} = [B_k, L]$ with $B_k = (L^k)^+ - (L^k)^-$.

For E outside an interval containing the spectrum $\Sigma(L)$ of L, there exists $\{u_n^+\} \in \mathbf{R}^{\mathbf{Z}}$ satisfying $L(x)u^+ = Eu^+$ and $u_n^+ \in l^2(\mathbf{N})$.

Claim. If $u(t)$ satisfies the differential equation

$$\dot{u} = B_k(x)u$$

with initial condition $u(0) = u^+$ then $L(t)u(t) = Eu(t)$.

Proof. In order to make sense of the differential equation $\dot{u} = Bu$, we can view u as living in a weighted Banach space $\mathcal{M}_K$ of sequences defined as

$$\mathcal{M}_K = \{c = (\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots) \mid \|c\| = \sum_n |c_n|K^{-|n|} < \infty\}.$$

172

If the positive real parameter K is chosen big enough, a solution u of the Schrödinger equation $Lu = Eu$ is in $\mathcal{M}_K$. The differential equation $\dot{u} = Bu$ has then by Cauchy's existence theorem a unique solution in $\mathcal{M}_K$. Using $\dot{L} = [B_k, L]$ and $\dot{u} = B_k u$ we get

$$\frac{d}{dt}(L(x)(t)u(t) - Eu(t)) = (B_k L - LB_k)u + L\dot{u} - E\dot{u}$$
$$= (E - L)B_k u + LB_k u - EB_k u = 0 \ .$$

With this and the fact that we assumed $L(t)u(t) = Eu(t)$ for $t = 0$ follows the claim.

The differential equation $\dot{u} = B_k u$ is an equation of the form

$$\dot{u}_n = F(u_{n-k}, \ldots, u_{n+k}).$$

Define $w_n = (u_{n+1}, u_n)$ and write $\dot{w}_0 = V_{k,E}w_0$, where $V_{k,E}$ is calculated as follows: use the equation $L(x)u = Eu$ which is a linear difference equation of the form $u_{n+1} = G(u_n, u_{n-1})$ to express (u_{n+1}, u_n) for $n \neq 0, 1$ as a linear function of (u_0, u_1). The so obtained matrix $V_{k,E}(x)$ does not depend on the choice of u. The integrability condition for the two equations

$$(\frac{\partial}{\partial t} - V_{k,E})w_0 = 0 \ , (A_E(x)\tau^*)w_0 = w_0 \ ,$$

(differentiate the second equation and plug in $\dot{w}_0$ into the first) gives the zero curvature equation

$$\dot{A}_E(x)\tau^* = [V_{k,E}(x), A_E(x)\tau^*] \ .$$

In general, if the flow is given by a Hamiltonian $H(L) = \sum h_n \text{tr}(L^n) \in C^\omega(\mathcal{L})$, we get also a zero curvature representation with

$$V(x) = \sum_n h_n V_{k,E}(x) \ .$$

The rough estimate $||V_{k,E}(x)|| \leq k||L(x) - E||^{2k}$ shows that the sum $V(x)$ converges. The mapping $x \mapsto V_E(x)$ is measurable. Define a curve of cocycles $U(t)$ by

$$\dot{U} = V_{h,E}U, \ U(0) = 1 \ .$$

We check

$$U^{-1}(t)A_E\tau^* U(t) = U^{-1}(t)A_E U(T^{-1})(t)\tau^* = A_E(t)\tau^*$$

and the Toda-motion is indeed an isospectral deformation of the cocycle A_E. All the deformed cocycles are cohomologuous. $\qquad\square$

Proposition 3.2 *Given two Hamiltonians F, G. Denote by t_F and t_G the time parameters for the Toda flows induced by F rsp G. Then the Sakharov-Shabat equation*

$$[\frac{\partial}{\partial t_F} - V_F, \frac{\partial}{\partial t_G} - V_G] = \frac{\partial V_F}{\partial t_G} - \frac{\partial V_G}{\partial t_F} + [V_F, V_G] = 0$$

holds in $L^\infty(X, sl(2, \mathbb{C}))$.

Proof. Because the Toda flows corresponding to the Hamiltonians F and G are commuting, the parallel transports

$$\frac{\partial}{\partial t_F} - V_F, \frac{\partial}{\partial t_G} - V_G$$

are also commuting and the commutation of the vector fields is equivalent to the Sakharov-Shabat equation. $\qquad\square$

4 Cohomology of SL(2,R) cocycles

Cocyles appearing as transfer operators of Jacobi operators are not as special as one might think. Every $SL(2,\mathbb{R})$ cocycle is conjugated to one of them:

Theorem 4.1 *Assume (X, T, m) is aperiodic. Every $A \in \mathcal{A}$ is cohomologuous to an element B in $\mathcal{B} = \{B \in \mathcal{A} \mid B_{22}(x) = 0\}$.*

Proof. Denote by $\tilde{v} = q/r$ the projective coordinate of a vector $v = (q, r)$ and with $\tilde{C}$ the projective transformation $\tilde{v} \mapsto \tilde{C}v$ belonging to a matrix $C \in SL(2, \mathbb{R})$. The requirement $[C(T)^{-1}AC]_{22} = 0$ is equivalent to

$$\tilde{C}(T)^{-1}\tilde{A}\tilde{c}_2 = \tilde{e}_1,$$

where e_i are the basis vectors in $\mathbb{R}^2$ and $c_i = Ce_i$. Multiplying from the left with $\tilde{C}(T)$ shows, that this is equivalent to

$$\tilde{A}\tilde{c}_2 = \tilde{c}_1(T).$$

Denote by $R(\phi)$ the matrix in $SO(2, \mathbb{R})$ which makes a rotation about an angle ϕ. There exists (see [Kni1]) cocycles B arbitrary near to A such that $BR(\frac{\pi}{2})$ has two different Lyapunov exponents. Oseledec's theorem implies then the existence of a projective field $\tilde{v}$ satisfying

$$\tilde{B}\tilde{R}(\frac{\pi}{2})\tilde{v} = \tilde{v}(T).$$

Define $c_1 = v$ and $c_2 = A^{-1}BR(\frac{\pi}{2})v$ where v are unit vectors having the projective value $\tilde{v}$. For B sufficiently near to A, the determinant of the cocycle C' satisfying $C'e_i = c_i$ for $i = 1, 2$ is bounded uniformly away from 0 because for $A = B$, the determinant of C' is 1. Define now

$$C(x) = C'(x)/\det(C'(x)) \in \mathcal{A}.$$

We have then

$$\tilde{A}\tilde{c}_2 = \tilde{B}\tilde{R}(\frac{\pi}{2})\tilde{c}_1 = \tilde{c}_1(T)$$

or $\tilde{A}\tilde{c}_2 = \tilde{c}_1(T)$ which is equivalent with $C(T)^{-1}AC \in \mathcal{B}$. $\qquad\square$

Remark. This result shows, that every cocycle $A \in \mathcal{A}$ can occur in a zero curvature equation if the dynamical system is aperiodic. In the case of a periodic dynamical system Theorem 4.1 is wrong. A counterexample is already $|X| = 1, A(x) = 1$.

Corollary 4.2 *If (X, T, m) is aperiodic, the Lyapunov exponent $\lambda(A)$ of a $SL(2, \mathbf{R})$ cocycle can be determined by calculation of a determinant of a random operator.*

Proof. The determinant of a random Jacobi operator $L = a\tau + (a\tau)^* + b$ is related with the Floquet exponent by

$$e^{-w(E)} = e^{\mathrm{tr}(\log(L-E))} = \det(L - E) .$$

The Thouless formula for the transfer cocycle A_E is

$$-\lambda(A_E) + i\rho(A_E) = w(E) + \log(M) ,$$

where $M = \exp(\int_X \log(a) \, dm$ and $\rho(A_E)$ is the rotation number. We obtain therefore

$$|\det(L)| = M \cdot e^{\lambda(A_0)} .$$

In the last theorem we have seen that every $SL(2, \mathbf{R})$ cocycle is cohomologuous to a transfer cocycle of a random Jacobi operator. $\qquad \square$

An other application of this conjugation result is

Corollary 4.3 *If (X, T, m) is aperiodic then*
a) $\mathcal{P} \cap \mathcal{B}$ is dense in $\mathcal{B}$.
b) $\mathcal{P} \cap \mathcal{B} \setminus \mathrm{int}(\mathcal{P} \cap \mathcal{B})$ is not empty.

Proof. Assume $C(T)AC^{-1} = B$. Then there exists a continuous mapping ϕ from a neighborhood $\mathcal{U}$ of A onto a neighborhood $\mathcal{V}$ of $B \in \mathcal{B}$

$$\phi(A) = C(A)(T)AC(A)^{-1} .$$

Given $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ cohomologuous to B and every such conjugation can be extended to a conjugation valid in a neighborhood of B.
a) Take a sequence $A_n \to A$, where $A_n \in \mathcal{P}$ and map it to a sequence $B_n \to B$ where $B_n \in \mathcal{B}$ is conjugated to A_n and $B \in \mathcal{B}$ is conjugated to A.
b) There exists a sequence $A_n \in \mathcal{A}$ with $\lambda(A_n) = 0$ and $A_n \to A$ and $A \in \mathcal{P}$. Conjugate this to a sequence $B_n \in \mathcal{B}$ with $B_n \to B \in \mathcal{B}$. $\qquad \square$

One could ask if there exists $C \in \mathcal{A}$ such that

$$C(T)AC^{-1}(x) = \begin{pmatrix} b(x) & -1 \\ 1 & 0 \end{pmatrix} .$$

We don't know the answer but the above proposition shows that this can be reduced to the question when

$$A = \begin{pmatrix} \tilde{b}(x) & -\tilde{a}(x) \\ \tilde{a}(x)^{-1} & 0 \end{pmatrix} .$$

In the case when $a = c(T)c$, we calculate with

$$C = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$$

$$C(T)AC^{-1} = \begin{pmatrix} b(x)c(T)c^{-1} & -1 \\ 1 & 0 \end{pmatrix} .$$

For the condition $a = c(T)c$ to be true, the equation $\log(a(T)a^{-1}) = \log(c(T^2)) - \log(c)$ must be true. This is a cohomology problem and c does not necessarily have to exist.

5 Discrete Zero curvature equations in more dimensions

Given a matrix group $G \subset M(k, \mathbf{R})$ with trace tr and a $\mathbf{Z}^d$ dynamical system determined by d commuting automorphisms $T_1, \ldots T_d$ of the probability space (X, m). For a *Gauge potential* or *connection* $A\tau = (A_1\tau_1^*, \ldots, A_d\tau_d^*)$ with $A_i \in L^\infty(X, G)$ is defined the *(additive) curvature*

$$F_{ij}\tau_i^*\tau_j^* = [A_i\tau_i^*, A_j\tau_j^*] = (A_iA_j(T_i) - A_jA_i(T_j))\tau_i^*\tau_j^*$$

for all $i, j \in \{1, \ldots d\}$. If this curvature is vanishing, we say the Gauge field $A\tau$ has *zero curvature*. In this case the Gauge potential $A\tau^*$ is also called a *cocycle*. We call

$$F_{ij}\tau_i^*\tau_j^* = \int_{\Gamma_{ij}} A = A_iA_j(T_i)A_i(T_j)^{-1}A_j^{-1}$$

the *multiplicative curvature* of the field $A\tau$. $F_{ij}(x)$ is the result of the parallel transport around a *plaquette* $\Gamma_{ij} = \{x, T_i(x), T_iT_j(x), T_j(x)\}$. The multiplicative curvature is the identity if and only if the field has zero additive curvature. A trivial possibility to construct zero curvature fields is to take a cocycle C and to form the *gradient*

$$(A_1, \ldots, A_d) = (CC^{-1}(T_1^{-1}), \ldots, CC^{-1}(T_d^{-1})) .$$

A map

$$(A_1, \ldots, A_d) \mapsto (UA_1U^{-1}(T_1^{-1}), \ldots UA_dU^{-1}(T_d^{-1})) ,$$

is called a *Gauge transformation*. The gradients are precisely the fields which are Gauge equivalent to the identity $1\tau^*$. An other trivial possibility to construct zero curvature fields is to take constant diagonal cocycles $A_i(x) = A_i$ which don't depend on $x \in X$. They also have zero curvature but they are no gradients. The moduli space of zero curvature equations is uncountable. The function $\text{tr}(F_{ij})(x)$ is invariant under gauge transformations. Define for each $1-$ form A the *Wilson action*

$$W(A) = \int_X \sum_{i<j} \text{tr}(F_{ij}(x)) \, dm(x) .$$

An aim would be to find the *moduli space* of two dimensional zero curvature gauge fields $(A_1, \ldots A_d)$ over the $\mathbb{Z}^d$- dynamical system (X, T, m). A more general problem is the determination of the moduli space of arbitrary $SL(2, \mathbb{R})$ fields over an ergodic $\mathbb{Z}^d$ action. If we would find a gauge invariant measure μ on the moduli space of 1− forms, this would be a starting point for a pure Gauge field theory with a *partition function*

$$Z = \int e^{-W(A)} \, d\mu(A)$$

and for any gauge invariant function $\phi : \Lambda^1 \to \mathbb{R}$, the expectation value

$$< \phi >= Z^{-1} \int \phi(A) e^{-W(A)} \, d\mu(A)$$

would be of general interest because it does not depend on the field A.

6 The moduli space of discrete zero curvature fields

We begin studying the *moduli space of zero curvature fields* over a two dimensional dynamical system (X, T_1, T_1, m), when the group G is abelian. We will see, that it is important to know the *cohomology group* $H^1(T_1, G)$ and the way, T_2 acts on this group, in order to understand the moduli space of zero curvature fields.

The Gauge group $\mathcal{G} = L^\infty(X, G)$ is acting on gauge fields $(A_1, A_2) \in \mathcal{G} \times \mathcal{G}$ by

$$(A_1, A_2) \mapsto (B(T_1) A_1 B^{-1}, B(T_2) A_2 B^{-1}) \, .$$

A gauge field (A_1, A_2) has zero curvature, if $A_1 A_2(T_1) = A_2 A_1(T_2)$ or equivalently $A_1(T_2) A_1^{-1} = A_2(T_1) A_2^{-1}$. The group of zero curvature fields has the subgroup of gauge fields cohomologuous to the identity

$$(A_1, A_2) = (B(T_1) B^{-1}, B(T_2) B^{-1}) \, .$$

We would like to determine the group

$$M(T_1, T_2, G) = \frac{\{A_1(T_2) A_1^{-1} = A_2(T_1) A_2^{-1}\}}{\{(B(T_1) B^{-1}, B(T_2) B^{-1})\}} \, ,$$

the *moduli space of two-dimensional zero curvature fields.*

Denote by $H^0(T_i, G)$ the zeroth cohomology group

$$H^0(T_i, G) = \{B \in \mathcal{G} \mid B(T_i) = B\} \, .$$

If T_i is ergodic, then $H^0(T_i, G)$ is isomorphic to G, because only constant fields have then the property that $B(T) = B$. The *first cohomology group* $H^1(T_i, G)$ of the dynamical system (X, T_i, m) with structure group G is

$$H^1(T_i, G) = \mathcal{G}/\{A \mid A = B(T_i)B^{-1}\} \ .$$

We denote with $\tilde{A}$ an equivalence class in $H^1(T_1, G)$. The multiplication in $H^1(T_i, G)$ is defined by $\tilde{A}\tilde{B} = \widetilde{AB}$. Because the automorphisms T_1, T_2 are commuting, the transformation T_1 acts also on $H^1(T_2, G)$. The fixed points of this action form a subgroup of $H^1(T_2, G)$, which we denote by $\text{Fix}(T_1, H^1(T_2, G))$. In the same way, $\text{Fix}(T_2(H^1(T_1, G)$ is defined. We can prove

Proposition 6.1

$$M_0(T_1, T_2, G) = \text{Fix}(T_2, H^1(T_1, G)) \times H^0(T_1, G) \ .$$

Proof. We can use the Gauge transformation to fix the first element A_1 in its cohomology class $\tilde{A}_1$. We have therefore

$$
\begin{aligned}
M_0(T_1, T_2, G) &= \{\tilde{A}_1 \mid \tilde{A}_1(T_2)\tilde{A}_1^{-1} = \tilde{1}\} \times \{B \mid B(T_1) = B\} \\
&= \{\tilde{A}_1(T_2) = \tilde{A}_1\} \times H^0(T_1, G) \\
&= \text{Fix}(T_2, H^1(T_1, G)) \times H^0(T_1, G)
\end{aligned}
$$

$\square$

Because of the symmetry $T_1 \leftrightarrow T_2$, we get as a corollary

$$\text{Fix}(T_1, H^1(T_2, G)) = \text{Fix}(T_2, H^1(T_1, G)) \ .$$

In the special case when $T_1 = T_2^k$, we obtain $\text{Fix}(T_2, H^1(T_1, G)) = H^1(T_1, G)$ and

$$M_0(T_1, T_2, G) = H^1(T_1, G) \times H^0(T_1, G) \ .$$

It would therefore be of advantage to know $H^1(T_1, G)$ in order to determine the moduli space of two-dimensional zero curvature fields. However, it could well be that there are situations when T_1 has only one fixed point in $H^1(T_2, G)$ which would imply that the moduli space of zero curvature fields is $H^0(T_1, G)$. If T_1 is also ergodic, the moduli space of zero curvature fields is isomorphic to G. In any case, we would like to understand the first cohomology group $H^1(T, G)$ and to understand how a second transformations commuting with T acts on this space.

Examples:
• Assume we have a $\mathbf{Z}^2$ dynamical system such that $|X| < \infty$ and

$$T_1(x) = x + 1 \pmod{|X|}, \ T_2(x) = x + p \pmod{|X|}$$

178

are ergodic. In this case, $H^0(T_i, G) = G$ and $H^1(T_i, G) = G$ and the moduli space of zero curvature fields is $G \times G$ because every cohomology class of T_1 is invariant under the action of T_1 and so under the action of $T_2 = T_1^p$.

• Let $G = Z_2$. A gauge field (A_1, A_2) is then represented by 2 measurable sets $Y_1, Y_2 \subset X$, where $Y_i = \{x \in X \mid A_i(x) = -1\}$. A gauge transformation is given by

$$(Y_1, Y_2) \mapsto (Y_1 \Delta(Z \Delta Z(T_1)), Y_2 \Delta(Z \Delta Z(T_2))) \ .$$

A field has zero curvature if $Y_1(T_2) \Delta Y_2 = Y_2(T_1) \Delta Y_1$. Already here, we are not able to determine the moduli space of zero curvature fields because we don't know

$$\mathcal{H}^1(T, Z_2) = \{Y \subset X \text{ measurable}\} / \{Y \Delta Y(T) \mid Y \subset X \text{ measurable}\} \ .$$

The determination of this group is *cohomology problem of measurable sets.*

7 Two dimensional Gauge fields and a random Harper model

We want to look now at *circle-valued gauge fields* $\mathcal{G} = L^\infty(X, \mathsf{T}^1)$ over a two-dimensional dynamical system (X, T_1, T_2, m), where T_i are two commuting ergodic transformations on a probability space (X, m). Such a system is called *aperiodic*, if for any $(n, m) \in Z^2$

$$m(\{x \in X \mid T_1^{n_1} T_2^{n_2} x = x\}) = 0 \ .$$

A gauge potential $A = (A_1 \tau_1, A_2 \tau_2)$ with $A_i \in L^\infty(X, \mathsf{T}^1)$ is a 1-form and the derivative gives an *electro-magnetic field tensor*

$$F = F_{12} \tau_1 \tau_2 = dA = A_1 A_2(T_1) A_1(T_2)^{-1} A_2^{-1} \ .$$

Zero curvature means $F_{12} = 1$. There is a result of Feldmann-Moore [Fel 77]:

Theorem 7.1 *Let (X, T_1, T_2, m) be an aperiodic Z^2 dynamical system. Given any curvature function $F \tau_1 \tau_2$, there exists a potential $A = (A_1 \tau_1, A_2 \tau_2)$ such that $dA = F$.*

Proof. The original formulation of Feldman and Moore is: for every $F \in L^\infty(X, \mathsf{T}^1)$, (the group operation in T^1 is written multiplicatively), there exists g, h in $L^\infty(X, \mathsf{T}^1)$, such that

$$F = \frac{g(T_1)}{g} \cdot \frac{h(T_2)}{h} \ .$$

Writing $A_1 = g, A_2 = h^{-1}$ gives the claim. $\qquad\qquad\qquad\qquad\qquad \square$

Physically speaking, every electro-magnetic field in two-dimensions has a potential.

Remark. This result in two dimensions is also true for $G = \mathsf{T}^N$ because one can solve the problem in each coordinate of T^n.

The result can be related to the description of electrons in a two dimensional crystal moving in a magnetic field as treated by Landau, Peierls, Bellisart and others (see [Bel 92], [Con 90] p.165). Harper takes the Hamiltonian $H = t(\cos K_1 + \cos K_2)$ with $K = (i\nabla - eA)/h$ which is a simplified model derived from previous work of Peierls. The constant t is the energy an electron needs to go from one site of the crystal to a neighboring one. The operators $U_i = e^{iK_i}$ satisfy

$$U_1 U_2 = e^{2\pi i \alpha} U_2 U_1 \ , \tag{1}$$

where $2\pi\alpha$ is the normalized *magnetic flux* and the operator H can be rewritten as

$$H = t(U_1 + U_1^{-1} + U_2 + U_2^{-1}) \ .$$

Belissard [Bel 92] calls any such operator H given by two unitary operators U_1, U_2 satisfying Eqn. 1 *Harper's model*. A special case is *Hofstadter's case of the Mathieu equation*, where U_i are acting on $L^2(\mathsf{R}/(2\pi\mathsf{Z}))$ by

$$U_1 f = f(x + 2\pi\alpha), \ U_2 f(x) = e^{2\pi i x} f(x) \ .$$

Hofstadter's Hamiltonian can be written as $H(x) = \cos(2\pi i \alpha \frac{d}{dx}) + \cos(x)$.
We take an ergodic aperiodic Z^2 action (X, T_1, T_2, m) and unitary operators $U_1 = a_1 \tau_i$, with $a_i \in L^\infty(X, \mathsf{T}^1)$ such that the curvature F of the Gauge field $A\tau$ satisfies

$$F = dA = a_1 a_2 (T_1) a_1 (T_2)^{-1} a_2^{-1} = e^{2\pi i \alpha} \ .$$

The Hamiltonian to the corresponding Harper Model is the two- dimensional Laplacian

$$L = a_1 \tau_1 + (a_1 \tau_1)^* + a_2 \tau_2 + (a_2 \tau_2)^* \ .$$

Moore and Feldmann's result imply that even if the normalized magnetic flux $2\pi\alpha$ is a measurable function of X, there exists a Gauge potential such that

$$U_1 U_2(x) = e^{2\pi i \alpha(x)} U_2 U_1(x) \ .$$

It would be interesting to say something about the spectrum of the operator L depending on the measurable function α. We call this random Laplacian with space-dependent magnetic flux a *random Harper model* or *random Harper Laplacian.*

8   Questions

We add some questions.

- What is the moduli space of zero curvature fields?

- What is the moduli space of all fields?

- Does the following generalization of the theorem of Feldmann-Moore hold over a $\mathbb{Z}^d$ dynamical system? Given the curvature

$$F = \sum_{ij} F_{ij} \tau_i^* \tau_j^* .$$

Does there exist a gauge potential $A\tau^*$ such that $dA = F$ which is written explicitly as

$$F_{ij} = A_i A_j (T_i) A_j^{-1} A_i (T_j)^{-1} ?$$

If this is not the case, we can say that the second multiplicative de Rham cohomology group in the dynamical system $(L^\infty(X,G), T_1, \ldots, T_d)$ is not trivial.

- Given a $\mathbb{Z}^2$ dynamical system (X, S, T, m). Can one find nontrivial solutions $q \in L^\infty(X)$ to the equation

$$q(S) - 2q + q(S^{-1}) = \log\left(\frac{1 + h^2 \cdot e^{q(T)-q}}{1 + h^2 \cdot e^{q-q(T^{-1})}}\right) ?$$

- Given $\alpha \in L^\infty(X, \mathbb{R}/2\pi\mathbb{Z})$. Can one *construct* measurable functions β_1, β_2 satisfying

$$e^{i\beta_1(x)} e^{i\beta_2(T_1(x))} = e^{i\alpha(x)} e^{i\beta_2(x)} e^{i\beta_1(T_2(x))} .$$

The result of Feldmann and Moore proved only existence.

- Can one say something about the spectrum of the operator L of the generalized Harper model depending on the measurable function α? Does the density of states of L determine the measurable function α?

References

[Bbgp 92] A.Bobenko,M.Bordemann,C.Gunn,U.Pinkall. *On two integrable cellular automata.*
SFB 288 Preprint No.25, Technische Universität Berlin

[Bel 92] J.Bellisard. *Le papillon de Hofstadter.*
Astérisque, 206 Séminaire Bourbaki 44ème année.

[Cap 91] H.Capel,F.Nijhoff, V.Papageorgiou. *Complete integrability of Lagrangian mappings and lattices of KdV type.*
Physics Letters A, 155:377-387, 1991

[Con 90] A.Connes. *Geometrie non commutative.*
InterEditions, Paris 1990

[Fad 86] L.D.Faddeev,L.A.Takhtajan. *Hamiltonian methods in the theory of Solitons.*
Springer, 1986

[Fel 77] J.Feldmann, C.Moore. *Ergodic equivalence relations, cohomology and von Neumann algebras I,II.*
Transactions of the Americal Mathematican Society, 234: 289-359, 1977

[Mac 63] G. W. Mackey. *Ergodic theory, group theory and differential geometry*
Proc. Nat. Acad. Sci. US, 50:1184-1191, 1963

[Qui 91] G.Quispel,H.Capel,V.Papageorgiou,F.Nijhoff. *Integrable mappings derived from soliton equations.*
Physica A, 173:243-266, 1991

[Sur 90] Y.Suris. *Discrete time generalized Toda lattices: complete integrability and relation with relativistic Toda lattices.*
Physics Letters A, 145:113-119, 1990

[Sur 91] Y.Suris. *Generalized Toda chains in discrete time.*
Leningrad Math. J., 2:339-352, 1991

Some additional results for random Toda flows

Contents

Abstract

We give a new short integration of the classical periodic Toda systems using the translation of the Toda flow as a Volterra flow describing the motion of the Titchmarsh-Weyl functions.

We make a remark that the complexified Toda lattice with two particles is equivalent to the complexified mathematical pendulum. This gives a Lax representation for the mathematical pendulum.

We rewrite the first Toda flow as a conservation law in variables F, G, where $G = (L - E)_0^{-1}$ is the Green function for the operator L.

We outline a functional calculus for abelian integrals. This is obtained by looking at an abelian integral on the hyperelliptic Toda curve as a Hamiltonian of a time-dependent Toda flow.

1 Introduction

The set $\mathcal{O}$ of real $N-$ periodic Jacobi operators $L = a\tau + (a\tau)^* + b$ having the same Floquet exponent

$$w(E) = -\text{tr} \log(L - E)$$

and and the same mass

$$M = \int_X \log(a(x)) \, dm(x)$$

forms a finite dimensional torus which can be identified with the *Jacobi variety* of the *hyperelliptic curve* $\mathcal{R}$

$$y^2 = R_{per}(E) = \prod_{n=1}^{2N} (\lambda_n - E) \, ,$$

where λ_n are the eigenvalues of the $2N$ periodic Jacobi matrix. The dimension of the torus is equal to the genus g of the curve $\mathcal{R}$ and $g < N$ counts the number of spectral gaps of the infinite periodic Jacobi operator acting on $l^2(\mathbf{Z})$. The set $\mathcal{O}$ is a group. The group operation

$$L, K \mapsto L \circ K$$

for the operators is calculated as follows: one first assigns to a given Jacobi operator L a *divisor* $\{\mu_i(L)\}$ given by the poles μ_i of $M = m^+ + m^-$, where $m^{\pm}$ are the Titchmarsh-Weyl functions. The invertible Abel-Jacobi map

$$\{\mu_i\}_{i=1}^g \mapsto h_n(L) = \sum_{i=1}^g \int_{\gamma_{2i}}^{\mu_i} \frac{E^{n-1}}{\sqrt{R_{per}(E)}} \, dE, \ n = 1, \ldots, g$$

allows then to perform the group operation on $\mathcal{O}$

$$h(L \circ K) = h(L) + h(K) .$$

An obvious question is whether and how this solution of the inverse spectral problem for periodic Jacobi operators can be generalised for general random Jacobi operators. It is not so clear what is a reasonable definition of the *isospectral set* $\mathcal{O}(L)$ of a random Jacobi operator L over an aperiodic dynamical system T. One possibility is to take the set of random Jacobi operators which have the same density of states and the same mass. It is not excluded that such operators could however have different spectral types. A narrower class of "isospectral operators" would be the set of Jacobi operators which are unitarily conjugated to L.

In infinite dimensions, algebraic geometry will probably have to be replaced by functional analysis. Infinite dimensional integrable systems have been treated already successfully with algebraic geometry (see for example [McK 76]). The classical theory survives in some infinite dimensional cases because the corresponding hyperelliptic curves are not "wild" which means that for example the branch points have no accumulation points. McKean [McK 89] has worked out a program to deal with a wider class of infinite dimensional situations with the aim to deal for example with KdV flows on the space $C^{\infty}(\mathsf{R})$.

In the case of random Jacobi operators however, the hyperelliptic curve can get nasty because the spectrum can be quite arbitrary.
If L is a periodic Jacobi matrix then the hyperelliptic curve can be defined by

$$y^2 = \det(L - E) .$$

For $|X|$ going to infinity, the usual determinant $\det(L - E)$ goes to infinity also and one has to normalize the determinant while performing the thermodynamic limit. In the random case, the trace allows the definition of a determinant

$$\det(L - E) = \exp(\mathrm{tr}(\log(L - E))) = e^{-w(E)} ,$$

where $w(E)$ is the Floquet exponent. In general $e^{-w(E)}$ is transcendental and we have to deal with a transcendental hyperelliptic curve

$$y^2 = e^{-w(E)} ,$$

which has in general accumulations of branch points on the spectrum of L. The spectrum can be very rich. It is in general a Cantor set and the spectral type can vary from absolutely continuous over singular continuous up to dense point spectrum.

There are indications that *signed measures* will play the role of divisors in this infinite dimensional case. We think so, because in the finite dimensional case, the

185

signed spectral measure $d\sigma$ of the function $M = m^+ + m^-$ is supported on a finite set of points on the hyperelliptic curve and the points belonging to poles of m^+ and m^- are lying on different sides of the curve. Of course, this signed measure contains more information then just the divisor. We will see in the next sections that the weight of each point measure gives just the velocity of the motion of this point under the first Toda flow. If we look for an infinite dimensional analogue of the notion of the divisor, the signed spectral measure $d\sigma$ of $m^+ + m^-$ still makes sense and we hope that we will be able to describe once the motion of this measure. We expect also that there is (at least for some examples) an analogue of the Jacobi map which assigns to a divisor a point on an infinite dimensional torus.

We don't deal here yet with such questions. Instead, we will give a relative short construction of the integration of all the periodic Toda flows. If one compares this with the existing linearisations (one can find it in [Mor 76],[Mor 78] and in more detail in the book of Toda [Tod 80] for the first Toda flow), our set-up is technically simpler. The idea is to translate the Toda flows into the motion of Titchmarsh-Weyl functions. This gives easily the motion of the divisors which are constructed from the poles of the Titchmarsh-Weyl functions.

The chapters following this integration are somehow unrelated. But they lead to other loose ends in the investigation of the random Toda flows.

We will first add a remark which seems not have been done so far, namely that the complexified pendulum equation $\ddot{\alpha} = \sin\alpha$ (with complex α) is corresponding to the complexified Toda flow with 2 particles. This gives a Lax representation for the pendulum equation.

We will also add a formula for the motion of the Green function $G(x) = [(L - E)^{-1}(x)]_{00}$ of a random Jacobi operator L under the first Toda flow $\dot{L} = [L^+ - L^-, L]$.

We will further describe a vision about a *functional calculus for abelian integrals*. The idea is to take an abelian integral for the hyperelliptic curve $y^2 = \det(L - E)$ and to interpret it as a time-dependent Hamiltonian for a Toda flow. We hope to get like this an entrance to a Jacobian variety in the space of operators and to find an definition of a Abel-Jacobi map in infinite dimensions. This investigation is still in the beginning but there is an interesting starting point: the functional calculus works quite well for the abelian integral defined by the differential of the third kind

$$-(\frac{d}{dE}w)\,dE\ ,$$

(which has simple poles at ∞ and ∞'). Here, the functional calculus gives the Bäcklund transformations. In finite dimensions, a path on the hyperelliptic curve going from ∞ to ∞' leads to the transformation $L \mapsto L(T)$.

2 Motion of the spectral measure under the first flow

We say, a closed interval I in R is a *spectral gap* if the boundary of I belongs to the spectrum and the interior is disjoint from the spectrum. There are at most countably many spectral gaps for a random operator L and we can index them arbitrarily $(I_n)_{n \in \mathsf{N}}$.

The Titchmarsh-Weyl functions $m^{\pm}$ are meromorphic outside the spectrum of L. There can occur poles in each spectral gap. Define

$$M := m^+ + m^-$$

and

$$G = (m^+ - m^-)^{-1} .$$

Because $m^- - m^+$ is Herglotz, also $-G^{-1}$ is Herglotz as the sum of two Herglotz functions and so G is a Herglotz function.

We can write the functions M, G in the representation

$$M = \int \frac{d\sigma(E')}{E - E'} , G = \int \frac{dg(E')}{E - E'}$$

where dg is a measure and $d\sigma$ is a signed measure. We see from the definitions that a pole of M corresponds to a zero of G.

Lemma 2.1 *In the interior of a spectral gap I_n, the function $M := m^+ + m^-$ has one or no pole.*

Proof. The Green function G is analytic in a spectral gap and the claim follows from the monotonicity of G which is

$$\frac{d}{dE}G > 0 .$$

An existing pole of m^+ or m^- corresponds therefore to a zero of G and must be simple. $\qquad \square$

We call μ_n the (when existing) pole of $M = m^+ + m^-$ in the gap I_n. Denote it with μ_n^+ if it is a pole of m^+ and μ_n^- if it is a pole of m^-. We can now determine the motion of $\mu_n^{\pm}$ under the first Toda flow.

Lemma 2.2 *If μ_n is in the interior of I_n, then*

$$\dot{\mu}_n = -2 \cdot \mathrm{Res}(M, \mu_n) .$$

Furthermore $\dot{\mu}_n^+ > 0$ and $\dot{\mu}_n^- < 0$.

187

Proof. From the formula $mn(T) = a^2$ follows that a pole μ of m is a zero of $n(T)$. We have therefore

$$0 = \frac{d}{dt}n(T) = \frac{d}{dE}n(T)\dot{\mu} + \dot{n}(T)(\mu)$$

and so

$$
\begin{aligned}
\dot{\mu} &= -\frac{\dot{n}(T)(\mu)}{\frac{d}{dE}n(T)} = -2\frac{n(T)(m - m(T^{-1})}{\frac{d}{dE}n(T)}(\mu) \\
&= -2a^2(\frac{d}{dE}n(T))^{-1}(\mu) \, .
\end{aligned}
$$

Because $m^+ \neq m^-$ inside a gap we have $\mathrm{Res}(M,\mu) = \mathrm{Res}(m,\mu)$, where m is either m^+ or m^-. The claim follows from

$$
\begin{aligned}
\mathrm{Res}(M,\mu) &= \mathrm{Res}(m,\mu) = \mathrm{Res}(mn(T)/n(T),\mu) = \mathrm{Res}(a^2/n(T),\mu) \\
&= a^2\mathrm{Res}(\frac{1}{n(T)},\mu) = a^2(\frac{d}{dE}n(T))^{-1} \, ,
\end{aligned}
$$

where we used the fact that μ is a simple zero of $n(T)$. Because m^- and $-m^+$ are Herglotz functions, we have

$$\mathrm{Res}(m^+,\mu) < 0, \ \mathrm{Res}(m^-,\mu) > 0 \, .$$

$\square$

We would like to determine in the general infinite dimensional case the motion of the measure $d\sigma$. In general, there is the problem that we don't know what happens with a point μ if it hits the boundary of the gap and what happens with the non-atomic rest of σ.

However, if $|X|$ is finite, the measure $d\sigma$ is a point measure. Denote in this case dk_{per} the spectral measure of L acting on the finite dimensional space of $2|X|$ periodic sequences, The two measures $d\sigma$ and dk_{per} allow a reconstruction of L, because we can determine both M and G and so both m^+ and m^-. In the next section we want to see how $d\sigma$ moves under the first Toda flow.

3 Integration of the periodic lattice for the first flow

We assume in this paragraph that $|X| = N$ finite. In this case the random Toda lattice reduces to the periodic Toda lattice and $L(x)$ is an infinite trigonal matrix which is $N-$ periodic. Our aim is a self-contained integration of all the Toda flows.

We begin with the first Toda flow and find explicit formulas for $G^{-1} = m^+ - m^-$, $M = m^+ + m^-$. It is known that $L(x)$ has *band spectrum*

$$\sigma(L(x)) = \bigcup_{i=1}^{g+1} [\lambda_{2i-1}, \lambda_{2i}] .$$

and that there exists $g \le N - 1$ gaps in the spectrum. Define

$$R(E) = \prod_{i=1}^{2g+2} (E - \lambda_i), \ l(E) = \prod_{i=1}^{g} (E - \mu_i) ,$$

where μ_i denote as before the zeros of G.

Lemma 3.1 *a) For $n = 1, \ldots g$,*

$$\mathrm{Res}(M, \mu_n^\pm) = \mathrm{Res}(m^\pm, \mu_n) = \frac{\pm\sqrt{R(\mu_n)}}{l'(\mu_n)} .$$

$$G^{-1} = m^+ - m^- = \frac{\sqrt{R}}{l} .$$

b)

$$d\sigma = \sum_{i=1}^{g} \mathrm{Res}(M, \mu_i)\delta(\mu_i) = \sum_{i=1}^{g} \frac{\pm\sqrt{R(\mu_n)}}{l'(\mu_n)}\delta(\mu_i) .$$

Proof.
a) From the *discrete Ricatti equation* $m(T) = E - b(T) - a^2/m$ and $m(T^N x) = m(x)$ we get a continued fraction expansion

$$m = m(T^N) = E - b(T^N) - \cfrac{a^2(T^{N-1})}{E - b(T^{N-1}) - \cfrac{a^2(T^{N-2})}{\cdots \cfrac{\cdots}{-b(T^2) - \cfrac{a^2(T)}{E - b(T) - \cfrac{a^2}{m}}}}} .$$

It follows that the functions $m^\pm$ satisfy a quadratic equation. We can write the solutions as

$$m^\pm = \frac{A \pm \sqrt{B}}{C} ,$$

where A, B, C are polynomials in E. Therefore

$$G^{-1} = m^+ - m^- = 2\frac{\sqrt{B}}{C} ,$$
$$M = m^+ + m^- = 2\frac{A}{C} .$$

189

Because $m^+ - m^-$ has zeros in λ_i and simple poles at the places μ_i, we get that R divides B and l divides C. We know

$$G^{-1} = m^+ - m^- = O(E), \ E \to \infty$$

and that there are maximal g zeros of l. One has at least $2g + 2$ zeros of C because each of the λ_i is a zero of C. Maximally $2g + 2$ zeros of C are possible because else the growth rate of G^{-1} at infinity would be too big. This implies $B = c_1 \cdot R$ and $C = c_2 \cdot l$ for two constants c_1, c_2. From the growth rate of $G^{-1} = 2\sqrt{B}/C$ at infinity we get $B = 4R$, $C = l$. Since

$$m(E) = O(E), \ E \to \infty \ ,$$

we must have exactly g zeros of G which are poles of M if μ is in the interior of a gap.

b) From

$$G^{-1} = \frac{\sqrt{R(E)}}{l(E)}$$

we deduce

$$\mathrm{Res}(M, \mu) = \frac{\sqrt{R(\mu)}}{l'(\mu)} \ .$$

$\square$

The next lemma shows that the integral of the measure $d\sigma$ with respect to the time of the first Toda flow is absolutely continuous with respect to Lebesgue measure. This follows from the fact that the absolute velocity of a point in the support of $d\sigma$ is twice the weight of the point.

Lemma 3.2 *There exists $h(t) \in L^\infty(I, 2\mathbb{N})$ such that*

$$d\rho(t) := \int_0^t d\sigma(s) \, ds = h \, dE \ .$$

Proof. Assume $\mu_i(0)$ is in the interior of a spectral gap. For small enough time t_0, the point $\mu_i(t_0)$ is still in the gap and has not yet reached the boundary of the gap. Because velocity of the point μ is equal to the weight of the measure

$$-2 \cdot \mathrm{Res}(M, \mu)\delta(\mu) \ ,$$

we get

$$\int_0^{t_0} -2 \cdot \mathrm{Res}(M, \mu(s))\delta(\mu(s)) \, ds = -2 \cdot 1_{[\mu(0), \mu(t_0)]} \, dE \ .$$

If the point μ reaches the boundary of the gap, the weight of the point measure changes sign and moves to the other side. Because also the sign of $R(\mu)$ changes,

the sign of $d\sigma(t)$ does not change. The function $h(E,t)/2$ is equal to the number of times μ has passed the point E in the time-interval $[0,t]$. □

Define for $n = 0, \ldots, g-1$ the functions

$$f_n = \frac{E^n}{\sqrt{R}} \cdot$$

They are in $L^1(I)$ and form a basis in the space of *differentials of the first kind* on the *hyperelliptic curve* given by $y = \sqrt{R}$. Alternatively we could also take the basis

$$\tilde{f}_n = \frac{w'(E)}{E - \rho_n},$$

where ρ_n is the unique zero of w' in the $n'th$ gap. All the functions $f_n, \tilde{f}_n$ are in $L^1(\mathbb{R}, dE)$. We conjecture that this stays true in the general random case.

Lemma 3.3 *The evolution of the first Toda flow can be written as*

$$\begin{aligned}
< f_n, h(t) > \; &= \; \int f_n h(t) \, dE \\
&= \; \begin{cases} < f_n, h(0) > +2t & \text{for } n = g-1 \\ < f_n, h(0) > & \text{else.} \end{cases}
\end{aligned}$$

Proof.

$$\begin{aligned}
\frac{d}{dt} < f_n, h > \; &= \; \int_I f_n \, d\sigma = \sum_{i=1}^g \frac{\mu_i^n}{\sqrt{R(\mu_i)}} \dot{\mu}_i \\
&= \; \sum_{i=1}^g \frac{2\mu_i^n}{l'(\mu_i)} = \frac{2}{2\pi i} \int_\Gamma \frac{E^n}{l(E)} \, dE \\
&= \; \begin{cases} 2 & \text{for } n = g-1 \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

where Γ is a curve in the complex plane which goes once around the spectrum of L. □

Lemma 3.4 (Jacobi) *The mapping*

$$\{\mu_n\}_{n=1}^g \mapsto < f_n, h(t) >_{n=1}^g$$

is invertible.

Proof. In the usual language this means that the *Abel-Jacobi map*

$$\{\mu_i\} \mapsto \sum_{i=1}^g \int_{\gamma_{2i}}^{\mu_i} \frac{E^n}{\sqrt{R(E)}} \, dE$$

is invertible. For a proof see [Sie 69] Theorem 1, p.56. □

The density of states can be written as $\int_X dg(x) = dk$. The knowledge of dk determines the points λ_n. From λ_n and μ_n the operator L can be reconstructed [Mor 76]. To summarize: the sequence of coordinate transformations

$$L \mapsto (M, G) \mapsto (d\sigma, dk) \mapsto (d\rho, dk) \mapsto (h, dk) \mapsto (\{< f_n, h >\}, dk)$$

straightens the flow of the periodic Toda lattice. From the vector

$$(< f_1, h >, < f_1, h >, \ldots, < f_g, h >)$$

and the density of states dk, the operator L can be reconstructed.

We hope that in some infinite dimensional cases there are also infinitely many functions $f_n \in L^1(\mathbb{R}, dE)$ and a function $h(t) \in L^\infty(\mathbb{R}, dE)$ such that

$$\frac{d}{dt} < f_n, h(t) > = \beta_n$$

is constant and that the sequence $< f_n, h(t) >$ together with the density of states dk allows a reconstruction of L.

4 Integration of the higher periodic Toda flows

We determine now the motion of the spectral measure of the Titchmarsh-Weyl functions m, n under the *higher Toda flows*

$$\dot{L} = [(L^n)^+ - (L^n)^-, L] = [B, L] .$$

We will see that there is a meromorphic function $\tilde{m}$ with the same poles then m such that the poles of m are moving with a velocity given by the residuum of $\tilde{m}$. We assume again $|X| = N$. Denote with g the number of gaps in the spectrum of L.

We take as in the integration of the first flow for every $x \in X$ two solutions u^+, u^- of the equation

$$Lu = Eu .$$

Consider as before the old Titchmarsh-Weyl functions

$$m^\pm = \frac{L^+ u^\pm}{u^\pm}, \quad n^\pm = \frac{L^- u^\pm}{u^\pm}$$

an define the *higher Titchmarsh-Weyl functions*

$$\tilde{m}^\pm = \frac{(L^n)^+ u^\pm}{u^\pm}, \quad \tilde{n}^\pm = \frac{(L^n)^- u^\pm}{u^\pm} .$$

The motion of the old Titchmarsh-Weyl functions m, n under the higher Toda flow can be determined.

Lemma 4.1

$$\dot{m} = 2m(\tilde{n} - \tilde{n}(T)) \,,$$
$$\dot{n} = 2n(\tilde{m}(T^{-1}) - \tilde{m}) \,.$$

Proof. From $Lu = Eu$ follows $L^n u = E^n u$ and so

$$m + n = E - b, \quad \tilde{m} + \tilde{n} = E^n - (L^n)_0 \,.$$

If we differentiate the equation $L^n u = Eu$ with respect to the motion of the higher Toda flow

$$\dot{L} = [B, L] = [(L^n)^+ - (L^n)^-, L]$$

we get

$$BL^n u - L^n Bu + L\dot{u} = E\dot{u}$$

or $(E - L^n)Bu = (E - L^n)\dot{u}$. Because $E - L^n$ is invertible by assumption, we obtain

$$\dot{u} = Bu \,.$$

We calculate

$$\frac{d}{dt}\log(u) = \frac{Bu}{u} = \frac{(L^n)^+ u}{u} - \frac{(L^n)^- u}{u} = E - (L^n)_0 - 2n \,,$$

$$\frac{d}{dt}\log(u(T)) = \frac{(L^n)^+ u(T)}{u(T)} - \frac{(L^n)^- u(T)}{u(T)} = E - (L^n)_0(T) - 2n(T) \,,$$

$$\frac{d}{dt}\log(a) = (L^n)_0(T) - (L^n)_0$$

and get

$$\frac{d}{dt}\log(m) = \frac{d}{dt}\log(a) + \frac{d}{dt}\log(u(T)) - \frac{d}{dt}\log(u)$$
$$= 2\tilde{n} - 2\tilde{n}(T) \,.$$

In the same way

$$\frac{d}{dt}\log(u) = \frac{(L^n)^+ u}{u} - \frac{(L^n)^- u}{u} = 2m - E + L_0^n \,,$$

$$\frac{d}{dt}\log(u(T^{-1})) = \frac{(L^n)^+ u(T^{-1})}{u(T^{-1})} - \frac{(L^n)^- u(T^{-1})}{u(T^{-1})}$$
$$= 2m(T^{-1}) - E + (L^n)_0(T) \,,$$

$$\frac{d}{dt}\log(a(T^{-1})) = (L^n)_0 - (L^n)_0(T^{-1})$$

and get

$$\frac{d}{dt}\log(n) = \frac{d}{dt}\log(a(T^{-1})) + \frac{d}{dt}\log(u(T^{-1})) - \frac{d}{dt}\log(u) = 2\tilde{m}(T^{-1}) - 2\tilde{m} \,.$$

□

As in the case of the first Toda flow we want to know the motion of the μ spectrum. Call μ_n the poles of m, n. They are also the poles of $\tilde{m}$ because both $m, \tilde{m}$ have the same denominator u.

Lemma 4.2 *In the interior of a gap, the motion of a divisor μ is*

$$\dot{\mu} = -2 \cdot \text{Res}(\tilde{m}, \mu) \ .$$

From $mn(T) = a^2$ follows that the poles of m are the zeros of $n(T)$. We get

$$0 = \frac{d}{dt} n(\mu) = \frac{d}{dE} n(\mu)\dot{\mu} + \frac{\partial}{\partial t} n(\mu) \ ,$$

and so

$$
\begin{aligned}
\dot{\mu} &= -\frac{\dot{n}(T)(\mu)}{\frac{d}{dE} n(T, \mu)} = -\frac{2n(T)(\tilde{m} - \tilde{m}(T))}{\frac{d}{dE} n(T, \mu)} \\
&= -2\text{Res}(\frac{n(T)(\tilde{m} - \tilde{m}(T))}{n(T, \mu)}, \mu) \\
&= -2\text{Res}(\tilde{m} - \tilde{m}(T), \mu) = -2\text{Res}(\tilde{m}, \mu) \ .
\end{aligned}
$$

In order to find an explicit integration like in the first case, we need to know $\text{Res}(\tilde{m}, \mu)$ explicitly. The strategy is like before: one has to show that the residuum is of the form

$$\text{Res}(\tilde{m}, \mu) = \sum_k \alpha_k \mu^k \frac{\sqrt{R(\mu)}}{l'(\mu)}$$

which leads to linear flow on the Jacobi variety of L like in the first case. We show now how one has to proceed. We have

$$\tilde{m} = \sum_{k=1}^{n} \alpha_k m(T^{k-1}) \dots m(T) m \ ,$$

where α_k are measurable functions independent of E. Because each m can be written as

$$m^{\pm} = \frac{A \pm \sqrt{B}}{C} \ , \qquad \circ$$

where A, B, C are polynomials in E having $x-$ dependent coefficients, we obtain also

$$\tilde{m^{\pm}} = \frac{\tilde{A} \pm \sqrt{\tilde{B}}}{\tilde{C}}$$

194

with $x-$ dependent polynomials $\tilde{A}, \tilde{B}, \tilde{C}$. Therefore

$$\tilde{G}^{-1} = \tilde{m}^+ - \tilde{m}^- \;=\; 2\frac{\sqrt{\tilde{B}}}{\tilde{C}}\,,$$

$$\tilde{M} = \tilde{m}^+ + \tilde{m}^- \;=\; 2\frac{\tilde{A}}{\tilde{C}}\,.$$

Because $m^+(E)$ and $m^-(E)$ coincide precisely at the points $E = \lambda_i$, also $\tilde{m}^+(E) = \tilde{m}^-(E)$ for $E = \lambda_i$ and $\tilde{C}$ is a constant times R. Because $\tilde{m}^\pm$ can only have poles at μ_i, it follows that there exists a polynomial

$$P(E) = \sum_{k=0}^{n-1} \alpha_k E^k$$

such that

$$\tilde{G}^{-1} = P(E) \cdot \frac{\sqrt{R(E)}}{l(E)}\,.$$

Therefore

$$\text{Res}(M,\mu) = \sum_{k=1}^{n-1} \alpha_k \mu^k \frac{\sqrt{R(\mu)}}{l'(\mu)}$$

and we obtain

$$\frac{d}{dt} < f_n, h > = \beta_k$$

with time-independent β_k. It turns out that the β_k are also independent of $x \in X$ because the Toda flows are commuting with space translation $x \mapsto T(x)$.

Like for the first Toda flow, the invertible map

$$L(t) \mapsto < f, h(t) > = < f, h(0) > + t \cdot \beta$$

straightens the higher Toda flows.

5   The first periodic Toda lattice with 2 particles

Usually, the study of the Toda lattice begins with the simplest nontrivial case when there are three particles. In the book of Toda [Tod 80], this case is treated in full generality. We didn't find in the literature the even simpler situation of 2 particles. We will just see that the complexified Toda flow for two particles and the complexified pendulum equation coincide.

The periodic Toda lattice with 2 particles is given by

$$\ddot{q}_1 \;=\; e^{q_2-q_1} - e^{q_1-q_2}\,,$$
$$\ddot{q}_2 \;=\; e^{q_1-q_2} - e^{q_2-q_1}\,.$$

If we introduce the variables $w = q_1 - q_2$ and $z = q_1 + q_2$, the system goes into

$$\ddot{z} = 0 \,,$$
$$\ddot{w} = 2e^{-w} - 2e^{w} = -4 \cdot \sinh w \,.$$

The second equation is an equation which corresponds to the real Toda lattice. It becomes with $w = i\alpha + i\pi$

$$\ddot{\alpha} = 4\sin(\alpha)$$

the *pendulum equation*. We define $\beta = iz$. The complexified periodic Toda lattice with two particles is thus equivalent to the complexified physical pendulum! It follows that the pendulum can be written as an isospectral deformation of the periodic 4×4 matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & a_2 \\ a_1 & b_2 & a_2 & 0 \\ 0 & a_2 & b_1 & a_1 \\ a_2 & 0 & a_1 & b_2 \end{pmatrix}$$

with

$$\begin{aligned}
a_1 &= e^{(q_2-q_1)/2} = e^{w/2} = i \cdot e^{i\alpha/2} \,, \\
a_2 &= e^{(q_1-q_2)/2} = e^{-w/2} = -i \cdot e^{-i\alpha/2} \,, \\
b_1 &= \dot{q}_1/2 = \frac{\dot{z}+\dot{w}}{2} = i \cdot \frac{\dot{\beta}+\dot{\alpha}}{2} \,, \\
b_2 &= \dot{q}_2/2 = \frac{\dot{z}-\dot{w}}{2} = i \cdot \frac{\dot{\beta}-\dot{\alpha}}{2} \,.
\end{aligned}$$

If we want to get rid of the uninteresting linear motion of z, we can restrict us to the invariant set $\dot{z} = 0$ which corresponds to

$$\int b \, dm = \frac{b_1 + b_2}{4} = 0 \,.$$

The pendulum motion

$$\ddot{\alpha} = 4\sin(\alpha)$$

goes then into the isospectral deformation

$$\dot{L} = [L^+ - L^-, L]$$

of the matrix

$$L = i \cdot \begin{pmatrix} \dot{\alpha}/2 & e^{i\alpha/2} & 0 & -e^{-i\alpha/2} \\ e^{i\alpha/2} & \dot{\alpha}/2 & -e^{-i\alpha/2} & 0 \\ 0 & -e^{-i\alpha/2} & \dot{\alpha}/2 & e^{i\alpha/2} \\ -e^{-i\alpha/2} & 0 & e^{i\alpha/2} & \dot{\alpha}/2 \end{pmatrix} \,.$$

The spectrum of L is

$$
\begin{aligned}
-\lambda_1 &= \lambda_4 = \sqrt{2 + H/2}\,, \\
-\lambda_2 &= \lambda_3 = \sqrt{-2 + H/2}\,,
\end{aligned}
$$

where

$$
H = -2 \cdot \operatorname{tr}(L^2) = \frac{\dot{\alpha}^2}{2} + 4\cos(\alpha)
$$

is the energy of the pendulum.

We describe the picture qualitatively. We call the spectrum of the periodic Jacobi operator acting on $l^2(\mathbb{Z})$ the *band spectrum* of the pendulum. The Titchmarsh-Weyl function has only one pole having the location $\dot{\alpha}$. It is lying in the band spectrum and the motion of this pole determines the motion of the pendulum. The *energy surface* is a one-dimensional circle or consists of two one-dimensional circles depending on the energy. The picture is as follows. The real band spectrum is the set of allowed velocities β. It consists of two intervals if the energy is larger then 4 and each interval corresponds to a pendulum which is rotating only in one direction. For the energy $H = 4$, the band spectrum is the interval $[-2, 2]$ and this situation corresponds to the *homoclinic situation*. For smaller energies $H \in (-4, 4)$, the band spectrum looks like a cross and is the union of an interval on the real axis and an interval on the imaginary axis. For $H = -4$, the spectrum is only imaginary. This is the situation when the pendulum is on the bottom and the real energy surface is only a point.

The integrals of motion of the Toda lattice are related to the integrals of motion of the pendulum: The *momentum* of the Toda lattice $\operatorname{tr}(L) = \int b\,dm$ and the *mass* of the Toda lattice $\int \log(a)\,dm$ are always vanishing and have no meaning for the pendulum.

6 Motion of the Green function G under the first flow

We have seen how the first Toda flow can be written in terms of Titchmarsh-Weyl functions m, n as

$$
\begin{aligned}
\dot{m} &= 2m(n - n(T))\,, \\
\dot{n} &= 2n(m(T^{-1}) - m)\,.
\end{aligned}
$$

We describe now the Toda flow in the coordinates

$$
G = \frac{1}{m^+ - m^-}, \quad F = 2^{-1}\log\frac{m^+}{m^-}\,.
$$

Proposition 6.1 *The random Toda flow looks in the* F, G *coordinates as*

$$\dot{F} = \frac{2}{G(T)} - \frac{2}{G},$$
$$\dot{G} = \coth(F) - \coth(F(T^{-1})).$$

Proof.

$$\dot{F} = \frac{d}{dt}(\log(m^+) - \log(m^-)) = 2(n^+ - n^+(T) - n^- + n^-(T)),$$
$$= 2(m^+(T) - m^-(T)) - 2(m^+ - m^-) = \frac{2}{G(T)} - \frac{2}{G}$$

and

$$\dot{G} = -\frac{\dot{m}^+ - \dot{m}^-}{(m^+ - m^-)^2} = -2\frac{m^+(n^+ - n^+(T)) - m^-(n^- - n^-(T))}{(m^+ - m^-)^2}$$
$$= 2\frac{m^- n^- - m^+ n^+}{m^+ - m^-)^2} = 2\frac{m^- n^- - m^+ n^+}{(n^- - n^+)(m^+ - m^-)}$$
$$= 2\frac{m^-/n^+ - m^+/n^-}{(1/n^+ - 1/n^-)(m^+ - m^-)} = 2\frac{m^- m^+(T^{-1}) - m^+ m^-(T^{-1})}{(m^+(T^{-1}) - m^-(T^{-1}))(m^+ - m^-)}$$
$$= \frac{m^+ + m^-}{m^+ - m^-} - \frac{m^+(T^{-1}) + m^-(T^{-1})}{m^+(T^{-1}) - m^-(T^{-1})} = \coth(F) - \coth(F(T^{-1})).$$

$\square$

Remark. There is a similar formula for a derivative of the Green function of a Schrödinger operator

$$L = -\frac{d^2}{dx^2} + q(q),$$

where $q \in C(\mathsf{R})$ (see [Cra 89]). If one looks at the path $L(x) \mapsto L(x+t)$ the motion of the Green function G can be expressed by the Weyl m-function also

$$\dot{G} = -\frac{m^+ + m^-}{m^+ - m^-}$$

which is called *Dubrovin equation.*

Remark. The equations of motion for G, F are *discrete conservation laws.* This is not surprising because the Toda equations themselves can be written as conservation laws. But there is a fundamental difference for the conservation laws of the Toda flows and the equations for F and G because the later (like the Volterra flows also) are true for each energy $E \notin \sigma(L)$. If we write F, G, m, n as power series

$$F = \sum_n F_n E^n, G = \sum_n G_n E^n,$$
$$m = \sum_n m_n E^n, n = \sum_n n_n E^n,$$

198

then $\int_X F_n \, dm$, $\int_X G_n \, dm$, $\int_X m_n \, dm$ and $\int_X n_n \, dm$ are integrals of motion. We can also choose an arbitrary path γ in the resolvent set of L. Then,

$$\int_X \int_\gamma F \, dE \, dm \, , \int_X \int_\gamma G \, dE \, dm$$

are integrals of motion.

7   A functional calculus for Abelian integrals

Given a Riemann surface

$$\mathcal{R} : y^2 = R(E)$$

and an *abelian differential*

$$f(E, y) \, dE$$

which can be *of the first kind* (regular on the whole surface) *of the second kind* (having at least one pole and zero residue at each pole) or *of the third kind* (having at least one pole with non-vanishing residue). Fix a point E_0 on the surface $\mathcal{R}$ and take a curve γ on the surface which connects E_0 with E (avoiding poles). The integral

$$F_\gamma(E) = \int_\gamma f(E', y) \, dE'$$

is called an *abelian integral*. If the curve γ is closed then the value

$$F(\gamma) = \int_\gamma f(E, y(E)) \, dE$$

is called a *period* of the integral. In the case when the genus g of $\mathcal{R}$ is finite, the vector space of abelian integrals of the first kind is g dimensional and there are $2g$ paths γ_k such that every period can be expressed in terms of $2g$ fundamental periods

$$F(\gamma) = \sum_{k=1}^{2g} \alpha_k F(\gamma_k) \, .$$

On the universal cover $\tilde{\mathcal{R}}$ of $\mathcal{R}$, the abelian integral $F(z)$ is a uniquely valued analytic function if the differential is of the first kind.

We consider now the *transcendental Riemann surface*

$$\mathcal{R} : y^2 = R(E) = \det(L - E) = e^{-w(E)} \, ,$$

where L is a fixed N-periodic Jacobi matrix acting on $l^2(\mathbb{Z})$ and det is the normalized determinant defined by the Floquet exponent w.
This Riemann surface is topologically the same as the Riemann surface $\mathcal{R}_{per}$

$$y^2 = R_{per}(E) = \det(L_{per} - E) = \prod_{n=1}^{2N+2} (\lambda_n - E) \, ,$$

where L_{per} is a finite dimensional $2N \times 2N$ matrix and det is the usual determinant. Call $g < N$ the genus of $\mathcal{R}$. The surface $\mathcal{R}$ does only depend on the isospectral class of L because an isospectral deformation of L does not change the value of y.

The abelian differential

$$w'(E)\, dE$$

is of the third kind because we can write it as $w'(E)dE = dw(E)$ for $E \neq \infty$. Because the residues at the simple poles ∞ and ∞' are 1 and -1, the differential is *the elementary differential of the third kind* with poles at ∞ and ∞'. We know from the explicit formula for $m^+ - m^-$ that

$$w'(E) = \int_X (m^+ - m^-)^{-1} = \frac{f(E)}{R_{per}(E)} \, ,$$

where $f(E)$ is some polynomial of order g.

Take the point E_0 near $-\infty$. The abelian integral obtained by integrating the differential $-\frac{1}{2}w'(E)dE$ along a curve from E_0 to a point E is

$$-\frac{1}{2}w(E) - C(E_0) = \frac{1}{2}\text{tr}(\log(L - E)) - C(E_0)$$

with the constant $C(E_0) = -\frac{1}{2}w(E_0)$.

We interpret now this function as a function of L parametrised by E. We call it $F_E(L)$. The main point for the following is that it can also be viewed as the *Hamiltonian* of the *time-dependent Toda flow*

$$\begin{aligned}
\frac{d}{dE}L(E) &= \frac{1}{2}[\nabla F_E(L)^+ - \nabla F_E(L)^-, L] \\
&= \frac{1}{2}[(\frac{1}{L(E) - E})^+ - (\frac{1}{L(E) - E})^-, L(E)] \, ,
\end{aligned}$$

where $\nabla F_E(L)$ is the gradient of the function $F_E(L)$ (as a function of L). Starting at $E = \infty$, we obtain a family of Jacobi operators $U(E)^* L U(E) = L(E)$ parametrised by the *universal covering* $\tilde{\mathcal{R}}$ of the Riemann surface and we have a field of unitary operators $U(E)$ defined on $\tilde{\mathcal{R}}$.

Unlike the abelian integral which has a logarithmic singularity at ∞ and ∞', the Toda flow is well defined and can be continued through ∞ and ∞'. The reason is the surprising fact that the operators $L(E)$ can be calculated explicitly by

$$L(E) = BT^{\pm}(E) \, .$$

(We will show this in the chapter "Renormalisation of Jacobi operators".) The sign depends on which side of the Riemann surface we are. There is a *positive side*

$y > 0$ and a *negative side* $y < 0$ and the *spectrum* $y = 0$, where the two sides of the surface are glued together.) We call this isospectral deformation $E \mapsto L(E)$ *functional calculus for the abelian integral.*

Remark. For the KdV equation, there seems to exist a similar interpolation of the Bäcklund transformations by a time-dependent KdV flow

$$\frac{d}{dE} L = -2DG_x x(E, L)$$

where $G(E, L) = (L - E)^{-1}$ is the Green operator corresponding to $L = -D^2 + q$. (See [McK 89] p.31. where the motivation is to replace KdV flows (which do not exist for example in $C^\infty(\mathbb{R})$) by Bäcklund transformations.)

Starting with the operator $L = L(\infty)$ we can integrate along a curve γ connecting ∞ with ∞' and get the translated operator $L(\infty') = L(T)$.
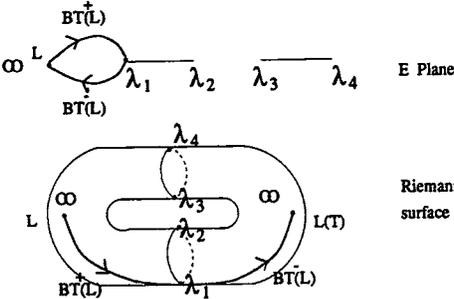
The Toda flow with Floquet exponent as Hamiltonian generates the Backlund transformations. Example of a functional calculus..

The Bäcklund transformed operator $L' = BT_E^\pm(L)$ can explicitely be given by

$$b' = b + n^\pm - n^\pm(T), \quad a'^2 = a^2 \frac{m^\pm(T)}{m^\pm} ,$$

where $m^\pm, n^\pm$ are the Titchmarsh-Weyl functions.
(The Titchmarsh-Weyl functions $m^\pm$ can be viewed as one function m defined on the Riemann surface $\mathcal{R}$. If L is real selfadjoint, the poles of m are located in the gaps of the spectrum.)

In principle, the Bäcklund transformation are defined for E on the whole Riemann surface $\mathcal{R}$. They give isospectral Jacobi matrices. Because they are not normal

201

any more in general for values of E not lying in the interval $[-\infty, \lambda_1]$, the norm of the transformed operators can blow up. Indeed, if E is at one of the poles or zeros of m, the Bäcklund transformed $BT(L)$ is infinite. We will not consider here this problem and view these unbounded operators as points in the boundary of the isospectral set of *complex Jacobi operators*. The unbounded operators could be added as compactification points of the complex Jacobi variety.

The explicit formula for the flow $E \mapsto L(E)$ means that we can in some sense regularize the flow through these points. This construction works also when $\mathcal{R}$ has infinite genus and is given by the hyperelliptic curve

$$y^2 = \det(L - E)$$

defined for a random Jacobi operator L over an ergodic dynamical system (X, T, m). There is however one difficulty: the abelian differential needs no longer to be regular at the bottom λ_1 of the spectrum and the flow is discontinuous there. Indeed, in general, the Titchmarsh-Weyl functions m^+ and m^- are not coinciding at λ_1.

In general, when the dynamical system (X, T, m) is not ergodic, we denote by (X, T, m_x) an ergodic fiber of (X, T, m), where m_x is defined by averaging the Dirac measure δ_x. The measure m_x can be constructed as a weak accumulation point of $\frac{1}{2n+1} \delta(T^n x)$.

We define a *random Riemann surface* $x \mapsto \mathcal{R}(x)$ as a map which assigns to each point x of the probability space (X, m) a Riemann surface

$$\mathcal{R}(x) : y^2 = R(x, E) \, ,$$

where we require that for fixed E the map $x \mapsto R(x, E)$ is measurable. The *universal cover* $\tilde{\mathcal{R}}$ of the random Riemann surface is obtained by assigning to each $x \in X$ the universal cover $\tilde{\mathcal{R}}(x)$ of $\mathcal{R}(x)$.

Fixing one random Jacobi operator L, the Hamiltonian $-\frac{1}{2}w = \log(y)$ generates a field of Jacobi matrices $L(E)$ defined on the random Riemann surface

$$\mathcal{R}(x) : y^2 = \det_x(L - E)$$

where $\det_x$ denotes the normalized determinant belonging to the measure m_x.

Moving on the Riemann surface $\mathcal{R}$ along a path $\gamma : t \mapsto E(t)$ corresponds to an isospectral deformation of the Jacobi matrix. We have $L(x) = L(\infty, x)$, $L(Tx) = L(\infty, Tx) = L(\infty', x)$.

Moving around the Riemann surface from ∞ to ∞' passing the lowest branch point λ_1 and continuing back to ∞ passing the highest branch point λ_{2N+2} is a closed

202

curve and the Riemann surface. The corresponding effect of this round on the operator is the identity.

It would be interesting to know what happens with other abelian differentials and with other closed curves while doing the functional calculus.

8 Questions

Some questions.

• In the integration of the periodic finite Toda lattice, we would like to calculate explicitly the velocity vector β on the Jacobian. We have also not shown that there are g linearly independent flows.

• How does the differential equation for the Green function for the higher Toda flows look like?

• Can we write the solutions of the random Toda flows in terms of generalized Theta functions? Can we find a space of differentials of the first kind and an element $g(t)$ in the dual space such that

$$L(t) \mapsto < f_i, h(t) >$$

straightens the Toda flow also in the infinite dimensional case? We could try with

$$\tilde{f}_i(E) = w'(E)/(E - \rho_i) ,$$

where ρ_i are the zeros of w' in the gaps of the spectrum and to take

$$h(t) \, dE = \int_0^t d\sigma(t) ,$$

where $\sigma(t)$ is the spectral measure of the function $M = m^+ + m^-$. If $\tilde{f}_i \in L^1(\mathbb{R}, dE)$ and $h(t) \in L^\infty(\mathbb{R}, dE)$, we could hope to get a linearisation

$$L(t) \mapsto < \tilde{f}_n, h(t) > = < \tilde{f}_n, h(0) > + t \cdot \beta_n$$

having also that the sequence $< \tilde{f}, h >$ allows the reconstruction of L.

• What gives the functional calculus for abelian integrals if we take the differentials

$$\tilde{f}_i(E) \, dE = E \frac{w'(E)}{(E - \rho_i)} \, dE \ ?$$

• We have seen that also one of the most simple integrable dynamical systems can be written as a Lax system. This supports the conjecture that every integrable

nonlinear system can be written as an isospectral deformation of some operator.

• Is there a Thouless formula for general Jacobi operators on the strip and an integration of the non-abelian periodic Toda flow?

• It would be interesting to investigate more the complex isospectral set of Jacobi operators corresponding to the pendulum. What do Bäcklund transformation do globally?

References

[Cra 89] W.Craig. The Trace Formula for Schroedinger Operators on the Line. Comm. Math. Phys., 126:379-407, 1989

[Deif2 80] P.Deift,L.C.Li,C.Tomei. *Toda flows with infinitly many variables.* J. of Funct. Analysis, 64:358-402, 1985

[McK 78] H.P.McKean. *Integrable systems and algebraic curves.* Lecture Notes in Math., No. 755, Springer 1978

[McK 76] H.P.McKean, E.Trubowitz. *Hill's Operator and Hyperelliptic Function Theory in the Presence of Infinitely many Branch Points.* Comm. Pure Appl. Math., XXIX, 143-226, 1976

[McK 89] H.P.McKean. *Is there an infinite dimensional algebraic geometry? Hints from KdV.* Proc. of Symposia in Pure Mathematics, 49:27-38, 1989

[Mor 76] P.v.Moerbeke. *The spectrum of Jacobi Matrices.* Invent. math., 37:45-81, 1976

[Mor 78] P.v.Moerbeke. *About isospectral deformations of discrete Laplacians.* Lecture Notes in Mathematics, No. 755, Springer 1978

[Rue 85] D.Ruelle. *Rotation numbers for diffeomorphisms and flows.* Ann. de l'Inst. Henri Poincare, Physique theoretique, 42:109-115, 1985

[Sie 69] C.L.Siegel. *Topics in complex function theory Vol 2.* Wiley-Interscience, New York, 1969

[Sim 88] S.Kotani, B.Simon. *Stochastic Schroedinger Operators and Jacobi Matrices on the Strip.* Comm. Math. Phys, 119:403-429, 1988

[Tod 80] H.Toda. *Theory of nonlinear lattices .* Springer. 1981

[Ven 89] S. Venakides. *The Continuum Limit of Theta Functions.* Comm. Pure Appl. Math., 42:711-728, 1989

Renormalisation of Jacobi matrices: Limit periodic operators having the spectrum on Julia sets

Contents

Abstract

We construct a Cantor set $\mathcal{J}_E$ of limit periodic Jacobi operators having the spectrum on the Julia set J_E of the quadratic map $z \mapsto z^2 + E$ for large negative real numbers E. The density of states of each of these operators is equal to the unique equilibrium measure μ_E on J_E.

The Jacobi operators in $\mathcal{J}_E$ are defined over the von Neumann-Kakutani system, where the later is a group translation on the compact topological group of dyadic integers. The Cantor set $\mathcal{J}_E$ is an attractor of the iterated function system built up by the two renormalisation maps $\Phi_E^{\pm} : L = (D^{(\pm)})^2 + E \mapsto D^{(\pm)}$. To prove this, we use an explicit interpolation of the Bäcklund transformations by Toda flows.

The potential theoretical Green function of the Julia set J_E turns out to be the Lyapunov exponent of an operator $L \in \mathcal{J}_E$. The Böttcher function conjugating the quadratic map $z \mapsto z^2 + E$ to the map $z \mapsto z^2$ near ∞ is given by the determinant $\det(L - E)$. There is a gap labelling for $L \in \mathcal{J}_E$.

We prove that the dyadic group on the attractor obtained by the symbolic labeling in the hyperbolic iterated function system is identical to the group obtained by all possible translations of L^+, the fixed point of Φ_E^+.

1   Introduction

Random Jacobi operators are *discrete one-dimensional Laplacians* and are discrete approximations of one-dimensional random Schrödinger operators (see [Cyc 87], [Car 90]). It would be desirable to have discrete Laplacians which are invariant under scale changes like a refinement of the lattice because such a self similarity is a realistic approximation to the continuum. A scale transformation of any physical system forces a renormalization of the energy in that a change of the scale must be done with a simultaneous adaptation of the energy or temperature. A simple scale transformation in one dimension is a doubling of the lattice spacing together with a squaring of the Laplacian and a simultaneous lowering of the energy $L \mapsto L^2 + E$. We have shown in [Kni 3] that the inverse of this map can be computed in the class of random Jacobi operators: one can find for a random selfadjoint Jacobi operator L two new selfadjoint random Jacobi operators $D^{(\pm)}$ defined over a "renormalized dynamical system" such that $(D^{(\pm)})^2 + E = L$. The entries of $D^{(\pm)}$ are constructed from the Titchmarsh-Weyl functions of L. The factorization $L = D^2 + E$ is the key for isospectral *Bäcklund transformations*, translations by one unit on the finer lattice. This work is a continuation of [Kni 3] and [Kni 4] with the aim to refine the analysis of the Bäcklund transformations and to study the iteration of the map $L = (D^{(\pm)})^2 + E \mapsto D^{(\pm)}$.

We outline now shortly the content of our results and compare them with earlier works of Baker, Barnsley, Bellisard, Bessis, Geronimo, Harrington, Mehta and Moussa [Bak 84], [Bel 82], [Bes 82], [Bar 83a], [Bar 88], who did similar constructions of semi-infinite Jacobi operators. We mention already now that despite some parallels, there is no overlapping of those results with the results discussed here. The mathematics of Jacobi operators in $\mathcal{B}(l^2(\mathsf{N}))$ and Jacobi operators in $\mathcal{B}(l^2(\mathsf{Z}))$ is in many respects quite different.

The automorphism of a probability space (X, m) form, when equipped with the uniform topology, a complete topological group $\mathcal{U}$. On this group is defined a map ϕ which assigns to a given dynamical system T its 2:1 integral extension S. This means that there exists a $S^2 = S \circ S$ invariant set $Y \subset X$ of measure 1/2 such that the induced system from S on Y is again T. The renormalisation of dynamical systems $T \mapsto S$ is a contraction on $\mathcal{U}$ and has exactly one fixed point, the *von Neumann-Kakutani system*, which is a group translation on the group of dyadic integers.

The set of random Jacobi operators forms a fiber bundle over the topological group $\mathcal{U}$. Over each dynamical system is defined the Banach space of random Jacobi operators which is a subspace of the crossed product of $L^\infty(X)$ with the dynamical system. The factorization result in [Kni 3] can be restated in saying that the renormalization given by the 2 : 1 integral extension on $\mathcal{U}$ can be lifted to two renormalization maps $\Phi_E^\pm$ defined on an open set of the bundle. A pair (T, L), where L is a Jacobi operator over the dynamical system (X, T, m), is mapped into a pair $(S, D^{(\pm)})$, where $(D^{(\pm)})^2 + E = L$ and S is an integral extension of T satisfying $S^2 = T$. We will show that for large enough real $-E$, both renormalization maps $\Phi_E^\pm$ are contractions on an open set of the bundle. This means that the two maps Φ_E^+, Φ_E^- form a *hyperbolic iterated function system* as defined by Barnsley. This iterated function system has an attractor $\mathcal{L}_E$ which is a Cantor set in the fiber $\mathcal{L}$ over the von Neumann Kakutani system.

The spectrum of L and the spectrum of D are related by $\sigma(D)^2 + E = \sigma(L)$ and the spectrum of each operator L in the attractor is the *Julia set* J_E of the quadratic map $z \mapsto z^2 + E$. Moreover, we will show that the *density of states* of L is the unique *equilibrium measure* on J_E. The Lyapunov exponent turns out to be the potential theoretical *Green function* of the Julia set and the determinant $\det(L - E)$ of an operator $L \in \mathcal{J}_E$ is the Böttcher function which conjugates the map $z \mapsto z^2 + E$ to $z \mapsto z^2$ in a neighborhood of ∞.

A main tool to prove our result is the following interpolation of Bäcklund transformations by a Toda flow by a time-dependent Hamiltonian $H_E(L) = \pm \mathrm{tr}(h_E(L)) = -\pm \frac{1}{2}w(E)$, where $w(E)$ is the *Floquet exponent* of L. The interpolating Toda flow is

$$\frac{d}{dE}BT_E^\pm(L) = -\pm \frac{1}{2}[(\frac{1}{L-E})^+ - (\frac{1}{L-E})^-, L] \ .$$

If we denote by L^+ the fixed point of Φ_E^+, we can form the set of all translates of

208

$L^+(T_x)$, where T_x is the translation belonging to any element x in the group X of dyadic integers. This set of translates of L^+ forms a group with the operation $L^+(T_x)L^+(T_y) = L^+(T_xT_y)$. The attractor of the iterated function system can be labeled by elements ω in the set $\Omega = \{-1, 1\}^{\mathsf{N}}$:

$$\Phi_E^\omega(L^+) = \lim_{n\to\infty} \Phi_E^{\omega_n} \Phi_E^{\omega_{n-1}} \ldots \Phi_E^{\omega_1} L^+ .$$

A change of alphabet $1 \mapsto 0, -1, \mapsto 1$ identifies the set $\Omega = \{-1, 1\}^{\mathsf{N}}$ with the group X and Ω inherits that group structure. We will prove that $\Phi_E^\omega(L^+) = L^+(T_{x(\omega)})$, where $T_{x(\omega)}$ is the group translation on X by the group element $x(\omega) \in X$ belonging to $\omega \in \Omega$. This means that each element of the attractor of the iterated function system can be obtained by an explicitly known translation of the fixed point L^+.

Jacobi operators $\tilde{L}_E$ in $\mathcal{B}(l^2(\mathsf{N}))$ with spectra on Julia sets J_E have been found earlier in [Bak 84], [Bel 82], [Bes 82], [Bar 83a], and [Bar 88]. Such operators satisfy $\tilde{L}_E^2 + E = \tilde{L}_E$ but they are different from the operators considered here. The side diagonal $d_n = [\tilde{L}_E]_{n,n+1}$ of $\tilde{L}_E$ begins with

$$d_0 = [\tilde{L}_E]_{01} = 0, d_1 = [\tilde{L}_E]_{12} = \sqrt{E}, d_2 = [\tilde{L}_E]_{23} = 1, \ldots$$

and the other entries d_k are defined recursively using

$$\begin{aligned} d_{2n+1}^2 &= -d_{2n}^2 + E , \\ d_n^2 &= d_{2n}^2 d_{2n-1}^2 , \end{aligned}$$

and are *algebraic functions* of E. In our case, there is no boundary condition at 0 and the entries of any element L_E in the attractor are in general *transcendental*. Moreover, there exists $\epsilon > 0$ such that all entries satisfy $|[L_E]_{ij}| < |\sqrt{E}| - \epsilon$ which is obviously not the case for the half-infinite operators $\tilde{L}_E$. It seems also that the renormalisation maps $\Phi_E^\pm$ can only be defined in $\mathcal{B}(l^2(\mathsf{Z}))$ and not in $\mathcal{B}(l^2(\mathsf{N}))$.

This chapter is organized as follows:
In the *second section*, we consider a map on the space of abstract dynamical systems. The dynamics of this renormalisation map is quite simple and has the von Neumann Kakutani system as a fixed point.
In the *third section*, the factorization of random Jacobi operators is reviewed slightly more general than in [Kni 4] because we don't restrict the analysis to selfadjoint operators. The two factorizations lead to the two renormalization maps $\Phi_E^\pm$ on the space of random Jacobi operators.
In the *fourth section*, we refine the analysis of Bäcklund transformations and give explicitly the Toda flow interpolating these isospectral transformations. This deeper understanding of Bäcklund transformations is necessary to prove a contraction property of $\Phi_E^\pm$ for large real $-E$.
In the *fifth section*, a version of a lemma of Barnsley about iterated function systems is proven.

In the *sixth section*, we review some results about the quadratic map $\tau_E : z \mapsto z^2 + E$. In the *seventh section*, it is shown that for large enough $-E$, the two maps $\Phi_E^{\pm}$ form a hyperbolic iterated function system leading to the existence of the attractor $\mathcal{J}_E$. In the *eighth section*, we will see that the attractor $\mathcal{J}_E$ is the same as the group of all translates of the fixed point L^+ of Φ^+.

In the *ninth section*, we prove that the density of states of an element in the attractor $\mathcal{J}_E$ is the unique equilibrium measure on the Julia set J_E and that the Lyapunov exponent of $L \in \mathcal{J}_E$ has a potential theoretical interpretation.

In the *tenth section*, we discuss shortly some generalizations.

In section *eleven*, we consider a matrix model for random Jacobi operators.

We collect some questions in section *thirteen*.

2 The von Neumann Kakutani system

The group $\mathcal{U}$ of *automorphism* of a standard probability space (X, m) is a complete topological group with the metric $d(T, S) = m\{x \in X \mid T(x) \neq S(x)\}$ giving the so called *uniform topology* on the space of automorphism. Every $T \in \mathcal{U}$ gives an abstract dynamical system (X, T, m) and we call $\mathcal{U}$ also the *group of dynamical systems*.

Given a measurable set $Y \subset X$ of positive measure, the *induced transformation* T_Y is an automorphism of (Y, m_Y) defined by $T_Y(x) = T^{n(x)}(x)$, where $n(x) = \min\{n > 0 \mid T^n(x) \in Y\}$. This map T_Y leaves the probability measure $m_Y = m/m(Y)$ invariant and is again an automorphism of (X, m), because all non-atomic standard probability spaces are isomorphic. We call $\Phi_Y : \mathcal{U} \to \mathcal{U}$ the map which brings T into the transformation T_Y. The later map is identified with the transformation on (X, m) by $\Phi_Y(T)(x) = s_Y \circ T_Y \circ s_Y^{-1}$, where $s_Y : Y \mapsto X$ is a measurable "identification function".

Dual to the formation of induced systems is the *integral extension*: if $f \in L^1(X)$ is a positive integer-valued function, then a new dynamical system (X^f, T^f, m^f) is defined as follows. Define $X^f = \{(x, i) \mid x \in X \text{ and } 1 \leq i \leq f(x)\}$ and a probability measure m^f on X^f by $m^f((Y, i)) = m(Y)/\int f \, dm$. This measure is preserved by the transformation

$$T^f(x, i) = \begin{cases} (x, i+1) & \text{, if } i+1 < f(x) \,, \\ (T(x), 1) & \text{, if } i+1 = f(x) \,. \end{cases}$$

The space X^f can be visualized as a tower, whose foundation is X and which has $f(x)$ floors over each point $x \in X$. Under the action of T^f, a point (x, i) is lifted vertically up one floor, if this is possible and else lowered down to the ground floor, where it takes the position of the point $(T(x), 1)$. This construction gives also a mapping defined on $\mathcal{U}$ because the integral transformation can again be viewed as

an automorphism of (X, m). Denote by

$$\Phi^f : \mathcal{U} \to \mathcal{U}$$

the transformation bringing an automorphism to its integral extension. Again an identification function $s^f : X \to X^f$ makes the correspondence between the Lebesgue spaces X and X^f.

We review without further use some properties of the maps $\Phi^f, \Phi_Y : \mathcal{U} \to \mathcal{U}$ (see [Hal 56], [Cor 82], [Den 76]). One knows for example that they preserve the dense set of periodic systems in $\mathcal{U}$ or the nowhere dense set of ergodic systems in $\mathcal{U}$ but they don't leave invariant for example the class of mixing systems. For the entropy $h_m(T)$ of the systems it is known by Abramov's formula that $h_m(\Phi_Y T) = m(Y)^{-1} \cdot h_m(T)$ and that $h_m(\Phi^f T) = (\int_X f(x) \, dm(x))^{-1} \cdot h_m(T)$.

We consider now the special case of an integral extension with $f \equiv 2$, where the identification function is given by

$$X = [0, 1] \to (X, \{1\}) \cup (X, \{2\}) = X^f = [0, 1/2] \cup [1/2, 1]$$

and where the Lebesgue space X is identified with the interval $[0, 1]$. In order to fix the ideas we can write a dynamical system as a measurable map $T : [0, 1] \to [0, 1]$ leaving invariant the Lebesgue measure on $[0, 1]$ and define Φ^f by

$$\Phi^f(x) = \begin{cases} x + 1/2 & , \text{if } x \in [0, 1/2) , \\ T(2x - 1)/2 & , \text{if } x \in [1/2, 1] , \end{cases}$$

on $[0, 1]$. For $Y = [0, 1/2]$ we have then $\Phi_Y \circ \Phi^f(T) = T$.

Proposition 2.1 *The renormalisation map Φ is a contraction on $\mathcal{U}$. Every automorphism T is attracted to a unique fixed point $\bar{T}$.*

Proof. Given two transformations $T_1, T_2 \in \mathcal{U}$. We have $d(\Phi T_1, \Phi T_2) = \frac{1}{2} d(T_1, T_2)$, because the transformed automorphism are equal on a set of measure $1/2$ and

$$m(\{T_1(x) \neq T_2(x)\}) = m(\{T_1(x) \neq T_2(x)\})/2 .$$

This proves that Φ is a contraction having a unique fixed point. $\qquad \square$

The next figure illustrates how the graph of a dynamical system (given by a bijective on the interval $[0, 1]$) is converging under the renormalisation to the graph belonging to the von Neumann-Kakutani system.

Renormalisation of dynamical systems
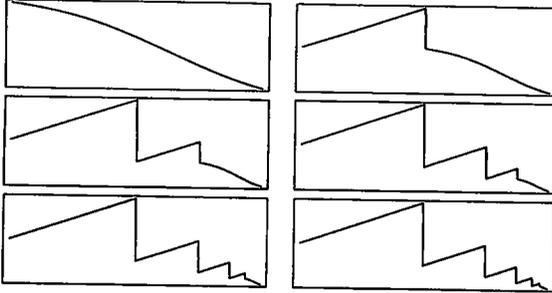
Remark. In the same way, one can show that any map Φ^f different from the identity is a contraction and has a unique fixed point. On the other hand, a map Φ_Y has an expanding character.

We review now some results about the fixed point T of Φ^f, with $f(x) = 2$, *the von Neumann-Kakutani system*. It is an automorphism of the unit interval X defined by a piecewise translation of intervals

$$T(x) = x + 1 - C_{n+1}, \text{ for } C_n \leq x < C_{n+1},$$

where $C_0 = 0$ and $C_n = \sum_{i=1}^{n} 2^{-n}, n > 0$. (See [Par 80], [Fri 92]). The following is known about the von-Neumann Kakutani system (X, T, m):

Proposition 2.2 (X, T, m) *is ergodic and has a discrete spectrum*

$$\hat{G} = \{e^{2\pi i k 2^{-n}} \mid k \in \mathbb{Z}, n > 0\}.$$

(X, T, m) *is conjugated to a group translation on the compact abelian group G of dyadic integers, the dual group of $\hat{G} \subset T^1$.*

Proof. The von Neumann Kakutani system is a group translation on the group of dyadic integers because this is the character group of a discrete subgroup of the circle formed by the spectrum $\hat{G}$ of the system. (See [Par 80] for the calculation of the spectrum and [Rud 62] for general properties on topological groups).

The dual group G of $\hat{G}$ is the group of dyadic integers which is the space of sequences $\omega = \{\omega_1, \omega_2, \ldots\}$ in $\{0,1\}^{\mathbb{N}}$ with the group operation

$$(\omega + \eta)_n = \omega_n + \eta_n + \rho_{n-1} \pmod 2,$$

212

where $\rho_0 = 0$ and $\rho_n \in \{0,1\}$ is equal to 1 if and only if $\omega_n + \eta_n + \rho_{n-1} \geq 2$. The group translation T on G is given by $\omega \mapsto \omega(T) = \omega + (1,0,0,\ldots)$ and is a special case of a so called *adding machine*. The map

$$\omega \mapsto \sum_{n=1}^{\infty} \omega_n 2^{-n} \in [0,1]$$

conjugates the group translation T to the map T on the interval I.

In order to see that G is the dual group of $\hat{G}$, we assign to each element $\omega \in G$ the character

$$\gamma_\omega(e^{2\pi i k 2^{-n}}) = \left(w_n \sqrt{w_{n-1}\sqrt{\cdots\sqrt{w_1}}} \right)^k ,$$

where $w_n = 1 - 2\omega_n \in \{-1,1\}$ is the multiplicative way of writing $\mathbb{Z}_2$. The map T_g is conjugated to

$$T_g(\gamma)(z) = z \cdot \gamma(z) .$$

The ergodicity of T_g can be seen in the character picture. The generator of the group translation is the character $\gamma_T(z) = z$. By Pontryagin duality, the group $\hat{G}$ is the character group of G. A necessary and sufficient condition for a group translation to be ergodic is that $g(\gamma_T) = \gamma_T(g) \neq 1$ for any nontrivial $g \in \hat{G}$ (see [Cor 82], p. 97) and this is true in our case. An other way, to prove ergodicity is to see that the functions

$$f_n(x) = \sum_{j=0}^{2^n-1} 1_{(0,2^{-n})}(T^j x) \cdot e^{\frac{2\pi i j}{2^n}}$$

form a complete set of eigenfunctions to the eigenvalues

$$e^{\frac{2\pi i}{2^n}} \in \hat{G} .$$

$\square$

3 Renormalization of random Jacobi operators

The *crossed product* $\mathcal{X}$ of $L^\infty(X)$ with the dynamical system (X,T,m) is a C^* algebra (it is even a von Neumann algebra ([Con 90])) that consist of operators $K = \sum_{n\in\mathbb{Z}} K_n \tau^n$ with convolution multiplication

$$KM = \sum_n (KM)_n \tau^n = \sum_{k+m=n} (K_k M_m(T^k))\tau^n .$$

The norm on $\mathcal{X}$ is given by $|||K||| = |\,||K(x)||\,|_\infty$, where $K(x)$ is the infinite matrix $[K(x)]_{mn} = K_{n-m}(T^m x)$. The adjoint of an operator is defined by

$$(K^*)_n(x) = \overline{K}_{-n}(T^n x) .$$

213

The algebra $\mathcal{X}$ has the trace $\mathrm{tr}(K) = \int_X K_0 \, d\mu$. A *random Jacobi operator* L is an element in $\mathcal{X}$ of the form

$$L = a\tau + a(T^{-1})\tau^* + b$$

with $a, b \in L^\infty(X, \mathbb{C})$. We denote by $\mathcal{L} \subset \mathcal{X}$ the complex Banach space of random Jacobi operators. We call $\exp(\int \log |a| \, dm)$ the *mass* of the operator. If $\log |a| \geq \delta > 0$ for some $\delta > 0$, we say, the operator has a *positive definite mass*. Notice that random Jacobi operators are only normal, if a, b are real. Denote by $\Phi(\mathcal{X})$ the von Neumann algebra corresponding to the renormalized system $(X, \Phi(T), m)$. As long as we consider only one renormalization step, we denote the renormalized dynamical system with (Y, S, n) and the von Neumann algebra with $\mathcal{Y}$ and elements in $\mathcal{Y}$ by $B = \sum_n B_n \sigma^n$, where σ is the symbol in $\mathcal{Y}$ corresponding to τ in $\mathcal{X}$. Call ψ the map $\Phi(\mathcal{X}) \to \mathcal{X}$

$$K = \sum_n K_n \sigma^n \mapsto \sum_n \tilde{K}_n \tau^n \ ,$$

where $\tilde{K}_n(x) = K_{2n}(x)$ for $x \in X_1 = X$. The mapping ψ gives for $x \in X_1$

$$[\psi(K)(x)]_{nm} = [K(x)]_{2n,2m} \ .$$

Let $L \in \mathcal{L}$ be a random Jacobi operator having strictly positive mass. For E outside a ball containing the spectrum of L, we can form the Titchmarsh-Weyl functions

$$m^+(x) = a(x)\frac{u^+(Tx)}{u^+(x)} \quad , \quad m^-(x) = a(x)\frac{u^-(Tx)}{u^-(x)} \ ,$$
$$n^+(x) = a(T^{-1}x)\frac{u^+(T^{-1}x)}{u^+(x)} \quad , \quad n^-(x) = a(T^{-1}x)\frac{u^-(T^{-1}x)}{u^-(x)} \ ,$$

where $u^\pm(x) \in \mathbb{R}^{\mathbb{Z}}$ are solutions of $L(x) = Eu(x)$ with $\sum_{\pm n > 0} |u_n^\pm(x)|^2 < \infty$. These functions are measurable according to the multiplicative ergodic theorem of Oseledec and are bounded when E is outside a ball containing the spectrum of L. Using

$$Lu^\pm = au^\pm(T) + a(T^{-1})u^\pm(T^{-1}) + bu^\pm = Eu^\pm$$

and the definition of $m^\pm, n^\pm$, we get

$$\begin{aligned} m^\pm + n^\pm &= E - b \ , \\ m^\pm \cdot n^\pm(T) &= a^2 \ . \end{aligned}$$

We define new random Jacobi operators

$$D^{(\pm)} = \sqrt{c^\pm}\sigma + \sqrt{c^\pm(S^{-1})}\sigma^* \in \mathcal{Y}$$

with functions $c^\pm$ defined on Y by requiring for $x \in X = X_1$,

$$c^\pm(x) = -m^\pm(x), \ c^\pm(S^{-1}x) = -n^\pm(x) \ .$$

The sign of $D^{(\pm)}$ is specified if we take the principal branch of the square root for $\sqrt{-m^\pm}$ and the branch $\sqrt{-n^\pm}$ such that $a = \sqrt{m^\pm n^\pm(T)}$. We get then

$$\begin{aligned}
c^\pm(x) + c^\pm(S^{-1}x) &= -E + b(x)\,, \\
c^\pm(x) \cdot c^\pm(Sx) &= a^2(x)\,.
\end{aligned}$$

As c is defined on Y, these formulas extend the functions $a, b \in L^\infty(X)$ to functions in $L^\infty(Y)$.

Proposition 3.1 *The random Jacobi operators*

$$D^\pm = \sqrt{c^\pm}\sigma + \sqrt{c^\pm(S^{-1})}\sigma^* \in \mathcal{Y}$$

are bounded for E outside a ball containing the spectrum of L and satisfy

$$\psi((D^\pm)^2) = L - E\,.$$

The operator $BT_E^\pm(L) := \psi((D^{(\pm)})^2(S) + E)$ is isospectral to L. The operators $D^\pm$ are selfadjoint if L is selfadjoint and E is real below $\Sigma(L)$.

Proof. The relation $\psi((D^\pm)^2) = L - E$ follows from the definition:

$$\begin{aligned}
\psi((D^\pm)^2) &= \psi(\sqrt{c^\pm \cdot c^\pm(S)}\sigma^2 + (c^\pm + c^\pm(S^{-1})) + \sqrt{c^\pm(S^{-2}) \cdot c^\pm(S^{-1})}\sigma^{-2}) \\
&= a\tau + b - E + a(T^{-1})\tau^* = L - E\,.
\end{aligned}$$

If E is real and below the spectrum of L, the functions $c^\pm$ are positive and $D^\pm$ are real and selfadjoint. The maps

$$L \mapsto BT_E^\pm(L) := \psi((D^{(\pm)})^2(S) + E)$$

are called *Bäcklund transformations*.

In order to prove that $BT_E(L)$ is isospectral to L we take first the periodic ergodic case, where $N = |X|$ is finite and where we can build for each periodic $N \times N$ Jacobi matrix L of positive mass a periodic $2N \times 2N$ Jacobi matrix D such that $D^2 + E$ is the direct sum of two $N \times N$ matrices L and $BT_E(L)$. The spectrum of periodic Jacobi operators is generically simple and the multiplicity of their eigenvalues is ≤ 2. (i) Assume therefore first that L has N simple eigenvalues. We want to show that $BT(L)$ has in this case the same spectrum as L. The Jacobi matrix D has a spectrum $\pm\lambda_1, \ldots, \pm\lambda_N$ symmetric with respect to the imaginary axis because if λ is an eigenvalue with the eigenvector $(u_1, u_2, \ldots, u_{2N-1}, u_{2N})$ then $-\lambda$ is an eigenvalue with the eigenvector $(u_1, -u_2, \ldots, u_{2N-1}, -u_{2N})$. The matrix $D^2 + E$ is the direct sum of the two Jacobi matrices $L, BT_E(L)$ and has the eigenvalues $\lambda_i^2 + E$, each with multiplicity exactly 2. As L has by assumption a simple spectrum we obtain

that $\sigma(L) = \{\lambda_i^2 + E \mid i = 1, \ldots, N\}$ and the operator $BT_E(L)$ must have the same spectrum as L because each eigenvalue of $D^2 + E$ has multiplicity 2.

(ii) In the case, when L is periodic with not necessarily simple spectrum, the claim follows because in the weak operator topology, the factorization is continuous, the spectrum depends continuously on the operator and matrices having a simple spectrum are dense in the finite dimensional vector space of N periodic Jacobi matrices.

(iii) In the general infinite dimensional case, we can approximate a Jacobi matrix $L(x)$ in the weak operator topology by periodic Jacobi matrices $L^{(N)}(x)$ and the spectra of these approximations converge for $N \to \infty$ to the spectrum of $L(x)$. The Bäcklund transformed matrices $BT_E(L^{(N)}(x))$ converge for $N \to \infty$ in the weak operator topology to $BT_E(L(x))$ because the Titchmarsh-Weyl functions depend continuously on the matrices. So, the spectrum of $BT_E(L)$ is the same as the spectrum of L. □

We got the two *renormalisation maps*

$$\Phi_E^{\pm} : \mathcal{L} \to \Phi(\mathcal{L}), \; L \mapsto D^{(\pm)}$$

parameterized by an energy $E \in \mathbb{C}$. The maps are defined on an open (possibly empty) set $\mathcal{V}_E$ of $\mathcal{L}$. Random Jacobi operators form a fiber bundle over the space $\mathcal{U}$ of dynamical systems. Over each dynamical system T is defined the fiber $\mathcal{L}$ of Jacobi operators over this system. Given $E \in \mathbb{C}$, there is an open (possibly empty) subset of this fiber bundle, where the renormalisation maps $\Phi_E^{\pm}$ make sense. A pair (T, L), where L is a Jacobi operator over the dynamical system (X, T, m) is mapped into a pair $(S, D^{(\pm)})$, where $\Psi((D^{(\pm)})^2 + E) = L$ and $\Phi(T) = S$ is the $2 : 1$ integral extension of T.

Illustration.
The following Mathematica program calculates numerically the fixed point of Φ_E^+ at $E = -5$.

```
m[i_Integer,n_Integer]:=Mod[i-1,n]+1;
a[n_Integer]:=Table[N[2+Sin[k 2 Pi/n]],{k,n}];

A[a_List,b_List,EE_]:=Table[1/a[[m[i-1,Length[a]]]]*
  {{EE-b[[i]],-(a[[m[i-1,Length[a]]]])^2},{1,0}},{i,1,Length[a]}];
Monodro[a_List,b_List,EE_]:=Block[{t=0,A=A[a,b,EE],B=IdentityMatrix[2]},
  Do[B=A[[i]].B;t=t+Re[Log[B[[1,1]]]];B=B/B[[1,1]],{i,Length[a]}];{B,t}];

mplus[a_List,b_List,EE_]:=Block[
  {M=Monodro[a,b,EE][[1]],s,m0,ad,n=Length[a]},ad=M[[1,1]]-M[[2,2]];
  m0=(ad-Sqrt[ad^2+4*M[[1,2]]*M[[2,1]]])/(2*M[[2,1]]);s={m0};
  Do[s=Prepend[s,a[[m[n-i,n]]]^2/(EE-First[s]-b[[m[n-(i-1),n]]])],{i,n-1}]; s];
cplus[a_List,b_List,EE_]:=Block[{mpl=mplus[a,b,EE],npl},
  npl=EE-b-mpl; Flatten[Join[Table[{-npl[[i]],-mpl[[i]]},{i,1,Length[a]}]]]];
```

216

```
RenormStep[d_List,EE_,n_Integer]:=Block[{c=d},
  Do[c=Sqrt[cplus[c,Table[-EE,{Length[c]}],0.0]],{n}]; c ];

Dirac[a_List,EE_]:=Block[{n=Length[a]},
  d[k_Integer,l_Integer,n_Integer]:=IdentityMatrix[n][[m[k,n],m[l,n]]];
  Table[N[d[k,j+1,n]*a[[m[j,n]]]+d[k,j-1,n]*a[[m[j-1,n]]]+d[k,j,n]*EE,3],
  {j,-4,+3},{k,-4,+3}]]];

RenormFixedPoint[E_,NumIter_Integer]:=Dirac[RenormStep[a[5],E,NumIter],E];
```

We calculate a fixed point of Φ^+ with $E = -5$ with

```
TeXForm[MatrixForm[N[RenormFixedPoint[-5.0,4],5]]]   >>fixedpoint1.tex;
```

Here is a part of the matrix of the fixed point of $\Phi^+(-5)$

$-5.$	0.48531	0	0	0	0	0	0
0.48531	$-5.$	2.0033	0	0	0	0	0
0	2.0033	$-5.$	0.99334	0	0	0	0
0	0	0.99334	$-5.$	2.2314	0	0	0
0	0	0	2.2314	$-5.$	0.145	0	0
0	0	0	0	0.145	$-5.$	2.0355	0
0	0	0	0	0	2.0355	$-5.$	0.92556
0	0	0	0	0	0	0.92556	$-5.$

4 Bäcklund transformations

For E outside a ball containing the spectrum $\Sigma(L)$, the Bäcklund transformations $BT_E^\pm$ are given by

$$L = a\tau + a(T^{-1})\tau^* + b \mapsto L' = a'\tau + a'(T^{-1})\tau^* + b ,$$

where

$$b' = b + n^\pm - n^\pm(T), \ a'^2 = a^2 \frac{m^\pm(T)}{m^\pm} .$$

We have shown in [Kni 3] that

$$\lim_{E \to -\infty} BT_E^+(L) = L(T) , \lim_{E \to -\infty} BT_E^-(L) = L$$

and that in the periodic case the transformations can be interpolated by time-dependent Toda flows in $\mathcal{L}$. Because we want to estimate the Fréchet derivative of the Bäcklund transformations near $-\infty$, we have to refine the analysis of Bäcklund

transformations and to determine explicitly the Hamiltonian Toda flow, which does the interpolation. We define the projections

$$K = \sum_{n=-\infty}^{\infty} K_n \tau^n \mapsto K^{\pm} = \sum_{\pm n > 0} K_n \tau^n$$

which yield the decomposition $K = K^- + K_0 + K^+$. Define $K \mapsto K^\Delta$ to be the projection from $\mathcal{X}$ to $\mathcal{L}$. For a *Hamiltonian*

$$H \in C^\omega(\mathcal{L}) := \{L \mapsto \mathrm{tr}(h(L)) \mid h \text{ analytic near the spectrum of } L\} ,$$

the differential equation

$$\dot{L} = [B_H(L), L] = [L^+ - L^-, h'(L)^\Delta] ,$$

with $B_H(L) = h'(L)^+ - h'(L)^-$ is called *a random Toda flow* [Kni 3]. In order to get local existence of the flow, the domain of analyticity of h must be sufficiently large. We will consider also complex time t as well as time-dependent Hamiltonians H.

The *Floquet exponent* $w(E)$ of $L = a\tau + a(T^{-1})\tau^* + b$ is

$$w(E) = -\mathrm{tr}(\log(L - E))$$

and is by the *Thouless formula* defined on $\{\mathrm{Im}(z) \geq 0\}$. The Lyapunov exponent $\lambda(E) = -\mathrm{Re}(w(E)) - \int \log|a(x)| \, dm(x)$ is defined for all $E \in \mathbb{C}$ and the derivative $w'(L)$ is bounded for all E in the resolvent set of L.

Proposition 4.1 *Bäcklund transformations can be interpolated by random Toda flows with the time-dependent Hamiltonians*

$$H_E^{\pm}(L) = \pm\mathrm{tr}(h_E(L)) = \pm\frac{1}{2}w(E) = \pm \int \log(m^{\pm}(x)) \, dm(x) .$$

This means, we have

$$\frac{d}{dE} BT_E^{\pm}(L) = - \pm \frac{1}{2}[(\frac{1}{L - E})^+ - (\frac{1}{L - E})^-, L] . \tag{1}$$

Proof. We prove the Proposition first in the finite-dimensional periodic case $|X| < \infty$ and under the condition that E is real below the spectrum of L. We know from [Kni 4] that Bäcklund transformations can then be interpolated by Toda flows. In the coordinates $(d, b) = (\log(a), b)$, the Toda flow can be written as

$$\begin{aligned}
\frac{d}{dt}d &= h'(L)_0(T) - h'(L)_0, \\
\frac{d}{dt}b &= e^d \cdot h'(L)_1 - e^{d(T^{-1})} \cdot h'(L)_1(T^{-1}) .
\end{aligned}$$

The Bäcklund transformations $BT_E^{(\pm)}$ are given in the coordinates $(d, b) = (\log(a), b)$ by

$$d \;\mapsto\; d(E) = d + \frac{1}{2}\log(m^{\pm})(T) - \frac{1}{2}\log(m^{\pm}) \,,$$
$$b \;\mapsto\; b(E) = b + n^{\pm} - n^{\pm}(T) \,.$$

Differentiation of these equations with respect to E gives

$$\frac{d}{dE}d \;=\; \frac{1}{2}\frac{\frac{d}{dE}m^{\pm}}{m^{\pm}}(T) - \frac{1}{2}\frac{\frac{d}{dE}m^{\pm}}{m^{\pm}},$$
$$\frac{d}{dE}b \;=\; \frac{d}{dE}n^{\pm} - \frac{d}{dE}n^{\pm}(T) \,.$$

Requiring $(\frac{d}{dE}d, \frac{d}{dE}b) = (\frac{d}{dt}d, \frac{d}{dt}b)$ gives (up to a L-independent constant function which we put to zero)

$$h'(L)_0 \;=\; \frac{1}{2}\frac{\frac{d}{dE}m^{\pm}}{m^{\pm}} = \frac{1}{2}\frac{d}{dE}\log(m^{\pm}), \tag{2}$$
$$h'(L)_1 \;=\; -\frac{\frac{d}{dE}n^{\pm}(T)}{a} \tag{3}$$

and so

$$\mathrm{tr}(h'(L)) \;=\; \int_X h'(L(E))_0 \, dm = \frac{d}{dE}\int \frac{1}{2}\log(m^{\pm}) \, dm = \pm\frac{1}{2}\frac{d}{dE}w(E)$$
$$=\; -\pm\frac{1}{2}\mathrm{tr}((L - E)^{-1}) \,.$$

Therefore

$$h'_E(L) = -\pm\frac{1}{2}(L - E)^{-1} \,, \quad h_E(L) = -\pm\frac{1}{2}\log(L - E)$$

which leads to

$$H_E(L) = \mathrm{tr}(h_E(L)) = \pm\frac{1}{2}w(E) \,.$$

We have proven equation 1 in the finite dimensional case with E real below the spectrum of L. The formulas (2) and (3) are true in general, if they hold in each finite dimensional case. Because one can approximate the operators $L(x)$ by periodic matrices $L^{(N)}(x)$ in the weak operator topology. For $N \to \infty$, we obtain $h(L^{(N)})(x) \to h(L)(x)$ and $(m^{\pm})^{(N)}(x) \to m^{\pm}(x)$ for almost all $x \in X$. (By analytic continuation, the formula (1) holds also for complex numbers E satisfying $|E| > |||L|||$.) $\qquad\qquad\Box$

We can use this interpolation to estimate the Fréchet derivative $\frac{d}{dL}BT_E^{\pm}(L)$ of the Bäcklund transformations near ∞.

Proposition 4.2 *For $|E| \to \infty$, we have $\|\frac{d}{dL}BT_E^{\pm}(L)\| \to 1$, uniformly for $L \in \mathcal{V}_E$.*

Proof. Since there is for $\||L\|| < |E|$ a uniform bound for the Fréchet derivative of

$$E^2 \cdot \frac{p}{d}BT_{\mp}^{E}(L) = -\mp \frac{1}{2}\sum_{m=1}^{\infty}[(\frac{E_{n-1}}{L^n})^{+} - (\frac{E_{n-1}}{L^n})^{-}, L]$$

on $\mathcal{V}_E$, we have

$$\frac{p}{d}\frac{p}{dE}BT_{\mp}^{E} = O(E^{-2})$$

for $|E| \to \infty$. Therefore

$$\||\frac{p}{d}BT_{\mp}^{E}(L) - I_{\mp}\|| = O(E^{-1}), \quad |E| \to \infty,$$

where $I^- : \mathcal{L} \to \mathcal{L}$ is the identity operator and $I^+(L) = L(T)$. $\square$

5 Iterated function systems

We state and prove here a version of Barnsley's result about hyperbolic iterated function systems [Bar 85]. Such a result holds in the general context of complete metric spaces. We formulate and use it in the case when the hyperbolic iterated function system is acting on a Banach space.

Lemma 5.1 Given a Banach space $(\mathcal{M}, \|\cdot\|)$ and two differentiable maps Φ^+, Φ^- : $\mathcal{V} \subset \mathcal{M} \hookrightarrow \mathcal{M}$ leaving invariant an open connected bounded subset $\mathcal{V}$ of $\mathcal{M}$ and assume there exists a common inverse T of both Φ^+, Φ^- on $\Phi^+(\mathcal{V}) \cup \Phi^-(\mathcal{V})$. Suppose, there exists $\lambda < 1$ such that for $L \in \mathcal{V}$

$$\|\frac{d}{dT}\Phi_{\mp}(L)\| \leq \lambda$$

and $\Phi_L^+ \neq \Phi_L^-$ for all $L \in \mathcal{V}$. Then there exists a $\Phi^{\mp}$ invariant Cantor set $\mathcal{J} \subset \mathcal{V}$ homeomorphic to $\Omega = \{-1,1\}^{\mathbb{N}}$ which is the image of the injective map

$$\Phi : \omega = (\omega_1, \omega_2, \cdots) \mapsto \lim_{k\to\infty} \Phi^{\omega_1} \circ \Phi^{\omega_2} \circ \cdots \circ \Phi^{\omega_k}(K) = \Phi_\omega(K),$$

where $K \in \mathcal{V}$ is arbitrary. The map T restricted to $\mathcal{J}$ is topologically conjugated by Φ to the one-sided Bernoulli shift σ on $\{-1,1\}^{\mathbb{N}}$: $T \circ \Phi = \Phi \circ \sigma$.

Proof. The contraction property. Because $\mathcal{V}$ is open and connected, we can connect two given points $K_0, K_1 \in \mathcal{V}$ by a piecewise differentiable path $t \mapsto K(t) \in \mathcal{V}$ so that $K_1 - K_0 = \int_0^1 \dot{K}_t\, dt$. We have by assumption

$$\frac{d}{dt}\Phi^{\mp}(K(t)) \leq \lambda < 1.$$

Therefore

$$\|\Phi^{\pm}(K_1) - \Phi^{\pm}(K_0)\| = \| \int_0^1 \frac{d}{dt} \Phi^{\pm}(K(t)) \, dt\| \le \| \int_0^1 \| \frac{d}{dL} \Phi^{\pm}(K(t))\| \|\dot{K}(t) \, dt\|$$

$$\le \lambda \cdot \| \int_0^1 \dot{K}(t) \, dt\| = \lambda \cdot \|K_1 - K_0\| \, .$$

Existence of the mapping Φ. Take $K \in \mathcal{V}$ and $\omega \in \Omega$. The sequence $\Phi^{\omega_1} \circ \Phi^{\omega_2} \circ \cdots \circ \Phi^{\omega_k}(K)$ is Cauchy in $\mathcal{V}$ because

$$\text{diam}(\Phi^{\omega_1} \circ \Phi^{\omega_2} \circ \cdots \circ \Phi^{\omega_n}(\mathcal{V})) \le \lambda^n \cdot \text{diam}(\mathcal{V}).$$

The limit $\Phi(\omega)$ of this Cauchy sequence exists, because $\mathcal{M}$ is complete and the limit is independent of $K \in \mathcal{V}$. The map is continuous as

$$\|\Phi(\omega)K - \Phi(\eta)K\| \le \text{diam}(\mathcal{V}) \cdot \lambda^{-\min\{k \in \mathbb{N} \,|\, \omega_k \ne \eta_k\}} \, .$$

Injectivity of $\Phi : \Omega \to \mathcal{M}$. Assume $\Phi^{\omega} = \Phi^{\nu}$ so that $\omega \ne \nu$ and k is the smallest index with $\omega_k \ne \nu_k$. Since Φ^+, Φ^- have a common inverse, we obtain from $\Phi^{\omega} = \Phi^{\nu}$ that for all $k \in \mathbb{N}$

$$\Phi^{\sigma^k(\omega)} = \Phi^{\sigma^k(\nu)} \, ,$$

where σ is the shift $\omega = (\omega_1 \omega_2 \ldots) \mapsto (\omega_2, \omega_3, \ldots)$. With $\omega_n \ne \nu_n$ and the assumption $\Phi^+(L) \ne \Phi^-(L)$ for all $L \in \mathcal{M}$, we get

$$\Phi^{\omega_n} \Phi^{\sigma^{n+1}(\omega)} \ne \Phi^{\nu_n} \Phi^{\sigma^{n+1}(\omega)}$$

in contradiction to the fact that both sides are equal to $\Phi^{\sigma^n(\omega)} = \Phi^{\sigma^n(\nu)}$.

Conjugation to a Bernoulli shift. The map $\Phi : \Omega \to \mathcal{J}$ is a continuous bijection. Since Ω is compact, Φ is a homeomorphism. As $\Phi^{\sigma\omega} = \mathcal{T}\Phi^{\omega}$, the map Φ conjugates the Bernoulli shift $\sigma : \Omega \to \Omega$ to the map $\mathcal{T} : \mathcal{J} \to \mathcal{J}$. $\square$

The maps Φ^+, Φ^- in the Lemma form a *hyperbolic iterated function system* and the invariant Cantor set $\mathcal{J}$ of the iterated function system is called the *attractor* of the iterated function system.

6 The quadratic map

We will need some facts about the dynamical system on the complex plane $\mathbb{C}$ defined by the *quadratic map* $\tau_E : z \mapsto z^2 + E$, where $E \in \mathbb{C}$ is a parameter. The inverse of τ_E is a correspondence $\phi_E : x \mapsto \phi_E^{\pm} = \pm\sqrt{x - E}$ having the two branches $\phi_E^{\pm}$. A map for measures $\mu \mapsto \phi^*(\mu)$ is defined by

$$\phi^*(\mu)(Y) = \mu(\tau_E(Y)) \, .$$

The *Julia set* $J_E \subset \mathbf{C}$ which is defined to be the closure of all repelling periodic orbits of τ_E is a nonempty, compact, τ_E invariant perfect set. For large $|E|$, the two maps $\phi^{\pm}$ form a hyperbolic iterated function system having the Julia set as the attractor. The Julia set is called *hyperbolic* or *expanding* if τ_E restricted to J_E is hyperbolic, i.e. if there exists $n \in \mathbf{N}$ and and $\lambda > 1$ such that for all $z \in J_E$

$$|d\tau_E^n(z)| \geq \lambda \ .$$

It is known that J_E is hyperbolic and also a completely disconnected Cantor set for E outside the *Mandelbrot set* $M = \{E \in \mathbf{C} \mid \tau_E^n(0) \not\to \infty\}$. In general, there exists a unique *electrostatic measure* μ_E with support on the Julia set J_E. This measure is ϕ^* invariant and is balanced [Bar 83]. This means that for each chosen branch $\phi^{\pm}$, one has

$$(\mu_E)(\phi^{\pm}(Y)) = \frac{1}{2}\mu_E(Y)/2 \ .$$

For $E \neq 0$, the map ϕ^* has the unique attractor μ_E in the space of probability measure on $\mathbf{C}$ so that $(\phi^*)^n(\mu) \to \mu_E$ for all probability measures μ on $\mathbf{C}$ [Bro 65], [Lju 83].
The dynamical system (J_E, τ_E, μ_E) is mixing [Bar 83] and μ_E is maximizing the metric entropy and is so an *equilibrium measure* [Lju 83]. A power of τ_E is as an measure theoretical dynamical system isomorphic to a one-sided Bernoulli shift [Man 85]. For large enough $|E|$, when $\phi^{\pm}$ form an iterated function system it follows already from Barnsley's Lemma 5.1 that τ_E is topologically conjugated to a one-sided Bernoulli shift.

Illustration:
A plot of the Mandelbrot set can be obtained with the 2-line program

```
M=Compile[{x,y},Module[{z=x+I y,k=0},While[Abs[z]<2.&&k<50,z=z^2+x+I y;++k];k]];
DensityPlot[50-M[x,y],{x,-2.,1.},{y,-1.5,1.5},PlotPoints->200,Mesh->False];
```
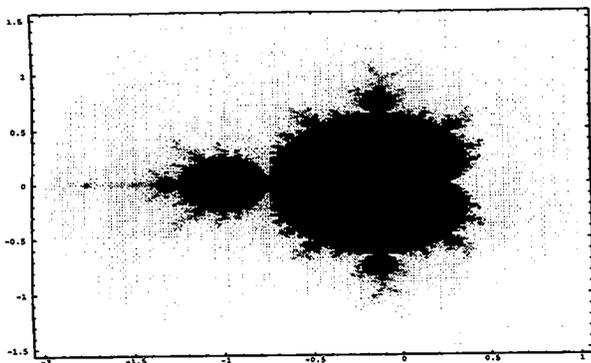
We can look at at an array $J_{E_0 + n*d + m*i*d}$ of Julia sets with the following Mathematica program

```
Complex2List[c_List]:=Table[{Re[c[[i]]],Im[c[[i]]]},{i,Length[c]}];
fL[c_,EE_Complex]:=Block[{d=Table[Sqrt[c[[i]]-EE],{i,Length[c]}]},Union[d,-d]];
fL[z_Complex,EE_Complex,n_Integer]:=Block[{d={z},b},
  Do[b=fL[d,EE];d=Union[b,-b],{i,n}];d ];
JuliaQ[z_Complex,EE_Complex,n_Integer]:=ListPlot[Complex2List[fList[z,EE,n]],
  DisplayFunction->Identity,Axes->False];
JuliaArray[EE_Complex,n_Integer,m_Integer,dd_Real]:=Show[GraphicsArray[
  Table[JuliaQ[0.3+0.45345*I,EE+k*dd+l*dd*I,n],{l,-m,m},{k,-m,m}]],
  DisplayFunction->$DisplayFunction,
  PlotLabel->FontForm["Julia sets of the quadratic family",{"Helvetica",12}]]
Display["!psfix -land -stretch > julia.ps",JuliaArray[0.0+0.0*I,10,2,0.3]];
```

223

Julia sets of the quadratic family
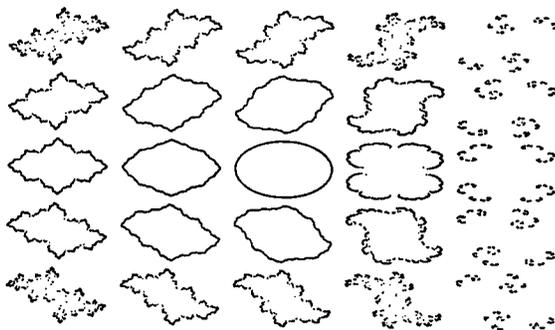
Lemma 6.1 *On the space of probability measure on* **C**, *the map* ϕ^* *has the unique attractor* μ_E *and* $(\phi^*)^n(\mu) \to \mu_E$ *for all probability measures* μ *on* **C**.

Proof. For $n \to \infty$, and $E \neq 0$ the images of any Dirac measure is converging to μ_E in the weak topology [Bro 65] [Lju 83]. (For the polynomial map $z \mapsto z^2 + E$ only ∞ is an exceptional point if $E \neq 0$). In the given references we could only find this result for Dirac measures μ. The result for general measures however follows immediately: given any continuous bounded function f on **C**. We want to show that for $n \to \infty$

$$< f, (\phi^*)^n \mu > \to < f, \mu_E > \ .$$

Define $g_n(z) = < f, (\phi^*)^n \delta_z >$. These measurable functions are bounded by $||f||_\infty$ and the sequence g_n converges pointwise to $< f, \mu_E >$ because $(\phi^*)^n \delta_z$ are converging to μ_E in the weak operator topology. But we have also

$$< f, (\phi^*)^n \mu > = < g_n, \mu > \ .$$

(One checks this first for Dirac measures μ.) Lebesgue's dominated convergence theorem assures the convergence of

$$< g_n, \mu > = \int g_n(z) \, d\mu(z)$$

to the constant $<< f, \mu_E >, \mu > = < f, \mu_E >$. We have therefore also

$$< f, (\phi^*)^n \mu > \to < f, \mu > \ .$$

We have shown that every measure μ is attracted by μ_E under the dynamics of ϕ^*. $\square$

7 Existence of the attractor

We return now to the *renormalisation maps* $\Phi_E^{\pm}$ acting on the Banach space $\mathcal{L}$ of Jacobi operators defined over the von Neumann Kakutani system.

Theorem 7.1 *For large enough real $-E$, the maps Φ_E^+, Φ_E^- form a hyperbolic iterated function system on an open non-empty set $\mathcal{V}_E \subset \mathcal{L}$. Each element in the attracting Cantor set consists of limit-periodic operators and has the spectrum on the Julia set J_E.*

Proof. *Fixing a neighborhood of the Julia set.* For large $-E$, there exists an open ϕ_E invariant neighborhood V_E of J_E that does not contain E and a constant $\mu < 1$ such that for $z \in V_E$, $|\frac{d}{dz}\phi_E(z)| \leq \mu$.

Fixing an open set of Jacobi operators. The open set

$$\mathcal{V}_E = \{ L \in \mathcal{L} \mid \sigma(L) \in V_E, \text{ L has positive definite mass } \}$$

is not empty, because we can take any selfadjoint Jacobi operator $L \in \mathcal{L}$ with positive definite mass and normalize it with suitable constants $\alpha > 0$, and $\beta \in \mathbb{R}$, to get

$$\sigma(\alpha L + \beta) \in V_E .$$

There exists a whole neighborhood $\mathcal{V}_E$ of $\alpha L + \beta$ such that $K \in \mathcal{V}_E$ has positive definite mass and satisfies $\sigma(K) \subset V_E$.
The renormalisation maps have a common inverse. The inverse of $\Phi_E^{\pm}$ is given by

$$\mathcal{T}_E D^{(\pm)} = \psi((D^{(\pm)})^2) + E .$$

The two renormalisation maps have no common image. For large enough $|E|$ and $L \in \mathcal{V}_E$,

$$\Phi_E^+(L) \neq \Phi_E^-(L)$$

because $\Phi_E^+(L) = \Phi_E^-(L)$ would imply $m_E^+ = m_E^-$ and E would be an eigenvalue. This is not possible, since we have assumed E to be outside V_E.

Decomposition of the renormalisation maps. In order to estimate the Fréchet derivative of $\phi_E^{\pm}$, we make the decomposition

$$\Phi_E^{\pm} = \varphi_E \circ \eta_E^{\pm} \circ \theta ,$$

where $\varphi_E : L \mapsto +\sqrt{L-E}$ and $\theta(L) = L \oplus L \in \mathcal{X}$ is the unique operator which satisfies

$$\psi(\theta(L)) = L, \ \psi(\theta(L(T))) = L$$

and

$$\eta_E^{\pm}(L \oplus K) = L \oplus BT_E^{(\pm)}K .$$

225

The mapping $\varphi_E(K) = \sqrt{K - E}$ is defined on the manifold $\eta_E^\pm \circ \theta(\mathcal{L}) \subset \mathcal{X}$.

The derivative of θ. Since $\theta : \mathcal{L} \mapsto \mathcal{X}$ is a linear isometry, we have $||\frac{d}{dL}\theta|| = ||\theta|| = 1$.

The derivative of $\eta_E^\pm$. We know by Proposition 4.2 that for $|E| \to \infty$

$$||\frac{d}{dL}BT_E^{(\pm)}(L)|| \to 1 \, ,$$

uniformly for $L \in \mathcal{V}_E$. We obtain therefore also

$$\lim_{E \to -\infty} ||\frac{d}{dL}\eta_E^\pm(L)|| = 1 \, .$$

The derivative of $\varphi_E^\pm$. The derivative of the map $L \mapsto \sqrt{L - E}$ from $\eta_E^\pm \circ \theta(\mathcal{L})$ to $\mathcal{X}$ is given by

$$\frac{d}{dL}\varphi_E^\pm(L)U = \frac{1}{2}(L - E)^{-1/2}U \, .$$

Because $|||(L - E)^{-1/2}||| \to 0$, for $|E| \to \infty$, we get

$$\lim_{-E \to \infty} ||\frac{d}{dL}\varphi_E^\pm(L)|| = 0 \, .$$

The derivative of $\Phi_E^\pm : \mathcal{V} \to \mathcal{L}$. It follows from the four previous steps that for $|E| \to \infty$

$$||\frac{d}{dL}\Phi_E^\pm|| = ||\frac{d}{dL}(\varphi_E \circ \eta_E^\pm \circ \theta)|| \leq ||\frac{d}{dL}(\varphi_E)|| \cdot ||\frac{d}{dL}\eta_E^\pm|| \cdot ||\frac{d}{dL}\theta|| \to 0 \, .$$

The hyperbolic iterated function system. We have checked the existence of a common inverse, the contraction property and the disjointness of the two maps $\Phi_E^\pm$. The Lemma of Barnsley 5.1 is thus applicable and we have shown that for large enough $-E$, a hyperbolic iterated function system has a unique attractor $\mathcal{J}_E$ in $\mathcal{L}$.

Limit periodicity. We can define the iterated function system on an open set of the fiber bundle of random Jacobi operators on the group $\mathcal{U}$. If we begin with a periodic system T, every Jacobi operator L over T is periodic. Under the iteration of the renormalisation maps, the periodic operators $\Phi_E^\varphi L$ converge in norm to a point of the attractor. Such a point is a limit periodic operator. $\qquad\qquad\square$

8   The renormalisation limit

We will assume in this paragraph that $-E$ is large enough such that the maps Φ_E^+, Φ_E^- are defined and a form a hyperbolic iterated function system. We first fix some notation.

Different notation for X and Ω. The topological space $\Omega = \{1, -1\}^{\mathbb{N}}$ (labelling the renormalisation sequence) and the topological group $X = \{0, 1\}^{\mathbb{N}}$ (the dyadic group) are trivially homeomorphic via the change of alphabet $1 \mapsto 0, -1 \mapsto 1$. We will use the notation $\omega = \omega(x)$ or $x = x(\omega)$ if x and ω correspond to each other by this change of alphabet. The addition in Ω is the group operation inherited from the group X. We also use the notation $x_0 = (0, 0, 0, \ldots)$ for the zero in X and $x_1 = (1, 0, 0, \ldots)$ for the unit in X.

The fixed points of $\Phi_E^{\pm}$. Call $L_E^{(\pm)}$ the unique fixed point of $\Phi_E^{\pm}$ in $\mathcal{J}_E$. By definition, we have

$$L_E^{(+)} = \Phi_E^{\omega(x_0)} K \;,\; L_E^{(-)} = \Phi_E^{-\omega(x_0)} K$$

where K is any operator in $\mathcal{V}_E$.

The group structure on the attractor $\mathcal{J}_E$. We use the notation

$$\Phi_E(\omega) = \Phi_E^{\omega} K \;,$$

where $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$ and $K \in \mathcal{V}_E$ is arbitrary. The homeomorphism $x \mapsto \omega(x)$ brings the group structure of X to Ω and so to $\mathcal{J}_E$ by

$$\Phi_E(\omega) \Phi_E(\eta) = \Phi_E(\omega + \eta) \;.$$

The group of all the translates of $L_E^{(+)}$.
Call T_x the group translation on X defined by $T_x(y) = x + y$ and by T_ω the analogous group translation on Ω. The group X is acting also on $\mathcal{L}$ by $L \mapsto L(T_x)$. The orbit

$$\mathcal{O}(L_E^{(+)}) = \{L_E^{(+)}(T_x) \mid x \in X\}$$

of $L_E^{(+)}$ is a compact set in $\mathcal{L}$ which becomes with the operation

$$L_E^{(+)}(T_x) \circ L_E^{(+)}(T_y) = L_E^{(+)}(T_{x+y})$$

a compact topological group.

Theorem 8.1 *The two sets $\mathcal{J}_E$ and $\mathcal{O}(L_E^{(+)})$ coincide and are as groups isomorphic by the isomorphism*

$$\Phi_E(\omega) = L_E^{(+)}(T_{x(\omega)}) \;.$$

The proof of this theorem needs some preliminary steps. Denote by ρ the involution $(T, L) \mapsto (T^{-1}, L(T^{-1}))$ on the bundle of random Jacobi operators.

Lemma 8.2

$$a) \quad \rho \circ \Phi_E^+ = \Phi_E^- \circ \rho \;,$$
$$b) \quad L^{(-)}(T^k) = L^{(+)}(T^{k-1}), \; \forall k \in \mathbb{Z} \;.$$

Proof.

a) Given (T, L). We write $m^{\pm}_{(T,L)}$ for the Titchmarsh-Weyl functions of the operator L. Using the definitions of these functions, we obtain

$$
\begin{aligned}
n^{+}_{(T,L)} &= m^{-}_{(T^{-1},L(T^{-1}))} , \\
m^{+}_{(T,L)} &= n^{-}(T)_{(T^{-1},L(T^{-1}))}
\end{aligned}
$$

which is equivalent to

$$
d^{+}_{(T,L)}(S^{-1}) = d^{-}_{(T^{-1},L(T^{-1}))} .
$$

Because $\Phi^{\pm}_{E}(L) = d^{(\pm)}\sigma + d^{(\pm)}(S^{-1})\sigma^{*}$, this can be rewritten as

$$
\rho \circ \Phi^{+}_{E}(L) = \Phi^{-}_{E} \circ \rho(L) .
$$

b) Using a), we get

$$
\Phi^{-}_{E}(\rho L^{(+)}) = \Phi^{-}_{E} \circ \rho(L) = \rho \circ \Phi^{+} L^{(+)} = \rho L^{(+)}
$$

which shows that $\rho L^{(+)}$ is the fixed point of Φ^{-}. Therefore $L^{+}(T^{-1}) = L^{-}$. The claim follows by applying T^{k} on both sides. $\qquad\square$

Define $X_{00} = X = [0,1]$ and for $i = 0, 1, \ldots, 2^{k} - 1$ the sets

$$
X_{ki} = [i \cdot 2^{-k}, (i+1) \cdot 2^{-k}] .
$$

Given $L \in \mathcal{J}_{E}$, we define inductively

$$
L_{(0)} = L, \; L_{(k+1)} = L^{2}_{(k)} + E .
$$

Each operator $L_{(k)}$ is in $\mathcal{X}$ and can be written as

$$
L_{(k)} = d_{k}\tau^{(2^{k})} + d_{k}(T^{2^{-k}})\tau^{(-2^{k})} ,
$$

with $d_{k} \in L^{\infty}(X, \mathbb{C})$.

Lemma 8.3

$$
\begin{aligned}
&a) \quad L^{(+)}(T^{j}) \in \mathcal{J}_{E}, \; \forall j \in \mathbb{Z} , \\
&b) \quad \mathcal{O}(L^{(+)}) \subset \mathcal{J}_{E} .
\end{aligned}
$$

Proof.

a) The spectrum of $L_{(k)}$ is the Julia set J_{E} because this is true for $L_{(0)}$ and by the spectral theorem inductively for each $L_{(k)}$. Each $L_{(k)}$ is a random Jacobi operator over the dynamical system $(X, T^{2^{k}}, m)$ which has the 2^{k} measurable invariant sets X_{ki}. Each map $T^{(2^{k})}$ restricted to such a set X_{ki} is ergodic. This means that the

operator $L_{(k)}$ restricted to X_{ki} is an ergodic random Jacobi operator over the dynamical system $(X, T^{(2^k)}, m)$. By definition, $T_E^n(L(T^i))$ is defined as $L(T^i)_{(k)}$ restricted to X_{k0} and this is the same operator as $L_{(k)}$ restricted to X_{ki}. The spectrum of $T_E^n(L(T^i))$ is therefore also the Julia set J_E. Since $T_E^n(L(T^i)) \in \mathcal{V}_E$, we conclude that $L(T^i)$ is in the image of some Φ_E^ω, where ω is a word of length n. Because this is true for all $n \in \mathbb{N}$, we know that $L(T^i)$ is arbitrarily close to the closed set $\mathcal{J}_E$ and therefore $L \in \mathcal{J}_E$.

b) Follows immediately from $a)$ and the fact that $\mathcal{J}_E$ is closed. $\qquad\square$

Define for $L \in \mathcal{J}_E$ and $k \in \mathbb{N}, k > 0$

$$\omega_k(L) = -\text{sign} \int_{X_{k0}} \log |d_{k-1}|) \, dm(x) .$$

We call ω also the *code* for L. The next lemma justifies this name.

Lemma 8.4 *For all $L \in \mathcal{J}_E$, the sequence $\omega : k \mapsto \omega_k(L) \in \Omega$ satisfies $L_E(\omega) = L$.*

Proof. We know by definition that $T_E^k(L)$ is $L_{(k)}$ restricted to X_{k0}. For $x \in X_{k0}$,

$$|d_{k-1}(x)| = \sqrt{|m_{k-1}^\pm(x)|} ,$$

where $m_{k-1}^\pm$ are the Titchmarsh-Weyl functions of $T_E^{k-1}(L)$. The upper-script $\pm$ in $m_{k-1}^\pm$ is in correspondence with the fact that $T_E^{k-1}(L)$ is in the image of $\Phi^\pm$. We see that $T_E^{k-1}(L)$ is in the image $\Phi^{\omega_k(L)}$ for all k and so $L_E(\omega) = L$. $\qquad\square$

The involution ρ restricted to $\mathcal{J}_E$ is also an involution on Ω. This involution is a replacement of the two letters in the alphabet:

Lemma 8.5

$$
\begin{aligned}
a) & \quad \omega(\rho(L)) = -\omega(L) , \\
b) & \quad \omega(L^+(T^n)) = -\omega(L^-(T^{-n})) .
\end{aligned}
$$

Proof.
a) Lemma 8.3 implies
$$\rho \circ \Phi_E^\omega \circ \rho = \Phi_E^{-\omega}$$
for all $\omega \in \Omega$. Let L have the code ω such that $L = \Phi_E(\omega)$. It follows from

$$\rho L = \rho \Phi_E(\omega) = \rho \circ \Phi_E^\omega(K) = \Phi_E^{-\omega} \rho(K) = \Phi_E(-\omega)$$

that $\rho(L)$ has the code $-\omega$.

b) can be deduced from $\rho(L^{(-)}(T^{-n})) = L^{(+)}(T^n)$ and a). $\qquad\square$

Proof of the theorem. We know that the orbit $L^{(+)}(T^n)$ of $L^{(+)}$ is contained in $\mathcal{J}_E$. Our aim is to show that $L_E(n \cdot \omega_1) = L_E^{(+)}(T^n)$ for all $n \in \mathbb{Z}$, where $\omega_1 = \omega(x_1) = (-1, 1, 1, 1, \ldots)$ is the unit in the dyadic group. In order to determine the action of T on the subset of $\Omega = \{-1, 1\}^{\mathbb{N}}$ labeling the points of $\mathcal{O}(L_E^{(+)}) \subset \mathcal{J}_E$, we define the matrix

$$M_{ki} := \omega_k(L^{(+)}(T^i)), k > 0, i \in \mathbb{Z} .$$

Knowing this matrix, we can read of the code $\omega = \{M_{ki}\}_{k \in \mathbb{N}}$ of $L^{(+)}(T^i) = L^+(\omega)$ from the columns of the matrix M.

We build up the matrix M beginning at the top first row and determine inductively one row after the other. The first row is given by

$$M_{1i} = \omega_1(L^{(+)}(T^i)) = (-1)^i$$

because $T(X_{01}) = X_{11}$ and $T(X_{11}) = X_{01}$ and

$$\text{sign}(\int_{X_{10}} \log|d(x)| \, dm(x)) = -\text{sign}(\int_{X_{11}} \log|d(x)| \, dm(x)).$$

Claim: if

$$(\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)$$

is the k-th row of M then the $(k+1)$-th row of M is

$$(\ldots, -a_{-2}, a_{-2}, -a_{-1}, a_{-1}, -a_0, a_0, -a_1, a_1, -a_2, a_2, \ldots) .$$

Proof. The $(k+1) - th$ row can be constructed from the $k - th$ row using

$$\omega_{k+1}(L^{(+)}(T^{2i})) = \omega_k(L^{(+)}(T^i)) , \tag{1}$$

$$\omega_k(L^{(+)}(T^i)) = -\omega_k(L^{(+)}(T^{-(i+1)})) . \tag{2}$$

Proof of formula (1): we know $\Phi^+ L^+ = L^+$ and that d_{k-1} restricted to X_{ki} is equal to d_k restricted to $X_{k+1,2i}$. Therefore

$$
\begin{aligned}
w_{k+1}(L^{(+)}(T^{2i})) &= -\text{sign} \int_{X_{k+1,0}} \log|d_k(T^{2i})| \, dm(x) \\
&= -\text{sign} \int_{X_{k+1,2i}} \log|d_k| \, dm(x) = -\text{sign} \int_{X_{k,i}} \log|d_{k-1}| \, dm(x) \\
&= -\text{sign} \int_{X_{k,0}} \log|d_{k-1}(T^i)| \, dm(x) = w_k(L^{(+)}(T^i)) .
\end{aligned}
$$

Formula (2) follows from

$$w_k(L^{(+)}(T^i)) = -w_k(L^{(-)}(T^i)) = -w_k(L^{(+)}(T^{-i+1}))$$

which is a consequence of Lemma 8.5 b) and Lemma 8.2 b). We have confirmed the following picture of the matrix M.

$$M = \begin{pmatrix} \cdot & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & \cdot \\ \cdot & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & \cdot \\ \cdot & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

The matrix M has the property that if

$$b = (b_1, b_2, b_3, \ldots)$$

is the $i - th$ column of the matrix M then the addition in the dyadic group

$$b + \omega_1 = (b_1, b_2, b_3, \ldots) + (-1, 1, 1, \ldots)$$

gives the $(i+1) - th$ column. To prove this, one checks that if a matrix $\tilde{M}$ is built up the rule for the columns, then $M = \tilde{M}$.

We know now the action of T on $\mathcal{J}_E$:

$$T^n \Phi_E(\omega) = \Phi_E(\omega + n \cdot \omega_1)$$

Because the orbit of $L^{(+)}$ can get arbitrarily close to every point in $\mathcal{J}_E$, it follows that $\mathcal{O}(L_E^{(+)}) = \mathcal{J}_E$. Moreover $\Phi_E(\omega) = L_E^{(+)}(T_{x(\omega)})$ holds for all $\omega \in \Omega$ because the relation holds on a dense orbit of the monothetic group X. $\qquad \square$

9 The density of states

The functional calculus for a normal element K in the C^* algebra $\mathcal{X}$ defines $f(K)$ for a function $f \in C(\Sigma(K))$. The mapping $f \mapsto \text{tr}(f(K))$ is a bounded linear functional on $C(\Sigma(K))$ and by Riesz representation theorem, there exists a measure dk on $\Sigma(K)$ with $\text{tr}(f(K)) = \int f(E) \, dk(E)$. This probability measure dk is called the *density of states* of K. The density of states can also be defined for not normal operators. By the analytic functional calculus, one can define $f(K)$ for any function, which is holomorphic in a neighborhood of the spectrum of K. Such functions form a dense linear subspace H in the Banach space of continuous functions on $\sigma(K)$. The functional $f \mapsto \text{tr}(f(K))$ is bounded on H and can by Hahn-Banach be extended in a unique way to a bounded linear functional on $C(\sigma(K))$ having the same norm. With Riesz representation theorem, one obtains again a measure on $\sigma(K)$, the density of states.

The next lemma gives the relation between the density of states of the renormalized Jacobi operator $\Phi_E^{\pm} L$ and the density of states of the operator $L = d\tau + d(T^{-1})\tau^*$.

Lemma 9.1

$$dk(\Phi_E^\pm L) = \phi_E^* \, dk(L) .$$

Proof.

Assume first that $|X| = N$ is finite so that L is a N-periodic Jacobi matrix. Denote by $\tilde{dk}(L)$ the Dirac measure $\frac{1}{N} \sum_{i=1}^N \delta(\lambda_i)$, where λ_i are the eigenvalues of L acting on the finite dimensional vector space of $N-$ periodic sequences in $l^2(\mathbb{Z})$. The $2N \times 2N$ periodic Jacobi matrices $D^{(\pm)} = \Phi_E^\pm(L)$ have the eigenvalues $\{\pm\sqrt{\lambda_i - E}\}_{i=1}^N$. This implies

$$\tilde{dk}(\Phi^\pm(L)) = \phi_E^* \, \tilde{dk}(L) .$$

In the general case, let $L^{(N)}(x)$ be a N-periodic approximation of $L(x)$ such that for $-N/2 \le i,j < N/2$,

$$[L^{(N)}(x)]_{i+N,j+N} = [L^{(N)}(x)]_{ij} = [L(x)]_{ij} .$$

By the Lemma of Avron-Simon (see [Cyc 87]), $\tilde{dk}(L^{(N)}(x)) \to dk(L)$ for almost all $x \in X$. The claim follows from the fact that $\Phi_E^\pm(L^{(N)}(x)) \to \Phi_E^\pm(L(x))$ for $N \to \infty$. the density of states for $\Phi_E^\pm L$ is equal to $\phi_E^* \, dk(L)$. $\qquad\square$

Proposition 9.2 *The density of states of $\Phi(\omega)$ is the unique equilibrium measure μ_E on J_E.*

Proof. We know that for any probability measure μ on $\mathbb{C}$ and $n \to \infty$

$$(\phi_E^*)^n \mu \to \mu_E$$

holds, where μ_E is the unique equilibrium measure on the Julia set J_E. Let $\mu = dk$ be the density of states of a Jacobi operator L. Lemma 9.1 implies that

$$dk(\Phi_E^{\omega_1} \circ \ldots \circ \Phi_E^{\omega_n}(L)) = (\phi_E^*)^n(\mu) \to \mu_E ,$$

for all $\omega \in \Omega$. We know therefore that the density of states of $\Phi_E^\omega(L)$ must coincide with μ_E. $\qquad\square$

Lemma 9.3 *Every operator $L \in \mathcal{J}_E$ has mass $M = 1$.*

Proof. Because by Theorem 8.1 every element in $\mathcal{J}_E$ has the same mass, it is enough to show that the fixed point $L^{(+)} = d^+\tau + d^+(T^{-1})\tau^*$ has mass $M(L^{(+)}) = 1$. We calculate

$$
\begin{aligned}
\int_X \log|d^+| \, dm &= \frac{1}{2} \int_{X_1} \log|m^+| \, dm + \frac{1}{2} \int_{X_1} \log|n^+| \, dm \\
&= \frac{1}{2} \int_{X_1} \log|(d^+)^2| \, dm = \int_{X_1} \log|d^+| \, dm \\
&= \frac{1}{2} \int_X \log|d^+| \, dm
\end{aligned}
$$

232

which implies $\log(M(L^{(+)})) = \int_X \log|d^+|\, dm = 0.$ □

The *potential theoretical Green function g* of a compact set $K \subset \mathbb{C}$ is defined to be a function $g : \mathbb{C} \to \mathbb{R}$ which is harmonic on $\mathbb{C} \setminus K$, vanishing on K and such that $g - \log(z)$ is bounded near $z = \infty$. The Green function is existing for the Julia set J_E of τ_E (see [Mil 91]).

Proposition 9.4 *The Lyapunov exponent $\lambda(E)$ of an operator $L \in \mathcal{J}_E$ is equal to the Green function $g(E)$ of the Julia set J_E. The Lyapunov exponent $\lambda(E)$ of $L \in \mathcal{J}_E$ is vanishing exactly on the spectrum J_E of L.*

Proof. The density of states of the Jacobi operator L_E is the equilibrium measure (Proposition 9.2) and gives the capacity 1 for the Julia sets (see [Bro 65]). The integral

$$u(E) = -\lambda(E) = -\int \log|E - E'|\, dk(E')$$

is called the *conductor potential*. The relation between the conductor potential, the Green function and the capacity is given by the formula $g(E) = -u(E) + \log(\gamma)$, where γ is the capacity. See ([Tsu 58] Theorem III,37). Because by Brolin, the capacity of a Julia set J_E satisfies $\gamma(J_E) = 1$, it follows that the Lyapunov exponent $\lambda(E)$ is equal to the Green function g of the Julia set J_E.
The potential theoretical Green function $g(z) = \lambda(z)$ is by definition vanishing on the Julia set. □
Because ∞ is a super-attractive fixed point of the polynomial map $\tau_E(z) = z^2 + E$, there exist new coordinates $w = \phi(z)$, near ∞ satisfying

$$\phi \circ \tau_E \circ \phi^{-1}(w) = w^2 .$$

The function ϕ is called the *Böttcher function* of the polynomial τ_E. (See for example [Mil 91].)

Corollary 9.5 *The Böttcher function ϕ for the polynomial τ_E satisfies $\phi(E) = \det(L - E)$ for E in a neighborhood of ∞. One has*

$$\det(L - (z^2 + E)) = \det(L - z)^2 .$$

Proof. The Green function g can be expressed as $g(z) = \log|\phi(z)|$. It follows from

$$|\phi(z)| = \exp(\lambda(z)) = |\exp(-w(z))| = |\det(L - z)|$$

that $\phi(z) = \det(L - z)$. The known identity $g(z) = g(z^2 + E)/2$ for the Green function g gives

$$\det(L - (z^2 + E)) = \det(L - z)^2 .$$

□

233

We end this section with a remark about the possible values of the integrated density of states, the *gap labelling*.

It follows from the structure of the Julia set that for the limit-periodic Jacobi operators in $\mathcal{J}_E$, there is an obvious gap labelling:

Proposition 9.6 *The integrated density of states takes exactly the values $l \cdot 2^{-n}$ with $n \in \mathsf{N}$ and $0 \leq l \leq 2^n$.*

Proof. We know from Proposition 9.2 that the density of states is the equilibrium measure on J_E and so a balanced measure. The inverse of the map τ_E^n has 2^n branches $\phi_E^{(n,j)}$ labelled by $0 \leq j \leq 2^n$. By the definition of balanced, each of the sets $\phi_E^{(n,j)}(J_E)$ has measure 2^{-n}.

If a gap of $L \in \mathcal{J}_E$ has l sets $\phi_E^{(n,j)}(J_E)$ to the left and $2^n - l$ such sets to the right, the integrated density of states of this gap is $l \cdot 2^{-n}$. $\qquad\square$

10   Generalisations

We discuss shortly some generalizations or extensions:

Complex values E.
Because an attractor of a hyperbolic iterated function system is structurally stable, the renormalisation can be extended to an open neighborhood of some set $\{E \in \mathsf{R} \mid E < -R\} \subset \mathsf{C}$. All the results about the density of states, the Green function etc. hold also here.

Julia sets of the anti-holomorphic quadratic map.
Operators with spectra on the Julia sets of

$$z \mapsto \overline{\tau_E}(z) = \bar{z}^2 + E$$

can be obtained by replacing $\Phi_E^\pm$ by $\overline{\Phi_E^\pm}$. The parameter set of $\overline{\tau}_E$ analogue to the Mandelbrot set is called the *Mandelbar set.*

Nonrandom Jacobi matrices.
The renormalisation scheme can also be done for general Jacobi matrices which have not to be random. This is however not so interesting because the same attractor consisting of random Jacobi matrices is obtained. An advantage of doing the renormalisation for random operators is that we get immediately the dynamical system over which the Jacobi operators are defined in the limit.

Jacobi operators on the strip.
The renormalization of Jacobi operators which we proposed can be generalized to

random Jacobi operators on the strip (see [Kot 88]), where the entries of the Jacobi matrix are finite-dimensional matrices. In the limit of renormalisation, these operators factorize into a direct sum of one-dimensional operators.

Higher dimensional Laplacians.
A direct generalization to higher dimensional random Laplacians is not possible without further modifications. The reason is that for a Laplacian

$$L = D^2 + E = \sum_i a_i \tau_i + a_i (T_i^{-1}) \tau_i^* + b \ ,$$

the connection $a_i \tau_i$ has to satisfy the *zero curvature condition*

$$[a_i \tau_i, a_j \tau_j] = (a_i a_j (T_i) - a_j a_i (T_j)) \tau_i \tau_j = 0$$

while the connection $d_i \sigma_i$ belonging to the *Dirac operator*

$$D = \sum_i d_i \sigma_i + d_i (S_i^{-1}) \sigma_i^*$$

must satisfy the anti-commutation relation

$$\{d_i \sigma_i, d_j \sigma_j\} = (d_i d_j (S_i) + d_j d_i (S_j)) \sigma_i \sigma_j = 0, \ i \neq j$$

which prevents a further factorization of D. Dirac operators play a role when doing isospectral deformations of higher dimensional Laplacians.

Operators with spectra on random Julia sets.
   The renormalization can be generalized in another way. Instead of taking a constant energy E, we can take a space dependent function $E(x)$ near a large constant E_0. We get again the same type of result as before. There exists an attractor above the von Neumann Kakutani system which consists of operators having the spectrum on *random Julia sets* (compare [For 91]).

The projected renormalisation on the complex plane is then no more the complex map

$$z \mapsto z^2 + E$$

but becomes a *random quadratic map* given by the skew product

$$(x, z) \mapsto (Tx, z^2 + E(x)) \ .$$

We can interpret this in the way that we have a fiber bundle with fiber C and the function $E(x)$ describes the quadratic map on the fibers.
Operators with spectra on Julia sets of higher degree polynomials.
Define for $a, b \in$ R the map $\Phi_{a,b} L = aL + b$. Given three vectors

$$E = (E_1, E_2, \ldots, E_d), \ a = (a_1, \ldots, a_d) \ , b = (b_1, \ldots, b_d)$$

235

in $\mathbb{R}^d$, we form the maps

$$\Phi_{E,a,b}^{\pm} = \Phi_{E_d}^{\pm} \circ \Phi_{a_d,b_d} \circ \ldots \circ \Phi_{E_1}^{\pm} \circ \Phi_{a_1,b_1} \ .$$

Define the linear polynomials $\tau_{a_i,b_i}(z) = a_i z + b_i$. If the polynomial

$$\tau_{E,a,b} = \tau_{E_d} \circ \tau_{a_d,b_d} \circ \ldots \circ \tau_{E_1} \circ \tau_{a_1,b_1}$$

has a real Julia set $J_{E,a,b}$ which is sufficiently expansive, we get operators having the spectrum on the Julia set $J_{E,a,b}$ of $\tau_{E,a,b}$.

More general values of E.
We don't know, how far one can explore the renormalization for general complex parameters E. For complex E, one has to deal with not normal operators. Another problem is that for smaller values of $|E|$, the norm of $\Phi^n(L)$ can blow up under the renormalisation steps even if the spectrum converges to the Julia set. The reason is that E can get closer and closer to the auxiliary spectrum of the matrices. Since the Titchmarsh-Weyl functions are singular at the auxiliary spectrum, the norm can get large or explode.
We think however that for all E outside the Mandelbrot set, there should exist a hyperbolic iterated function system having a Cantor set as an attractor. The scheme can not work for general $E \in \mathbb{C}$ because for $E = 0$, a fixed point of $D \mapsto \Phi_0^+ D = \psi(D^2)$ is not limit-periodic. Given a fixed point

$$D = d\tau + d(T^{-1})\tau^*$$

of Φ^+. The sequence d_n satisfies

$$\begin{aligned}
d_{2n+1}^2 &= -d_{2n}^2 \ , \\
d_n^2 &= d_{2n}^2 d_{2n-1}^2 \ .
\end{aligned}$$

We are free to choose $d_0^2 = -d_1^2 = \lambda \neq 0$. The second equation gives for $n = 0, n = 1$

$$d_2^2 = 1, d_{-1}^2 = 1 \ .$$

The other values of d_n are then determined inductively.

Lemma 10.1 *The sequence d_n is not periodic.*

Proof. We are free to choose $d_0^2 = -d_1^2 = \lambda \neq 0$. The second equation gives for $n = 0, n = 1$

$$d_2^2 = 1, d_{-1}^2 = 1 \ .$$

The other values are then determined. We take now $\lambda = 1$ and require $a_n := d_n^2$ to assume values in $\{-1, 1\}$. We want to show now, that the sequence a_n can't be periodic. We rewrite the fixed point condition as

$$\begin{aligned}
a_{2n+1} &= -a_{2n} \ , \\
a_{2n} &= a_m a_{2n-1} \ .
\end{aligned}$$

All the subsequent values are defined by $a_0 = 1$ and this recursion. We get

$$a_0 = 1, a_1 = -a_0 = -1, a_2 = a_1^2 = 1$$

and in general

$$a_{2n} = a_0 a_1 a_2 \cdots a_n (-1)^n .$$

For even n because of $a_0 a_1 = -1, \ldots, a_{n-2} a_{n-1} = -1$ also

$$a_{2n} = (-1)^{n/2} a_n .$$

Because $a_4 = -1$ this gives $a_{2^n} = -1$ for all $n \geq 2$. Because $a_0 = 1$ and $a_{2^n} = -1$, the period 2^n is excluded.

Form the hull X of all the iterates of the sequence $\{a_n\}_{n \in \mathbb{Z}}$ in $l^\infty(\mathbb{Z})$ equipped with the weak * (=product) topology. The shift transformation is continuous and has an ergodic invariant probability measure m. If the sequence a_n would be periodic, we would get a finite dynamical system, a cyclic permutation of a finite set. The continuous function $a(x) = x_0$ on X defines the random Jacobi operator $D = d\tau + d(T^{-1})\tau^*$ with $d = \sqrt{a}$ taking values in $\{1, i\}$. If we make the renormalisation of the dynamical system together with the operator, the operator doesn't change but the dynamical system converges to the von Neumann-Kakutani system (X, T, m). If the sequence a_n was periodic with period p, we would get a continuous function a on the dyadic group of integers X, which is T invariant and such that $a(T^p x) = a(x)$ for all $x \in X$. This is only possible for the period $p = 2^n$ which has been excluded before. $\square$

Because it takes only finitely many values, $d_n = d(T^n x)$ can't be almost periodic. If we do the renormalisation numerically for values E approaching 0, there are entries in the Jacobi matrix which begin to blow up.

The case $E = -2$ is interesting because it describes a situation, where the energy E is at the boundary of the Mandelbrot set. The spectrum $[-2, 2]$ is then absolutely continuous with respect to the Lebesgue measure and the corresponding operator is the free Laplacian. Numerically, the renormalisation maps work for all real values $E < -2$ and the attractor $\mathcal{J}_E$ approaches a single point in the limit $E = -2$.

11 Matrix models

Given any random operator $L \in \mathcal{X}$ over a dynamical system (X, T, m) having the density of states dk. The *energy* $I(L)$ of L is defined as

$$I(L) = \int_{\mathbb{R}} \int_{\mathbb{R}} \log |E' - E| \, dk(E') \, dk(E) .$$

In the case when L is a random Jacobi operator, one can write the energy with the *Thouless formula*

$$I(L) = \log(M) + \int_{\mathbb{R}} \lambda(E) \, dk(E) ,$$

where λ denotes the Lyapunov exponent. We write

$$
\begin{aligned}
\exp(I(L)) &= \exp(\int_{\mathbb{R}} \text{tr}(\log |L - E|) \, dk(E)) \\
&= \exp(\int_{\mathbb{R}} \log(\det |L - E|) \, dk(E)) \\
&= \text{tr}(\log(\det |L - L'|)) \\
&= \det'(\det |L - L'|) =: \Delta(L) ,
\end{aligned}
$$

where det and tr are the determinant and trace with respect to L and $\det'$ and tr' are the determinant and trace with respect to $L' = L$. From the last formula one would expect that $I(L)$ is always zero. However, as we will see below, there are examples of random operators which have positive Lyapunov exponents for all $E \in \mathbb{C}$ leading to positive energy. For finite matrices L with eigenvalues $\lambda_1, \ldots, \lambda_n$, we would get

$$\exp(I(L)) = \Pi_{i,j} |\lambda_i - \lambda_j|$$

which is of course vanishing. If we take out the self-interaction terms $(\lambda_i - \lambda_i)$, we end with

$$\exp(\tilde{E}(L)) = \Pi_{i<j} (\lambda_i - \lambda_j)^2 ,$$

the square of the *van der Monde determinant*. The problem of self-interaction in the finite dimensional case does not appear in the random Jacobi operators. If the mass is positive definite, the density of states dk is well behaved in the sense that

$$\int \log |E - E'| \, dk(E')$$

is finite for all $E \in \mathbb{R}$. The energy $I(L)$ is a generalization of the squared van der Monde determinant. This determinant appears also as a Jacobean in the so called *one matrix model* (See [Mar 91] [Alv 91] [Alv 91a] [Ger 91]). The number $\exp(I(L))$ is called the *capacity* of the set $\text{spec}(L)$ if the density of states measure dk is maximizing $\exp(I(\mu))$ among all probability measures μ on $\text{spec}(L)$, where

$$I(\mu) = \int_{\mathbb{R}} \int_{\mathbb{R}} \log |E - E'| \, d\mu(E) \, d\mu(E')$$

is the energy of the measure μ. Interesting are the measures which give the capacity. They are called *equilibrium distributions*.

In the finite dimensional case ($|X| < \infty$), which leads to periodic Jacobi matrices, the energy $I(L)$ is zero because the Lyapunov exponent is zero on the spectrum of L. The capacity is then always 1. In infinite dimensions, it is possible to find examples

with $I(L) > 0$ because the Lyapunov exponents can be positive for all $E \in \mathbf{C}$. This can happen even in the almost periodic case as an example of M.Herman shows: Take $(X, T, m) = (\mathsf{T}, x \mapsto x + \alpha, dx)$ with irrational α. Let

$$b(x) = \gamma \cdot \cos(2\pi x)$$

and $L = \tau + \tau^* + b$. Herman gives the estimate

$$\lambda(E) = -\mathrm{Re}(w(E)) \geq \log(\lambda/2) .$$

Therefore

$$|\det(L - E)| = e^{-Re(w)} = e^{\lambda(E)} \geq \gamma/2 .$$

If the measure μ is the density of states of L, the energy is

$$I(L) = \int \int \log|E - E'| \, d\mu(E') \geq \log(\gamma/2)$$

and the capacity is bigger then $\gamma/2$. We see also that the capacity does not depend continuously on the dynamical system. If α is rational, the capacity of the above operator L is 1 and for irrational α, it is $\geq \gamma/2$.

The operators $L \in \mathcal{J}_E$ constructed in this chapter are examples where the capacity of the spectrum is known to be 1, a result of Brolin [Bro 65]. An obvious question is to determine for which random operators the density of states is the equilibrium measure on the spectrum.

We end this paragraph with a little excurs. In nonperturbative *quantum field theory* have appeared so called *one-matrix models* [Mar 91],[Alv 91]. Given a probability measure μ on the space $M(N, \mathbf{R})$ of real $N \times N$ matrices and a potential

$$V(A) = \sum V_n \mathrm{tr}(A^n)$$

on $M(N, \mathbf{R})$. The aim is to calculate the *partition function*

$$Z = \int e^{-V(A)} \, d\mu(A) .$$

A natural choice is the measure $\mu = \Pi_{ij} \, dA_{ij}$ which can be simplified to

$$d\mu = \Pi_i \, d\lambda_i \Pi_{i<j} (\lambda_i - \lambda_j)^2 \, dU_{ij} ,$$

where

$$UAU^{-1} = \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$$

and U is unitary. Because the potential V is invariant under conjugation, the partition function can be written as

$$Z = \int \Pi e^{V(\lambda)} \Pi_{j<k} (\lambda_j - \lambda_k)^2 \, d\lambda$$

which is the partition function of a *one-dimensional Coulomb gas* of N equal charges (see [Gro 90]).

It is not clear how to make sense of this model when doing the thermodynamic limit $N \to \infty$. A starting point could be the following: take a measure μ in the C^* algebra $\mathcal{X}$. Given a potential $V \in C^\omega(\mathcal{X})$ one gets the partition function

$$Z = \int_{\mathcal{X}} \exp(V(L))\Delta(L) \, d\mu(L) = \int \exp(I(L) + V(L)) \, d\mu(L) \ .$$

The energy $I(L)$ should be interpreted as internal energy and $V(L)$ as the potential of an exterior field.

If the measure μ is sitting on the space of Jacobi operators which satisfy some bound in norm, this partition function is finite. The problem is now to find a natural choice for the measure μ on $\mathcal{L}$. It should be invariant under isospectral deformations in $\mathcal{L}$. Because

$$\exp(S(L)) = \exp(I(L) + V(L))$$

is only dependent on the density of states dk of L, one could take the measure μ on the set of probability measures on R which appear as density of states for random Jacobi operators.

12 Questions

• What is the maximal set in C, where the renormalisation mappings Φ_E make sense? What happens at the boundary of this maximal set?

• What is the limit set of a renormalisation map in the group $\mathcal{F}$ of renormalisations of abstract dynamical systems?

• Is the set of operators with the same density of states like the fixed point $L^+ \in \mathcal{L}_E$ a group?

More precisely: is $\mathrm{Iso}(L^{(+)})$ a topological group with respect to the weak operator topology such that $T : L(x) \mapsto L(Tx)$ is a group translation on this group? Can two points in $\mathrm{Iso}(L^{(+)})$ be connected by a Toda orbit? The high symmetry of the constructed operators in $\mathcal{J}_E$ could make the determination of the isospectral set easer. One could guess that the embedding of the dyadic group G in the infinite dimensional torus T^ω corresponds to an embedding of the attractor $\mathcal{J}_E$ in the isospectral set of L^+ which would then be an infinite dimensional torus.

• Is there a relation between the operators in $\mathcal{B}(l^2(\mathsf{N}))$ of Geronimo, et al. and the fixed point L_E^+ operators constructed here? There might exist an isospectral deformation of L_E^+ such that the auxiliary spectrum is lying at the boundary of all the gaps of the spectrum of L_E^+ and such that the operator having this auxiliary

spectrum is identical with the operator of Geronimo et al.

• Could it be in general that the isospectral set of a random operator is a compact topological group if and only if the operator is almost periodic?

• Is there for other polynomials p also a renormalisation scheme in a space of operators analogous to the quadratic map $z \mapsto z^2 + E$?

• Can one set up a renormalisation in higher dimensions? Such a renormalisation would need a renormalisation of commutation relations beside the renormalisation of the dynamical systems.

• Is the capacity $c(L)$ continuously depending on L? Is there even more regularity? Does the equation $\frac{\partial}{\partial L}c(L)$ make sense? For which operators is the density of states the equilibrium measure on the spectrum?

• Fix a dynamical system (X, T, m). Does there exist a maximum of the capacity under all random Jacobi operators with mass $M(L) = \text{const}$ and $\text{tr}(L^2) = \text{const}$? Does there exist a maximum if we allow also variations of the dynamical system?

References

[Alv 91] L.Alvarez-Gaume. *Integrability in Random Matrix Models.*
CERN preprint June, 1991

[Alv 91a] L.Alvarez-Gaume, C.Gomez, J.Lacki. *Integrability in random matrix models.*
Physics letters B, 253:56-62, 1991

[Bak 84] G.A.Baker, D.Bessis, P.Moussa. *A family of almost periodic Schrödinger operators.*
Physica A, 124:61-78, 1984

[Bar 83] M.F.Barnsley, J.S.Geronimo, A.N. Harrington. *Geometry, electrostatic measure and orthogonal polynomials on Julia sets for polynomials.*
Ergodic Th. Dynam. Sys., 3:509-520, 1983

[Bar 83a] M.F.Barnsley, J.S.Geronimo, A.N. Harrington. *Infinite-Dimensional Jacobi Matrices associated with Julia sets.*
Proc. Amer.Math.Soc, 88:625-630, 1983

[Bar 83b] M.F.Barnsley, J.S.Geronimo, A.N. Harrington. *On the Invariant Sets of a Family of Quadratic Maps.*
Comm. Math. Phys., 88:479-501, 1983

[Bar 85] M.F.Barnsley, J.S.Geronimo, A.N. Harrington. *Almost periodic Jacobi Matrices associated with Julia sets for Polynomials.*
Comm. Math. Phys., 99: 303-317, 1985

[Bar 88] M.Barnsley. *Fractals everywhere.*
Academic Press, London, 1988

[Bel 82] J.Bellisard, D.Bessis, P.Moussa. *Chaotic States of Almost Periodic Schrödinger operators.*
Phys. Rev. Lett., 49:700-704, 1982

[Bes 82] D.Bessis, M.L.Mehta, P.Moussa. *Orthogonal polynomials on a family of Cantor sets and the problem of iterations of quadratic mappings.*
Lett Math. Phys., 6: 123-140, 1982

[Bla 84] P.Blanchard. *Complex analytic dynamics on the Riemann sphere.*
Bull. (New Ser.) of the Amer. Math. Soc., 11:85-141, 1984

[Bro 65] H.Brolin. *Invariant sets under iteration of rational functions.*
Arkiv Math., 6: 103-144, 1965

[Car 87] R.Carmona, S.Kotani. *Inverse Spectral Theory for Random Jacobi Matrices.*
Journal Statistical Physics, 46:1091-1114, 1987

[Car 90] R.Carmona, J.Lacroix. *Spectral Theory of Random Schrödinger Operators.*
Birkhäuser, Boston, 1990

[Con 90] A.Connes. *Geometrie non commutative.*
InterEditions, Paris 1990

[Cor 82] I.P.Cornfeld,S.V.Fomin,Ya.G.Sinai. *Ergodic Theory.*
Springer, New York, 1982

[Cyc 87] H.L.Cycon, R.G.Froese, W.Kirsch, B.Simon. *Schroedinger Operators.*
Texts and Monographs in Physics, Springer, 1987

[Den 76] M.Denker,C.Grillenberg,K.Sigmund. *Ergodic Theory on Compact Spaces.*
Lecture Notes in Math., No. 527, Springer 1976

[Ere 90] A.Eremenko, M.Lyubich. *The dynamics of analytic transformations.*
Leningrad Math. J.,3:563-634, 1990

[For 91] J.Fornaess, N.Sibony. *Random iterations of rational functions.*
Ergod. Th. Dynam. Sys., 11:687-708, 1991

[Fri 92] N.A.Friedman. *Replication and Stacking in Ergodic Theory.*
Amer. Mathem. Monthly, January 1992

242

[Ger 91] A.Gerasimov, A.Marshakov, A.Mironov, A.Morozov, A.Orlov. *Matrix models of two dimensional gravity and Toda theory.*
Nuclear Physics B, 357:565-618, 1991

[Gro 90] D.Gross, A.Migdal. *Nonperturbative two dimensional quantum gravity.*
Physical review letters, 64: 127-130, 1990

[Hal 56] P.Halmos. *Lectures on Ergodic Theory.*
The mathematical society of Japan, Tokyo, 1956

[Kni 3] O.Knill. *Isospectral Deformations of Random Jacobi Operators.*
Commun. Math. Phys., 151:403-426, 1993

[Kni 4] O.Knill. *Factorisation of Random Jacobi Operators and Bäcklund transformations.*
Commun. Math. Phys., 151:589-605, 1993

[Kot 88] S.Kotani, B.Simon. *Stochastic Schroedinger Operators and Jacobi Matrices on the Strip.*
Comm. Math. Phys, 119:403-429, 1988

[Lju 83] M.Ju. Ljubich. *Entropy properties of rational endomorphisms of the Riemann sphere.*
Ergod. Th. Dynam.Sys., 3:351-385, 1983

[Man 85] R.Mané. *On the Bernoulli property for rational maps.*
Ergod. Th. Dynam.Sys., 5:71-88, 1985

[Mar 91] E.J.Martinec. *On the Origin of Integrability in Matrix Models.*
Commun. Math. Phys., 138:437, 1991

[Mil 91] J.Milnor. *Dynamics in one complex variable.*
Introductory Lectures, SUNY, 1991

[Par 80] W. Parry. *Topics in ergodic theory.*
Cambridge University Press, 1980

[Rud 62] W.Rudin. *Fourier Analysis on groups.*
Interscience tracts in pure and applied mathematics 12, Interscience Publishers, 1962

[Rue 82] D.Ruelle. *Repellers for real analytic maps.*
Ergod. Th. Dynam. Sys., 2:99-107, 1982

[Tsu 58] M. Tsuji. *Potential theory in modern function theory.*
Chelsea publishing company, New York, 1958

[Wol 91] Wolfram Research. *Mathematica Version 2.0.*
Illinois 1991

Isospectral deformations of discrete Laplacians

Contents

Abstract

We study higher dimensional discrete random Laplacians over an abstract $\mathbf{Z}^d$ dynamical system $(X, T_1, \ldots, T_d, \mu)$.

We prove that a sufficient condition for the existence of a Toda orbit through L is that L is not a stationary point of the first Toda flow and that it is possible to factor $L = D^2 + E$, where D is a random Jacobi operator defined over a $2 : 1$ integral extension of $(X, T_1, \ldots, T_d, \mu)$.

We find nontrivial critical points of a variational problem, whose critical points are discrete partial difference equations with random boundary conditions. Discrete random Laplacians appear as Hessians at such critical points. A generalized Morse index allows to prove the existence of many critical points. We show that for a generalized Harper Laplacian L, the curvature of a gauge field attached to L determines the density of states of L.

1 Introduction

The metric structure of a compact *spin Riemannian manifold M* is completely determined by the *Dirac operator D* on M [Con 89] because the geodesic distance between two points $P, Q \in M$ can be expressed by

$$d(P, Q) = \sup\{|a(P) - a(Q)| \mid a \in C(M), \|[D, a]\| \leq 1\} .$$

Isospectral deformations of the Dirac operator correspond therefore to deformations of the *Riemannian metric.* The question which Riemannian manifolds allow such a deformation is related to the inverse spectral problem for Laplacians the determination of the set of manifolds having a Laplacian with a fixed spectrum.

If a Dirac operator is defined in a more abstract sense as the "square root" of a Laplacian, one can ask the above problem also in other contexts.

We consider in this chapter discrete random Laplacians which are defined over a $\mathbf{Z}^d$ dynamical system. More precisely, we consider elements $L = \sum a_i \tau_i + (a_i \tau_i)^* + b$ in the crossed product $\mathcal{X}$ of the algebra $L^\infty(X, M(N, \mathbf{C}))$ with the $\mathbf{Z}^d$ action $(X, T, m) = (X, T_1, \ldots, T_d, m)$. For $d = 1$ these Laplacians are called random Jacobi operators and there exists a hierarchy of *random Toda flows* which consist of isospectral deformations of such Laplacians. For higher dimensional Laplacians, isospectral deformations are in general no more possible. We refer to a result of Mumford [Mor 78], who showed that generically, two dimensional periodic Laplacians do not allow isospectral deformations. We study the question of isospectral deformation with more primitive tools again but in a wider context. First of all we work in an infinite dimensional context if the dynamical system is not periodic. Second, we

allow the entries of the Laplacian to be matrices. We hope that this opening of the setup gives more freedom to do isospectral deformations. We will see that this question has a relation to the question if there exists a abstract Dirac operator for L.

There is no Laplacian D in $\mathcal{X}$ such that $D^2 + E = L$ is possible. We have to take a new dynamical system $(Y, S, n) = (Y, S_1, \ldots, S_d, n)$ called a 2:1 *integral extension* which satisfies $S_i^2 = T_i$. The system is constructed in such a way, that a Toda orbit of (X, T, m) appears as a lattice with a grid-spacing which is twice as large as the grid-spacing of the lattice obtained from (Y, S, n). The analogous crossed-product construction with (Y, S, n) is called $\mathcal{Y}$.

A Laplacian D over the finer dynamical system (Y, S, n) is in some sense a *first order difference operator*, when looked at in the scale of the old system. At the finer scale, it is a *second order difference operator*. It is again a Laplacian.

We would like to know what are the precise conditions for that the operator L can be deformed in $\mathcal{X}$ by Toda flows *while keeping the property of being a Laplacian*. We show that a sufficient condition for that is the existence of a Dirac operator $D \in \mathcal{Y}$ satisfying $D^2 + E = L$. We mean this in the way that if the operator L satisfies this condition and if the operator is not a stationary point of the first Toda flow then the orbit passing through L consists of Laplacians.

We would like to mention that it is trivial to deform a selfadjoint Laplacian $L = \sum_i a_i \tau_i + (a_i \tau_i)^* + b \in \mathcal{L}$ in an isospectral way if we don't insist in Toda deformations: take any curve $g_t \in L^\infty(X, SU(N))$ satisfying $g(0) = 1$. Then the curve $L(t) = g(t) L g(t)^{-1}$ consists of selfadjoint isospectral Laplacians.

The construction of the Dirac operator D for the Laplacian $L = \sum_i a_i \tau_i + (a_i \tau_i)^* + b$ satisfying $D^2 + E = L$ is not always possible if $d \geq 2$. Necessary is for example that a *zero curvature condition* $[a_i \tau_i, a_j \tau_j] = 0$ is fulfilled. If this condition is not satisfied, one cannot expect isospectral deformation in the space of Laplacians because the tangent vector of a Toda flow passing through L is already no longer a Laplacian.

But also if L can be factored, this does not yet mean that we get an isospectral curve of Laplacians through L. The reason is that L could be a stationary point of the Toda flow. This is for example the case if the entries of the Laplacian L are real or unitary valued.

Beside isospectral deformations we will consider also *discrete random partial difference equations*. These equations play the role of partial differential equation and each classical partial differential equation has formally such a random discrete analogue.

For example, the classical *Sin-Gordon equation*

$$u_{tt} - u_{xx} = \gamma \cdot \sin(x)$$

with some real parameter has the formal discrete counter-part

$$q(T_1) + q(T_1^{-1}) - q(T_2) - q(T_2)^{-1} = \gamma \cdot \sin(q)$$

over the two-dimensional system (X, T_1, T_2, m). The existence of solutions $q \in L^\infty(X)$ to this equation is by no means trivial. We will show how the *anti-integrable limit* of Aubry and Abramovici can be used to get existence of solutions for γ large enough. More general systems are obtained by taking a random Laplacian $L = \sum_{i=1}^d a_i(\tau_i - 2 + \tau_i^*)$ with $a_i \in \mathbf{R}$ and a smooth potential V. The random difference equation

$$Lq = \gamma V'(q) \, ,$$

is a nonlinear Schrödinger equation. We will prove that it has nontrivial solutions if γ is large enough. The difference equation is the Euler equation to the variational problem defined by a functional

$$\mathcal{L}(q) = \int_X \sum_i a_i \cdot \frac{(q(T_i) - q)^2}{2} + V(q) \, dm \, .$$

The second variation is at a critical point is the *random Laplacian*

$$L + V''(q) \, .$$

One can interpret a solution of the random difference equation $Lq = \gamma V'(q)$ as a *random discrete surface* in $\mathbf{Z}^d \times \mathbf{R}$. For each point $x \in X$, one has a discrete surface

$$(n_1, n_2, \ldots n_d) \in \mathbf{Z}^d \mapsto q(T_1^{n_1} T_2^{n_2} \ldots T_d^{n_d}) \in \mathbf{R} \, .$$

2   Discrete random Laplacians

Let $T_1, T_2, \ldots, T_d$ be commuting automorphisms of the probability space (X, μ). We call the $\mathbf{Z}^d$ action (X, T, μ) a *dynamical system* and write $T^n x = T_1^{n_1} T_2^{n_2} \ldots T_d^{n_d}(x)$ for $n \in \mathbf{Z}^d$. Denote by $\mathcal{X}$ the *crossed product* of the von Neumann algebra $\mathcal{A} = L^\infty(X, M(N, \mathbf{C}))$ with the dynamical system (X, T, μ). The group $\mathbf{Z}^d$ acts on $\mathcal{A}$ by automorphisms $f \mapsto f(T^n)$ where $f(T^n)(x) = f(T^n x)$ and the algebra $\mathcal{X}$ is obtained by completing the algebra of all polynomials in the variables $\tau_1, \ldots, \tau_d$ with coefficients in $\mathcal{A}$

$$K = \sum_{n \in F \subset \mathbf{Z}^d} K_n \tau^n, \quad (KL)_n = \sum_{l+m=n} K_l L_m(T^l) \tau^n$$

with respect to the norm $|||K||| = |\, \|K(x)\| \,|_\infty$, where $K(x)$ is the bounded linear operator on $l^2(\mathbf{Z}^d)$ defined by $(K(x)u)_n = \sum_m K_m(x) u_{n+m}$ and where $\| \cdot \|$ is the operator norm on $\mathcal{B}(l^2(\mathbf{Z}^d))$ and $|\cdot|_\infty$ the essential supremum norm. With the involution on $\mathcal{X}$ defined by

$$\left(\sum_n K_n \tau^n \right)^* = \sum_n K_n^*(T^{-n}) \tau^{-n}$$

$\mathcal{X}$ becomes a von Neumann algebra. It has the *trace*

$$\mathrm{tr}(K) = \int_X \mathrm{Tr}(L_0(x)) \, d\mu(x) \,,$$

where Tr denotes the usual trace on the matrix algebra $M(N, \mathbb{C})$. We call the elements in $\mathcal{X}$ *discrete random operators*.

In $\mathcal{X}$ lies the set of *discrete random Laplacians*

$$\mathcal{L} = \{L = a\tau + (a\tau)^* + b = \sum_{i=1}^{d} a_i\tau_i + (a_i\tau_i)^* + b\} \,,$$

where we assume $b = b^*$ and a_i to be invertible. We call the vector $(a_1, \ldots a_d)$ the *gauge potential part* and b the *scalar potential part* of the Laplacian. Discrete random Laplacians are by definition selfadjoint.

Given a normal element $K \in \mathcal{X}$. The functional calculus defines $f(K)$ for every function $f \in C(\sigma(K))$, where $\sigma(K)$ denotes the spectrum of K. According to Riesz's representation theorem, the bounded linear functional $f \mapsto \mathrm{tr}(f(K)) \in \mathbb{C}$ on $C(\sigma(K))$ defines a measure dk on $\sigma(K)$ which is called the *density of states* of K. It satisfies $\mathrm{tr}(f(K)) = \int f(E) \, dk(E)$ for all $f \in C(\sigma(K))$.

Given $L \in \mathcal{L}$ and $d \geq 2$. The *(multiplicative) curvature* of the Laplacian $L = a\tau + (a\tau)^* + b \in \mathcal{L}$ is defined as

$$F = \sum_{i<j} F_{ij} \, \tau_i\tau_j \,,$$

where the value $F_{ij} = a_i a_j(T_i) a_i(T_j)^{-1} a_j^{-1}$ can be considered as the result of the parallel transport with the connection $a\tau$ around the *plaquette* $P_{ij}x = \{x, T_ix, T_iT_jx, T_jx\}$. The *additive curvature* given by

$$\sum_{ij} F_{ij}^{+}\tau_i\tau_j = \sum_{ij}[a_i\tau_i, a_j\tau_j]\tau_i\tau_j = \sum_{ij}(a_ia_j(T_i) - a_ja_i(T_j))\tau_i\tau_j \,.$$

is vanishing if and only if $F_{ij} = 1$. We say in this case, a Laplacian has *zero curvature*. This means that parallel transport around each closed curve in the lattice $\mathbb{Z}^d$ gives the identity.

Examples of discrete random Laplacians are:

• *Random Jacobi operators*
If $d = N = 1$, we get random Jacobi matrices $L = a\tau + (a\tau)^* + b$. Special cases are *discrete random Schrödinger operators* $L = \tau + \tau^* + V$. Such operators have been studied much in the last years. (See [Cyc 87], [Car 90].)

248

- *Harper Laplacians*

  Let $d = 2$ and $a_1, a_2 \in L^\infty(X, SU(N))$ satisfy $a_1(x)a_2(T_1x) = e^{2\pi i\alpha}a_2(x)a_1(T_2x) \cdot 1$, where α is an irrational number. In this case

$$L = a_1\tau_1 + a_2\tau_2 + (a_1\tau_1)^* + (a_2\tau_2)^*$$

is called a *Harper Laplacian*. It has by definition a constant curvature $e^{2\pi i\alpha}$ and $2\pi\alpha$ has a physical interpretation as a normalized magnetic flux. A special case of a Harper Hamiltonian is given by $(X, T_1, T_2, \mu) = (\mathbb{T}^1, x \mapsto x + \alpha, x \mapsto x, dx)$ with $a_1 = 1$, $a_2(x) = e^{2\pi ix}$, which leads to the Hofstadter case of the *discrete Mathieu operator* $L = \tau_1 + \tau_1^* + V$, where $V(x) = 2\cos(2\pi x)$. See [Las 93] for recent results on the spectrum.

- *D'Alembert operator with Dirac operator*

  Let $d = 4$. The operator $L = \sum_{i=1}^4 g_i(\tau_i + 2 + \tau_i^*)$ with $g_1 = -1, g_2 = g_3, g_4 = 1$ is a discrete version of the flat d'Alembert operator $\square = \sum_{i=1}^4 g_i\frac{\partial^2}{(\partial x_i)^2}$. Define the Dirac operator $D = \sum_{i=1}^4 \gamma_i(\tau_i + \tau_i^*)$, where γ_i be the Dirac matrices, satisfying $\{\gamma_i, \gamma_j\} = 2g_i\delta_{ij}$. The square

$$D^2 = \sum_{i=1}^4 g_i(\tau_i^2 + 2 + (\tau_i^*)^2)$$

is a Laplacian over the $\mathbb{Z}^d$-dynamical system $(X, T^2 = (T_1^2, T_2^2, \ldots, T_d^2), \mu)$.

- *Periodic Laplacians*

  In the case $|X| = M < \infty$, the automorphisms T_i are just finite permutations of X. Ergodic T_i lead to periodic Laplacians which are mostly studied in the case $N = 1, a_i = 1, b = V$. If $d = N = 1$, we get periodic Jacobi matrices. The case $d = 2, N = 1, a_i = 1$ is the subject of the book [Gie 93].

We think about elements in $\mathcal{X}$ as discrete versions of *differential operators*. They can also be considered as the *Hamiltonian* of a quantum mechanical particle or describing the *geometry* of a discrete manifold. We think of the a_i as components of a *connection* or a *gauge potential* or a *one-form* and of the F_{ij} as the components of the *curvature* or a *gauge field* or a *two form*, always depending on geometrical, physical or algebraic preferences.

3 Discrete random Dirac operators

The classical d'Alembert operator $\square = \sum_{i=1}^4 g_i\frac{\partial^2}{(\partial x_i)^2}$ can be factored as $L = D^2$ with the Dirac operator $D = \sum_i \gamma_i\delta_{x_i}$, where the Dirac matrices γ_i satisfy the anti-commutation relation $\{\gamma_i, \gamma_j\} = \pm 2g_i\delta_{ij}$. Our aim is to construct a discrete random Dirac operator, which is by definition the square root of a discrete random Laplace

operator L. For doing so, it is necessary to construct a new dynamical system from the given dynamical system (X, T, μ). We get a new probability space (Y, ν) by defining $Y = \bigcup_{I \subset \{1,...,d\}} X_I$ to be the union of 2^d copies X_I of (X, μ) and letting ν be the normalized measure on Y, such that for $Z \subset X = X_I$, $\mu(Z) = 2^d \nu(Z)$. Define

$$S_i : X_I \to X_{I \Delta \{i\}}, S_i(x) = \begin{cases} x\, , i \notin I\, , \\ T_i(x)\, , i \in I\, . \end{cases}$$

$(Y, S_1, S_2, \ldots, S_d, \nu)$ is again a $\mathbb{Z}^d$ dynamical system and because $S_i^2(x) = T_i(x)$ for $x \in X_I$, we get the old system back by restricting S_i^2 on X_I. We call (Y, S, ν) a $2:1$ *integral extension* of (X, T, μ).

The dynamics of the 2:1 integral extension is illustrated as follows. The sets Y_I are ordered vertically according to the cardinality of the sets I. At the bottom is the set $I_\emptyset$ and at the top is the set $Y_{\{1,2,...,d\}}$. When going "up" one applys the identity map Id.
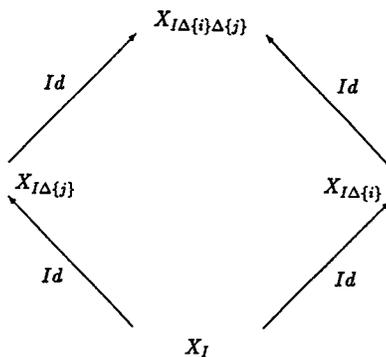
When going "down", one uses the transformation T_i, where i is the index which is deleted when going down:

$$X_{I \Delta \{i\} \Delta \{j\}}$$

T_i T_j

$$X_{I \Delta \{j\}} \qquad\qquad X_{I \Delta \{i\}}$$

T_j T_i

$$X_I$$
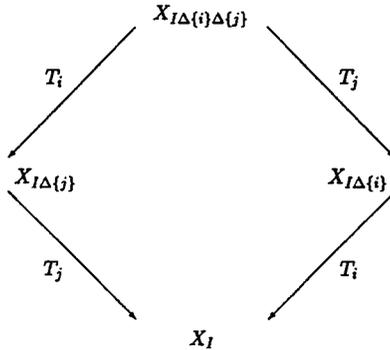
Remark. The space of all $\mathbb{Z}^d$ dynamical systems (X, T, m) forms a complete metric space $\mathcal{U}$ with distance

$$d(T, S) = \sum_{i=1}^{d} d(T_i, S_i) \, ,$$

where

$$d(T_i, S_i) = m\{x \in X \mid T_i(x) \neq S_i(x)\}$$

is the usual uniform topology on the space of automorphisms. Unless $d = 1$, the space $\mathcal{U}$ is not a group.

For all $x \in Y = \bigcup_I X_I$ we get $S_i^2 = T_i$. We can identify the probability space Y after a normalization of the measure with the probability space (X, m). Starting with any $\mathbb{Z}^d$ dynamical system and performing the renormalisation map again and again, the system will converge to a $\mathbb{Z}^d$ version of the *von Neumann- Kakutani system*. This fixed point depends only on the way, the identification of the probability space Y with the probability space X is done in each step.

Proposition 3.1 *The $2 : 1$ integral extension Φ is a contraction on the complete metric space $\mathcal{U}$:*

$$d(\Phi(T), \Phi(V)) \leq \frac{1}{2} d(T, V) \, .$$

Proof. Denote by Y_i the set, where $T_i(x) \neq V_i(x)$. By definition $d(T, V) = \sum_{i=1}^{d} m(Y_i)$. When building the new systems (Y, S, m) and (Y, U, m), where $S = \Phi(T), U = \Phi(V)$, there is a set $\bigcup_{i \notin I} X_I$ of measure $1/2$, where the maps S_i and U_i are coinciding because they are there the identity from $X_I \to X_{I \cup \{i\}}$. The new set

$$Z_i = \{x \in X \mid S_i(x) \neq U_i(x)\}$$

has thus half of the measure of the set

$$Y_i = \{x \in X \mid T_i(x) \neq V_i(x)\} \, .$$

This means $d(S_i, U_i) \leq \frac{1}{2}d(T_i, V_i)$ and $d(S, U) \leq \frac{1}{2}d(S_i, U_i)$. $\qquad\qquad\square$

By Banach's fixed point theorem, there exists a unique fixed point of Φ in $\mathcal{U}$. We call it the *von Neumann-Kakutani system*.

Define $\mathcal{Y}$ to be the crossed product of $\mathcal{B} = \mathcal{L}^\infty(\mathcal{Y}, \mathcal{M}(\mathcal{N}, \mathbb{C}))$ with the dynamical system (Y, S, ν). We will write elements in $\mathcal{Y}$ as $C = \sum_n C_n \sigma_n$, where $\sigma_i f = f(S_i)\sigma_i$. Let

$$\mathcal{D} = \{D = \sum_i d_i \sigma_i + (d_i \sigma_i)^* \mid d_i \in \mathcal{B}\} \subset \mathcal{Y}$$

be the set of discrete random Laplacians in $\mathcal{Y}$ which have zero scalar part. Define $\psi : \mathcal{Y} \to \mathcal{X}$ by $\psi(K)_n(x) = K_{2n}(x)$ for $x \in X = X_\emptyset$, where we use the notation $2n = (2n_1, \ldots, 2n_d)$. We say, an operator $D \in \mathcal{D}$ is a *discrete random Dirac operator*, if there exists $E \in \mathbb{C}$ such that $\psi(D^2 + E) \in \mathcal{L}$. In [Kni 2], we have constructed Dirac operators to every one-dimensional random Jacobi operator. In [Kni 3] we showed how the iteration of the factorization $L \mapsto D$ satisfying $L = D^2 + E$ leads to operators with spectra on Julia sets.

If $D = d\sigma + (d\sigma)^*$ is a Dirac operator to L, we can rewrite $\psi(D^2 + E) = L = a\tau + (a\tau^*) + b$ as

$$\begin{aligned}
\{d_i\sigma_i, d_j\sigma_j\} &= \delta_{ij}2a_i\ , \\
\{d_i\sigma_i, (d_j\sigma_j)^*\} &= \delta_{ij}b_i\ ,
\end{aligned}$$

with $\sum_{i=1}^d b_i = b - E$. This implies

$$\begin{aligned}
[a_i\tau_i, a_j\tau_j] &= 0\ , &&(1)\\
[a_i\tau_i, (a_j\tau_j)^*] &= \delta_{ij}c_i\ . &&(2)
\end{aligned}$$

and the Laplacian L must have zero curvature.

4　Isospectral Toda deformations of discrete Laplacians

Define the cones $(\mathbb{Z}^d)^+ := \{n \in \mathbb{Z}^d \mid n_i > 0\}$ and $(\mathbb{Z}^d)^- = -(\mathbb{Z}^d)^+$ in the lattice $\mathbb{Z}^d$. We define in $\mathcal{X}$ the projections

$$K = \sum_n K_n \tau^n \mapsto K^\pm = \sum_{n \in (\mathbb{Z}^d)^\pm} K_n \tau^n$$

and denote the images of these projections in $\mathcal{X}$ by $\mathcal{X}^\pm$. We remark that if $K \in \mathcal{X}^+$ then $K^* \in \mathcal{X}^-$. Denote by $C^\omega(\mathbb{R})$ the set of all entire functions $\mathbb{C} \to \mathbb{C}$ which map $\mathbb{R}$ into itself and define the space of Hamiltonians

$$C^\omega(\mathcal{X}) = \{H : \mathcal{X} \to \mathbb{C} \mid K \mapsto H(K) = \mathrm{tr}(h(K)),\ h \in C^\omega(\mathbb{R})\}\ .$$

To such a Hamiltonian $H \in C^\omega(\mathcal{X})$ belongs the *Toda* differential equation

$$\dot{K} = [h'(K)^+ - h'(K)^-, K]$$

which gives a globally defined isospectral flow in $\mathcal{X}$. It is isospectral because $B = h'(K)^+ - h'(K)^-$ is a skew symmetric operator. The *first Toda flow* is obtained with the Hamiltonian $H(L) = \operatorname{tr}(L^2/2)$. The random Toda flows do not leave $\mathcal{L}$ invariant unless $d = 1$. We have shown in [Kni 1] how one can linearize in the case $d = 1$ these infinite dimensional integrable dynamical systems. For finite $|X|$, the flows are then the classical periodic Toda lattices which can be linearized explicitly using algebraic geometry [Mor 76]. In more dimension, an isospectral Toda deformation in $\mathcal{L}$ is in general not possible. Already for finite $|X|$, there is a result of Mumford [Mor 78] which dais that, generically, there exist no isospectral deformations of higher dimensional real Laplacians (N=1). In the complex case, it is however trivial to deform a selfadjoint Laplacian $L = \sum_i a_i \tau_i + (a_i \tau_i)^* + b \in \mathcal{L}$. Take any curve $g_t \in L^\infty(X, SU(N))$ satisfying $g(0) = 1$. Then $L(t) = g(t)Lg(t)^{-1}$ consists of selfadjoint isospectral Laplacians.

5 A sufficient condition for isospectral deformations

Theorem 5.1 *If there exists $D \in \mathcal{D}$ and $E \in \mathsf{R}$ such that $L = \psi(D^2 + E)$, the orbit of the isospectral deformation*

$$\frac{d}{dt}L = [L^+ - L^-, L] =: [B, L]$$

stays in $\mathcal{L}$. If L is not a stationary point of this Toda flow, there exists a curve of isospectral Laplacians through L. The deformation can be written as

$$\dot{a}_i = a_i b(T_i) - b a_i ,$$
$$\dot{b} = \sum_{i=1}^{d} 2(a_i a_i^* - (a_i^* a_i)(T_i^{-1})) ,$$

The differential equation for $L = \psi(D^2 + E)$ is equivalent to

$$\dot{D} = [(D^2)^+ - (D^2)^-, D]$$

in $\mathcal{Y}$ and this is a decoupled system of one-dimensional random Volterra systems

$$\dot{d}_i = [d_i^2, d_i^*] = d_i \cdot (d_i d_i^*)(T_i) - (d_i^* d_i)(T_i^{-1}) \cdot d_i .$$

Proof. The differential equation $\frac{d}{dt}L = [B, L]$ has a solution in $\mathcal{X}$. We want to show that for an initial condition L having a factorization $L = \psi(D^2 + E)$, the solution stays in $\mathcal{L}$. It follows from Equation (1) and (2) that

$$\begin{aligned}[L^+ - L^-, L] &= [a\tau - (a\tau)^*, a\tau + (a\tau)^* + b] \\ &= [a\tau, b] - [(a\tau)^*, b] + [a\tau, (a\tau)^*] - [(a\tau)^*, a\tau]\end{aligned}$$

253

is in $\mathcal{L}$. This holds true as long as $L = \psi(D^2) + E$. To see that the factorization $L = \psi(D^2) + E$ is preserved by the flow we have to show that the Clifford structure

$$\begin{aligned}
\{d_i\sigma_i, d_j\sigma_j\} &= \delta_{ij}2a_i, \\
\{d_i\sigma_i, (d_j\sigma_j)^*\} &= \delta_{ij}b_i .
\end{aligned}$$

stays invariant under the flow. The differential equation

$$\dot{D} = [(D^2)^+ - (D^2)^-, D]$$

in $\mathcal{Y}$ can be written as a decoupled set of Volterra systems

$$\dot{d}_i = [d_i^2, d_i^*] = d_i \cdot (d_i d_i^*)(T_i) - (d_i^* d_i)(T_i^{-1}) \cdot d_i .$$

We write from now on simply d instead of $d\sigma$ and d^* instead of $(d\sigma)^*$ in order to simplify the writings. With the differential equations

$$\dot{d}_i = [d_i^2, d_i^*] , \dot{d}_i^* = -[(d_i^*)^2, d_i] ,$$

we get for $i \neq j$ using $\{d_i, d_j^*\} = 0$

$$\begin{aligned}
\frac{d}{dt}\{d_i, d_j\} &= \dot{d}_i d_j + d_i \dot{d}_j + \dot{d}_j d_i + d_j \dot{d}_i \\
&= [d_i^2, d_i^*]d_j + d_i[d_j^2, d_j^*] + [d_j^2, d_j^*]d_i + d_j[d_i^2, d_i^*] \\
&= d_i^2 d_i^* d_j - d_i^* d_i^2 d_j + d_i d_j^2 d_j^* - d_i d_j^* d_j^2 \\
&\quad + d_j^2 d_j^* d_i - d_j^* d_j^2 d_i + d_j d_i^2 d_i^* - d_j d_i^* d_i^2 \\
&= d_i^2(d_i^* d_j + d_j d_i^*) + d_j^2(d_i d_j^* + d_j^* d_i) \\
&\quad - (d_i^* d_j + d_j d_i^*)d_i^2 - (d_j^* d_i + d_i d_j^*)d_j^2 \\
&= d_i^2\{d_i^*, d_j\} + d_j^2\{d_i, d_j^*\} - \\
&\quad \{d_i^*, d_j\}d_i^2 - \{d_j^*, d_i\}d_j^2 = 0 .
\end{aligned}$$

Similarly, using $\{d_i, d_j\} = \{d_i^*, d_j^*\} = 0$ for $i \neq j$, we get for $i \neq j$

$$\begin{aligned}
\frac{d}{dt}\{d_i, d_j^*\} &= \dot{d}_i d_j^* + d_i \dot{d}_j^* + \dot{d}_j^* d_i + d_j^* \dot{d}_i \\
&= [d_i^2, d_i^*] d_j^* - d_i[(d_j^*)^2, d_j] - [(d_j^*)^2, d_j] d_i + d_j^*[d_i^2, d_i^*] \\
&= (d_i^2 d_i^* d_j^* + d_j^* d_i^2 d_i^*) - (d_i(d_j^*)^2 d_j + (d_j^*)^2 d_j d_i) \\
&\quad - (d_i^* d_i^2 d_j^* + d_j^* d_i^* d_i^2) + (d_i d_j(d_j^*)^2 + d_j(d_j^*)^2 d_i) \\
&= d_i^2(d_i^* d_j^* + d_j^* d_i^*) - (d_j^*)^2(d_i d_j + d_j d_i) \\
&\quad - (d_i^* d_j^* + d_j^* d_i^*)d_i^2 + (d_i d_j + d_j d_i)(d_j^*)^2 = 0 .
\end{aligned}$$

$\square$

Remarks.

• The condition in the proposition is always satisfied in one dimension, where every Laplacian can be deformed.

• The zero curvature conditions $[a_i\tau_i, a_j\tau_j] = [a_i\tau_i, (a_j\tau_j)^*] = 0$, $i \neq j$ for L do not imply in general that we can factor $L = D^2 + E$. We need that the zero curvature holds true also after deformation. This would need for example that the equation

$$\frac{d}{dt}[a_i\tau_i, (a_j\tau_j)^*] = 2[[a_i\tau_i, b], (a_j\tau_j)^*] = 0$$

is valid which implies that also a condition for b must be satisfied in order to have an isospectral deformation.

• Operators with unitary a_i and constant b are stationary points of the first Toda flow. Especially random Harper operators can not be deformed in this way.

• The necessary conditions

$$\begin{aligned} a_i a_j(T_i) &= a_j a_i(T_j)\,, \\ a_j a_i &= a_i(T_j) a_j(T_i),\ \ i \neq j \end{aligned}$$

for isospectral deformations have only constant solutions if we require the a_i to be complex valued cocycles and if the maps T_i are ergodic. (To see this, we divide the first equation by the second one giving $a_j(T_i)^2 = a_j^2$ which implies that a_j is constant if T_i is ergodic.) We conclude that we need the entries of the Laplacian to be *matrix-valued* in order to obtain interesting isospectral deformations.

6   Discrete random partial difference equations

Given a $\mathbb{Z}^d$ dynamical system (X, T, μ) and a *Lagrangian*

$$l(q_0, q_1, \ldots, q_d) = -\sum_{i=1}^{d} g_i \frac{(q_i - q_0)^2}{2} + \gamma \cdot V(q_0)\,,$$

where $g_i \in \mathbb{R} \setminus \{0\}$ and $V \in C^2(\mathbb{T}^1, \mathbb{R})$ is a real-valued potential and γ is a real coupling constant. We define on the space $L^\infty(X, \mathbb{T}^1)$ the *functional*

$$S(q) = \int_X l(q(x), q(T_1 x), q(T_2 x), \ldots, q(T_d x))\ d\mu(x)$$

on $L^\infty(X, \mathbb{T}^1)$. A critical point q of this functional satisfies the *discrete random partial difference equation*

$$-\sum_{i=1}^{d} g_i(q(T_i) - 2q + q(T_i^{-1})) - \gamma \cdot V'(q) = \delta S(q) = 0\,.$$

255

The second variation at a critical point (the Hessian) is the random Laplacian

$$L = \sum_{i=1}^{d} a_i \tau_i + (a_i \tau_i)^* + b$$

with $a_i = -g_i$ and $b = \gamma \cdot V'' - \sum_{i=1}^{d} 2g_i$.

It is a nontrivial problem to find nontrivial critical points of the above functional. For large coupling constants γ, it is however quite easy using the anti-integrable limit idea of Aubry [Aub 92]. Denote with

$$\Sigma = \{\sigma \in \mathsf{T}^1 \mid V'(\sigma) = 0, \ V''(\sigma) \text{ invertible}\}$$

the set of non-degenerate critical points of V.

Theorem 6.1 *Assume that Σ has at least 2 elements. Then there exists $\gamma_0 \in \mathsf{R}$ such that for coupling constants $|\gamma| > \gamma_0$, there exist nontrivial critical points of the functional S.*

Proof. Write $\gamma = 1/\epsilon$ and take the equivalent functional

$$S(q) = \int_X - \sum_{i=1}^{d} \epsilon g_i \frac{(q(T_i x) - q(x))^2}{2} + \gamma \cdot V(q(x)) \, d\mu(x) \, .$$

In the limit $\epsilon = 0$ every function $q \in L^\infty(X, \Sigma)$ is a critical point. The Hessian at such a point is $L = V''(q)$ which is by definition of Σ invertible. Applying the implicit function theorem, there exists also a critical point q_ϵ for small $|\epsilon|$. $\square$

We introduced the above functional S in [Kni 1] for the one-dimensional case, where the problem of finding critical points is equivalent to embed a factor of the dynamical system in a *monotone twist map* defined by the generating function l. In the case, when the dynamical system is the irrational rotation on T^1, the functional S is called the Percival functional. In [Kni 4], we used the functional together with a theorem of Krieger and the anti-integrable limit of Aubry to show that every ergodic dynamical system with finite metric entropy can be embedded into a monotone twist map. For two-dimensional systems $d = 2$ with $a_1 = -a_2 = 1$ and $V(q) = \cos(q)$ a critical point satisfies a discrete version of the Sine-Gordon equation.

One can not only establish the existence of critical points but also prove that there exist uncountably many different critical points. This can be done with the help of a *generalized Morse index* at a critical point q which is defined to be the value of the integrated density of states $k(0) = \int_{-\infty}^{0} dk(E')$ of the Hessian $L = \delta^2(S)$ at the energy $E = 0$. This number is lying in the interval $[0, 1]$ and measures, how much of the spectrum of L is below 0. One could say that it measures the "dimension" of the infinite dimensional unstable manifold, which is passing through the critical

point. If two critical points have a different index, then they must be essentially different in the sense that it is then excluded that one critical point is a translation of the other one.

.

Proposition 6.2 *If there exist two critical points σ_1, σ_2 of V with $V''(\sigma_1) > 0$ and $V''(\sigma_2) < 0$, then there exist uncountably many different critical points of S for $|\gamma|$ large enough. The Morse index at a critical point near $q \in L^\infty(X, \Sigma))$ is*

$$m(\{x \in X \mid V''(q(x)) < 0\}) .$$

Proof. In the anti-integrable limit, the Hessian is an invertible diagonal operator and the density of states is a Dirac measure located on the set

$$\{V''(\sigma) \mid \exists Y(\sigma) \subset X, \ m(Y(\sigma)) > 0, \ \forall x \in Y(\sigma), \ \sigma = q(x) \in \Sigma\} .$$

Each point σ in this set has the mass $m(Y(\sigma))$. The integrated density of states at 0 is given by $m(\{x \in X \mid V''(q(x)) < 0 \})$. Because the Hessian in the anti-integrable limit is invertible, the mass of the spectrum below 0 does not change under perturbations of the operator or (by the implicit function theorem) under perturbations of the critical point. This means that the Morse index is constant for critical points near the anti-integrable limit and it is given by the same formula. □

7 Suris-Bobenko-Pinkall maps

A discretisation of the pendulum equation leads to the *Standard map*

$$\ddot{q} = \sin(q) \rightsquigarrow q(T) - 2q + q(T^{-1}) = \sin q .$$

The integrable ordinary differential equation has then become a map which is in general not integrable. An analogue discretisation of the sin Gordon equation leads to a *discrete partial difference equation* which seems not have been studied so far.

$$q_{tt} - q_{xx} = \sin q \rightsquigarrow q(T) + q(T^{-1}) - q(S) - q(S^{-1}) = \gamma \cdot \sin(q) .$$

Such equations above the lattice $\mathbb{Z}^2$ instead above an orbit of a $\mathbb{Z}^2$ action has been studied in recent work of Bobenko et al. [Bob 93], who showed that if the nonlinearity $\sin(q)$ is replaced by $4 \cdot \arg(1 + \gamma \cdot e^{iq})$, the equation becomes integrable and the solutions $q_{nm} = q(T_1^n T_2^m x)$ above an orbit of the $\mathbb{Z}^2$ action describes a discrete surface being the analogue of the pseudo sphere. In our case, we have a random version of such surfaces above a $\mathbb{Z}^2$ action. The anti-integrable limit implies the existence of nontrivial solutions.

Remark. It has been found out by Bobenko and Pinkall [Bob 93] that the right discretisation of the pendulum is

$$q(T_1) + q(T_1^{-1}) - q(T_2) - q(T_2)^{-1} = 2\arg(1 + k \cdot e^{iq(T_2)}) + 2\arg(1 + k \cdot e^{iq(T_2)^{-1}})$$

if one wants also in the discrete case to have an integrable equation. If $T_2 = Id$, Bobenko et al. [Bob 93] remarked that this equation reduces to an integrable family of twist maps

$$(q, p) \mapsto (q + p + \gamma \cdot 4 \cdot \arg(1 + k \cdot e^{iq}), p + \gamma \cdot 4 \cdot \arg(1 + k \cdot e^{iq})) .$$

Already Suris [Sur 89] has found a family of integrable twist maps among other families. We call the above family *Suris-Bobenko-Pinkall-maps*.

8   Zero curvature and the density of states

We consider in this section Laplacians in $\mathcal{L}_{SU} = \{L \in \mathcal{L} \mid a_i \in L^\infty(X, SU(N)), b = 0\}$. Call

$$\mathcal{G} = \{G \in \mathcal{L} \mid G_0 \in L^\infty(X, SU(N)), G_n = 0, n \neq 0\}$$

the *group of $SU(N)$ gauge fields*. For $G \in \mathcal{G}$, the map

$$L \mapsto GLG^{-1}$$

on $\mathcal{L}_{SU}$ is called a *gauge transformation*. The $a_i \tau_i$ are transformed as

$$a_i \tau_i \mapsto G a_i G(T_i)^{-1} \tau_i .$$

Gauge transformations leave the set of zero-curvature Laplacians invariant. Also the density of states is invariant under such transformations so that gauge transformations are isospectral deformations. We will just see that the density of states decides also, if the operator has zero curvature or not. In the special case of a Harper Hamiltonian $d = 2, N = 1$ with constant curvature, the density of states determines the curvature and so the normalized magnetic flux:

Proposition 8.1 *a) The operator $L \in \mathcal{L}_{SU}$ has zero curvature if and only if*

$$\operatorname{tr}(L^4) = \int_{\mathbf{R}} E^4 \, dk(E) = N P_4(d) ,$$

where $P_4(d)$ is the number of closed paths of length 4 starting at $0 \in \mathbf{Z}^d$.
b) For a Harper operator $L = a_1\tau_1 + a_2\tau_2 + (a_1\tau_1)^* + (a_2\tau_2)^*$ with $N = 1$ satisfying

$$a_1 a_2(T_1) a_1(T_2)^{-1} a_2^{-1} = e^{2\pi i \alpha} ,$$

the density of states determines the normalized magnetic flux $2\pi i \alpha$:

$$\operatorname{tr}(L^4) = \int_{\mathbf{R}} E^4 \, dk(E) = 16 + 8\cos(2\pi i \alpha) .$$

c) Harper Hamiltonians with different $\cos(2\pi\alpha)$ are not isospectral.

Proof.
a) If we know the density of states, we can calculate

$$\text{tr}(L^n) = \int E^n \, dk(E) \, .$$

The operator has zero curvature if and only if $a_i a_j (T_i)(a_i(T_j))^{-1}(a_j)^{-1} = 1$ and this is the case if and only if

$$\text{Tr}(a_i a_j (T_i)(a_i(T_j))^{-1}(a_j)^{-1}) = N$$

for all i, j. Because the trace of an element a in $SU(N)$ is always $\leq N$ with equality in the case of zero curvature, this is equivalent to

$$\text{tr}(L^4) = N P_4(d) \, ,$$

where $P_4(d)$ is the number of closed paths of length 4 in $\mathbb{Z}^d$ starting at one point.
b) L^4 contains 4 curvature terms like $a_1 a_2(T_1)(a_1(T_2)^{-1})^*(a_2^{-1})^*$ for each positively oriented plaquette at $x \in X$ and 4 curvature terms like $a_1 a_2(T_1 T_2^{-1})^* a_1(T_2^{-1})^* a_2(T_2^{-1})$ for each negatively oriented plaquette. There are additionally 16 constant summands 1 belonging to closed paths of length 4 which are not passing around a plaquette.
c) As a corollary of b), we obtain that Harper Laplacians with different $\cos(2\pi\alpha)$ are not isospectral because isospectral Laplacians have the same $\text{tr}(L^4)$. $\qquad\qquad$ □

The functional
$$S_{gauge}(L) = g \cdot (\text{tr}(L^4) - N P_4(d))$$

on $\mathcal{L}_{SU}$ is the lattice action of pure lattice $SU(N)$ gauge field on a d dimensional infinite lattice. For finite X, the lattice is periodic and the functional is a finite sum. In general it is an averaged sum.
If we don't assume α to be constant for the Harper Hamiltonian, we can make the following remark about the relation between the curvature and the density of states.

Proposition 8.2 *Given $L \in \mathcal{L}_{SU}$. The curvature $F = a_1 a_2(T_1) a_1(T_2)^* a_2^*$ determines the density of states dk. On the other hand, the density of states dk does not determine the curvature F.*

Proof. If F is known, we can calculate $\text{tr}(L^n)$ for each $n \in \mathbb{N}$ because $\text{tr}(L^n)$ contains summands labeled by paths of length n and each path summand is the product of all the curvatures belonging to the plaquettes which are surrounded by this path. We can determine therefore also $\text{tr}(\log(L - E))$ for $\text{Im}(E) > 0$ and so the integrated density of states
$$k(E) = \text{Im}(\text{tr}(\log(L - E))) \, , \; E \in \mathbb{R}$$

which determines the density of states $dk = \frac{d}{dE} k(E) dE$. The curvature function $F \in L^\infty(X)$ and a translated function $F(T^n)$ belong both to the same density of

states. It is thus in general not possible to determine dk from F. $\qquad\square$

It can be seen in the same way that also for higher dimensional Laplacians $L = \sum_{i=1}^{d} a_i \tau_i + (a_i \tau_i)^*$ with $a_i \in L^\infty(X, SU(N))$, the curvature $F = \sum_{ij} F_{ij} \tau_i \tau_j$ determines the density of states.

9 Some questions

We formulate some open points.
• Is the sufficient condition in the Theorem 5.1 also necessary? In other words, can one factorize an operator if an isospectral deformation is possible?

• Assume we have two unitarily equivalent Jacobi operators $L_1 = U^* L_2 U$ and assume we can deform $L_2(t)$ in an isospectral way with a Toda flow. Is then also $U^* L_2(t) U$ an isospectral deformation of Jacobi operators?

• Is the property of factorization a spectral property in the sense that the density of states decides whether there exists D with $\psi(D^2) + E = L$? (We know that for special Jacobi operators $L = a\tau + (a\tau)^* + b$ with $a \in SU(N, \mathbb{C})$, the density of states decides whether the curvature $[a_i \tau_i, a_j \tau_j]$ is zero or not.)

References

[Aub 92] S.Aubry. *The concept of anti-integrability: definition, theorems and applications to the standard map.*
In "Twist mappings and their Applications", IMA Volumes in Mathematics, Vol. 44., Eds. R.Mc Gehee, K.Meyer, 1992

[Bel 92] J.Bellisard. *Le papillon de Hofstadter.*
Astérisque, 206 Séminaire Bourbaki 44ème année.

[Bob 93] A.Bobenko,N.Kutz,U.Pinkall. *The Discrete Quantum Pendulum.*
SFB 288 Preprint No.42, 1993

[Con 89] A.Connes. *Compact metric spaces, Fredholm modules, and hyperfiniteness.*
Ergod. Th. Dynam. Sys 9:207-220, (1989)

[Car 90] R.Carmona, J.Lacroix. *Spectral Theory of Random Schrödinger Operators.*
Birkhäuser, Boston, 1990

[Cyc 87] H.L.Cycon,R.G.Froese,W.Kirsch,B.Simon. *Schroedinger Operators.*
Texts and Monographs in Physics, Springer, 1987

[Gie 93]   Gieseker, Knörrer, Trubowitz. *The Geometry of algebraic Fermi curves.*
Perspectives in Mathematics, Academic Press, San Diego, 1993

[Kni 1]   O.Knill. *Isospectral Deformations of Random Jacobi Operators.*
Commun. Math. Phys., 151:403-426, 1993

[Kni 2]   O.Knill. *Factorisation of Random Jacobi Operators and Bäcklund trans-
formations.*
Commun. Math. Phys., 151:589-605, 1993

[Kni 3]   O.Knill. *Renormalization of random Jacobi operators.*
To be submitted.

[Kni 4]   O.Knill. *Embedding abstract dynamical systems in monotone twist maps.*
To be submitted.

[Las 93]   Y.Last. Zero measure spectrum for the almost Mathieu operator.
Technion preprint March, 1993

[Mor 76]   P.v.Moerbeke. *The spectrum of Jacobi Matrices.*
Invent. Math., 37:45-81, 1976

[Mor 78]   P.v.Moerbeke. *About isospectral deformations of discrete Laplacians.*
Lecture Notes in Mathematics, No. 755, Springer 1978

[Sur 89]   Y.B.Suris. *Integrable mappings of the standard type.*
Funct. Analy. Appl., 23:74-76, (1989)

Infinite particle systems

Contents

Abstract

We consider differential equations in $L^\infty(X)$ of the form

$$\dot{u} = F(u(T^{-1}), u, u(T))$$

where T is an automorphism of the probability space (X, m). Such systems form a thermodynamic limit of cyclic systems of ordinary differential equations.

We define random generalizations of dynamical systems of the form

$$\dot{u}(y) = \int F(u(x) - u(y)) \, dm(x)$$

which describe infinite particle motion with pairwise interaction.

The motion of random point vortex distributions can sometimes be described by a motion of random Jacobi operators.

1   Introduction

The description of *infinite particle motion* is a branch of *non-equilibrium statistical mechanics*. The problem of existence and uniqueness of the infinite particle motion is not easy. We refer to [Lan 75] for more information and references. But already for finitely many particles, there are outstanding problems. In the Newtonian N-body problem for example, it is still not known whether the set of initial configurations (in the $6N$ dimensional phase) having global solutions, has full Lebesgue measure (see problem 1A in [Sim 84]).

A general problem is to find and investigate solutions of infinitely many particles located at places $q_i \in \mathsf{R}^d$ which move according to

$$\ddot{q_i} = \sum_{i \neq j} F(q_i - q_j),$$

where F is the inter-particle force. In order to prevent blow up in finite time, one is either forced to restrict the set of initial phase points and prove existence and uniqueness for initial conditions in the restricted phase space or to find a Gibbs state which assigns probability one to a set of initial phase points with good regularity properties. A prototype of such a result with restricted phase space in one dimension is a theorem of Lanford [Lan 68] which states that if F is Lipschitz continuous, there is a subset of the phase space, where the system has a unique global solution. An initial point (q, p) in the allowed set satisfies bounds on the momentum $p_i = \dot{q_i}$ and bounds on the mean density of particles in any interval.

If one wants a thermodynamic limit of the above problem in which the particles are

assumed to be in a finite region of $\mathbb{R}^d$ (which implies that infinitely many particles in any bounded region) one is forced to rescale the force when going with the number of particles to infinity. This can be done as follows:

Let $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ be smooth and having compact support. The differential equation

$$\ddot{q}(x) = \int_X F(q(x) - q(y)) \, dm(y)$$

in the Banach space $L^\infty(X, \mathbb{R}^d)$ defines then the evolution of a *density field q*. This differential equation describes in some sense an infinite particle system: let (X, T, m) be an abstract ergodic dynamical system: $T : X \to X$. Using Birkhoff's ergodic theorem, the above differential equation can also be written for almost all $x \in X$ as

$$\ddot{q}_n = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{k=n-N}^{n+N} F(q_n - q_k),$$

where $q_n = q(T_n x)$. (The inclusion of the self interaction $F(q_n - q_n)$ is not relevant in the thermodynamic limit.) The finite particle approximation is then

$$\ddot{q}_n = \frac{1}{2N+1} \sum_{k \neq n, k=n-N}^{n+N} F(q_n - q_k)$$

which is (after a rescaling of time) equivalent to the original problem. In contrary to the original problem, the evolution of the infinite particle system is in the thermodynamic limit also guaranteed in the case when all particles are at the same position which corresponds to a constant q.

In one dimensions, there are also interesting systems of particles which are situated in a chain. In this case, not all the particles are interacting but only particles which are neighbors. The systems look in general as

$$\ddot{q}_n = F(q_{n-1}, q_n, q_{n+1}) \, .$$

An example is a *chain of harmonic oscillators* $\ddot{q}_n = q_{n+1} - 2q_n + q_{n-1}$ or the Toda system $\ddot{q}_n = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}}$. Again we can consider a thermodynamic limit of such systems which is defined by an abstract dynamical system. We consider flows on the Banach space $\mathcal{A} = L^\infty(X)$ defined by a differential equation

$$\dot{u}(x) = F(u(T^{-1}x), u(x), u(T(x))) \, ,$$

where $F : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function. We call such a system a *random limit of ordinary differential equations*. In the special case when X is finite, such a system is a *cyclic system* of ordinary differential equations. We will call this shortly the *finite cyclic case*.

2 Examples

We consider first some examples of random limits of cyclic systems of differential equations over an underlying dynamical system (X, T, m). In order to simplify the notation we write $u, u(T), u(T^{-1})$ instead of $u(x), u(Tx), u(T^{-1}x)$. Examples of such random limits are.

• The discrete wave equation
We call the system
$$\ddot{u} = u(T) - 2u + u(T^{-1}) =: \Delta(u)$$

the *random discrete wave equation*. This system has not yet the form of a random system but if we assume that $\dot{u}$ is an additive coboundary $\dot{u} = v(T) - v$ then one can write

$$\begin{aligned} \dot{u} &= v(T) - v , \\ \dot{v} &= u - u(T^{-1}) . \end{aligned}$$

Necessary for this is for example that $\int \dot{u}\, dm = 0$. We make an integral extension (Y, S, n) of (X, T, m) by taking two copies X_1, X_2 of X with union $Y = X_1 \cup X_2$ and defining $S : X_1 \to X_2, x \mapsto x$ and $X : X_2 \to X_1, x \mapsto Tx$. If we define for $x \in X$ $w(x) = u(x)$ and $w(S^{-1}x) = v(x)$, then the above two equations can be written as one equation
$$\dot{w} = w(S) - w(S^{-1}) .$$

This is a random system above the dynamical system (X, S, n) in the sense of the introduction. The system is linear and there exists an explicit formula for $w(t)$:

$$w(t, x) = (e^{\sigma \cdot t} - e^{\sigma^* \cdot t})w(0, x) = \sum_n \frac{t^n}{n!}(w(S^n x) - w(S^{-n}x)) ,$$

where $\sigma u(x) = u(Sx)$.

• The random Toda lattice
We system

$$\begin{aligned} \dot{a} &= a(b(T) - b) , \\ \dot{b} &= 2a^2 - 2a^2(T^{-1}) , \end{aligned}$$

or

$$\ddot{q} = e^{q(T) - q} - e^{q - q(T^{-1})}$$

is integrable and we know that the Titchmarsh-Weyl functions evolve according to the random Kac-Moerbeke system

$$\dot{c} = c(c(T) - c(T^{-1}))$$

which has for $c \to 1$ as a limit the discrete wave equation.

• Random Sine-Gordon system
The system
$$\ddot{q}(x) = q(Tx) - 2q(x) + q(T^{-1}x) + \gamma \cdot \sin(q(x))$$
has some interesting limits. The case when q is time-independent corresponds to a stationary solution called the *Standard map*. In the case when q is constant in space we have the *mathematical pendulum*. In the case when $\gamma = 0$ we have the *random wave equation*. On a fixed orbit $T^n x$, the system describes an infinite chain of pendulums which have a harmonic coupling. An obvious question is whether the system is integrable.

• Discrete wave, heat and Schrödinger equation with Riemannian metric
There is the following generalization of the wave equation: take a *random Jacobi operator* $L = a\sigma + (a\sigma)^* + b$ with $a, b \in L^\infty(X)$ and consider the *random wave equation* $\ddot{q} = Lq$ or the *random heat equation* $\dot{q} = Lq$ or the *Schrödinger equation* $ih\dot{q} = Lq$. All these equations are linear with respect to q and they can be solved in principle. Non-trivial is the study of the long-time behavior. We look now more closely at the random wave equation

$$\ddot{u} = Lu$$

when there is a factorization $L = D^2$ with a Jacobi operator D over an integral extension (X, S, n) which satisfies $S^2 = T$. Such a factorization can always be done, if L is positive definite. We have then also a factorization of the d'Alembert operator

$$\partial_t^2 - L = (\partial_t - D)(\partial_t + D)$$

and a solution of $\ddot{u} = Lu$ can be written as a linear combination of solutions of the heat equations

$$u_t = Du, \ u_t = -Du \ .$$

There is a connection between the solution of the heat rsp. Schrödinger equation and the random Toda lattice. The solution of the heat equation $\dot{u} = Lu$ can be written as $u(t) = e^{Lt} \cdot u(0)$ and the solution exists for all complex t in contrary to the differential operator case, where only a semi-flow exists. We can find a QR decomposition

$$e^{Lt} = QR = R^*Q^* \ ,$$

where $Q^*Q = Id$ and R is upper trigonal. It follows from an observation of Symes, that one gets

$$e^{Lt} = Q(t)R(t)$$

where $Q(t)$ is obtained from the solution $L(t)$ of the random Toda equation

$$L = [L^+ - L^-, L]$$

which can be written as $L(t) = Q^*(t)L(0)Q(t)$. If $L(t)\tilde{u}(t) = E\tilde{u}(t)$ then $\tilde{u}(t) = Q^*(t)\tilde{u}(0)$ and so $e^{Lt} = R(t)\tilde{u}(t)$.

• Gradient flows for twist maps

Given a *monotone twist map* with generating function h. Embedding a dynamical system (X, T, m) as a factor in the twist map is equivalent to find critical points of a functional

$$q \mapsto \mathcal{L}(q) = \int_X h(q(x), q(T(x))) \, dm(x) \ .$$

A critical point q satisfies the Euler equations

$$\delta\mathcal{L}(q) = h_1(q, q(T)) + h_2(q(T^{-1}), q) = 0 \ .$$

The second variation is a Jacobi operator. Critical points of the gradient flow

$$\dot{q} = \delta\mathcal{L}(q)$$

correspond to solutions of the Euler equations. Interesting are invariant sets of this gradient flows and their relation with the twist maps.

• Orszag-McLaughlin flow

The *Orszag-McLaughlin flow* [Str 89] has the following generalization as a random flow. Given three real constants a, b, c which satisfy $a + b + c = 0$. Look at

$$\dot{u} = a \cdot u(T)u(T^2) + b \cdot u(T^{-1})u(T^{-2}) + c \cdot u(T)u(T^{-1}) \ .$$

If X is a finite set, this is a differential equation and the system has measured positive metric entropy. (see [Str 89]). Because $\int_X u^2 \, dm$ is a constant of motion, it lives in the finite dimensional case $|X| < \infty$ on a sphere and the Lebesgue measure on these spheres is invariant. In general, the balls in $L^2(X)$ are invariant by the flow.

• Arnold-Beltrami-Childress flow (see [Str 89])

Given $a \in L^\infty$, the flow

$$\dot{u} = a(T) \cdot \sin(u(T)) + a(T^{-1}) \cdot \cos(u(T^{-1}))$$

is called *Arnold-Beltrami-Childress flow*. Because $u(x)$ can be taken modulo 2π, this is a differential equation in the space $L^\infty(X, \mathsf{T}^1)$ of circle-valued cocycles. When X is finite, the system is defined on a finite dimensional torus and is leaving invariant the Lebesgue measure. The metric entropy is also measured to be positive.

• The discrete Burger equation

See [Sha 90]. The discrete analogue of the Burgers equation $v_t = v_{xx} + 2vv_x$ is

$$p_t = p(p(T) - p) \ . \tag{1}$$

It looks similar to the Kac-van Moerbeke system. Because the integrable linear system $q_t = q(T)$ goes with the substitution $p = q(T)/q$ over into Equation (1), the later system is also integrable.

3 Other random systems

We want to exploit now another random generalizations of classical integrable dynamical systems. The aim is to describe a limit of particle motion with pair interaction when the number of particles goes to infinity. The potential has to be normalized suitably in the limit in order that a energy stays finite. In the special case of a finite probability space, we want to get the classical systems back. The differential equation in the finite case will go over into a differential equation on a Banach space. One can also look at the infinite particle system as the motion of a field or a density of particles.

Consider a classical dynamical system consisting of N particles with coordinates $q_n \in \mathbb{R}^d$ which are interacting by a potential $U(r)$. Call $p_n = \dot{q}_n$ the momentum. With the Hamiltonian

$$H(q, p) = \sum_i \frac{p_i^2}{2} + \sum_{i \neq j} U(|q_i - q_j|)$$

we get the Hamilton equations

$$\dot{q}_i = p_i, \; \dot{p}_i = \sum_{i \neq j} -\nabla U(|q_i - q_j|) \;.$$

We rewrite these equations a little bit: define the probability space $X = \mathbb{Z}_N$ with the Haar measure and a cyclic transformation $T : x \to x + 1 \pmod{N}$. Look now at the motion of two bounded functions $q, p : X \to \mathbb{R}$

$$\dot{q} = p, \; \dot{p} = -\frac{1}{N} \sum_{i=0}^{N-1} F(|q - q(T^i)|) \;,$$

where $F = \nabla(U) : \mathbb{R} \to \mathbb{R}^d$ is a smooth function with compact support. This motivates to define also for an aperiodic abstract dynamical system (X, T, m) a differential equation for $q, p \in L^\infty(X, \mathbb{R}^d)$ by

$$\dot{q} = p, \dot{p} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{i=-N}^{N} F(q - q(T^i)) \;.$$

By Birkhoff's ergodic theorem, this equation can be rewritten as

$$\ddot{q}(x) = \int_X F(q(x) - q(y)) \, dm(y)$$

and the Hamiltonian becomes the functional

$$H(q, p) = \int_X \frac{p(x)^2}{2} + \left(\int_X U(q(x) - q(y)) \, dm(y) \right) \, dm(x)$$

on $L^\infty(X, \mathbf{R}^d)^2$. Using the functional derivative, the Hamilton equations are

$$
\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p} = p \,, \\
\dot{p} &= -\frac{\partial H}{\partial q} = -\int_X \nabla U(q(x) - q(y)) \, dm(y) \,.
\end{aligned}
$$

If we understand the integrals as $\int_{\{x \neq y\}}$, which doesn't change anything in the aperiodic case, we get back the old ordinary differential equations for finite X. We interpret $q(x), p(x)$ as a *position and momentum density field* of particles. The Hamiltonian is an invariant of the given dynamical system. The self-interaction of each particle with itself (this is excluded in the finite case by taking the integral only over disjoint points) disappears in the aperiodic case. We have only to take care that

$$
q \mapsto F(q)(x) = \int_X \nabla U(q(x) - q(y)) \, dm(y) \in L^\infty(X)
$$

is Lipschitz continuous in order to get local existence of the flow. A natural problem is to find potentials U for which the flow exists for all times. The function space for f is another free parameter of this set-up.

We give now a situation, where we can prove that the flow exists locally at least:

Proposition 3.1 *Assume $F : L^\infty(X, \mathbf{R}^d) \mapsto L^\infty(X, \mathbf{R}^d)$ is differentiable, then the flow in $L^\infty(X, \mathbf{R}^d)$ defined by*

$$
\ddot{q}(x) = \int_X F(q(x) - q(y)) \, dm(y)
$$

exists locally.

Proof. The map

$$
q \in L^\infty(X) \mapsto f(q)(x) = \int_X F(q(x) - q(y)) \, dm(y) \in L^\infty(X)
$$

is Fréchet differentiable. (The derivative is

$$
Df(q)u(x) = \int F'(q(x) - q(y))(u(x) - u(y)) \, dm(y) \,,
$$

where $x \mapsto F'(x)$ is a $d \times d$-matrix valued function. Cauchy's existence theorem (see [Die 68]) implies then that the flow exists locally. $\qquad\square$

Examples:

• *Harmonic oscillator*

For $U(q) = \frac{1}{2} < q, q >$, we get $F(q) = q$ and

$$\ddot{q}(x) = \int_X F(q(x) - q(y))\, dm(x) = \int_X -(q(x) - q(y))\, dm(y)$$
$$= -q(x) + \int_X q(y)\, dm(y)$$

can easily been integrated because the center of mass $I(q) = \int_X q(y)\, dm(y)$ is an integral.

• *Exponential potential*

Take $F(q) = -e^q$. This gives

$$\ddot{q}(x) = \int -e^{q(x) - q(y)}\, dm(y) .$$

We believe that this flow is existing for all times.

The following examples are *formal* because we are dealing with singular potentials which lead to the problem that we have to choose the right function spaces and to show the existence of the flows.

• *The Calogero system*

We are in one dimension. Take $U(r) = r^{-2}$. We get the differential equation

$$\ddot{q}(x) = \int_X (q(x) - q(y))^{-3}\, dm(y) .$$

We assume that q is chosen in such a way that the right hand side is in $L^\infty(X)$. Is this condition preserved by the flow? It is not clear if the system can also be written as a Lax pair $\dot{L} = [B, L]$ in infinite dimensions like in the finite dimensional case.

• *The Sutherland system*

If we identify particles which have coordinate difference 2π, the particles are located on a circle. One is lead to the potential $U(r) = \sin^{-2}(x)$ and has the differential equation

$$\ddot{q}(x) = \int_X \cot(q(x) - q(y))\sin^{-2}(q(x) - q(y))\, dm(y) ,$$

where $q \in L^\infty(X, \mathsf{T}^1)$.

• *Vortex motion in the plane*

For $U(r) = \log(r)$, we get infinite vortex motion. We discuss this separately.

4   Random Vortices

We want to describe here a possibility to treat the motion of infinitely many vortices in the plane. The idea is to treat the *vortex distribution* as a spectral distribution

of a random Jacobi operator and to find the differential equation for the operator which defines then a motion of the spectrum. This motion of the spectrum is a thermodynamic limit of infinitely many vortices each having infinitesimal vorticity. For information on vortex motion in two dimension we refer to Aref's review article [Are 83].

Given a dynamical system (X, T, m) which is given by an automorphism T of the probability space (X, m). To a normal operator L in the crossed product $\mathcal{X}$ of $L^\infty(X)$ with the dynamical system is attached a measure dk, the *density of states* defined by the requirement

$$\operatorname{tr}(f(L)) = \int_X f(x) \, dm(x)$$

for every continuous function f on $\mathbb{C}$ and where tr is the trace in $\mathcal{X}$. If $L = a\tau + a(T^{-1})\tau^* + b$ is a random Jacobi operator with $|a(x)| \geq \delta > 0$ for all $x \in X$, the energy

$$H(L) = \int_{\mathbb{C}} \int_{\mathbb{C}} \log |E - E'| dk(E') \, dk(E)$$

is finite. The *Thouless formula* allows to write the energy using the Floquet exponent

$$w_L(E) = -\operatorname{tr}(\log(L - E))$$

which has as the real part the negative of the Lyapunov exponent.

$$
\begin{aligned}
H(L) &= \int_{\mathbb{C}} \int_{\mathbb{C}} \log |E - E'| \, dk(E') \, dk(E) \\
&= \int \log |a| \, dm + \int \lambda(E') \, dk(E') \\
&= \int -w(E') \, dk(E') = \operatorname{tr}(w_L(L)) \, .
\end{aligned}
$$

Heuristically we look first at the following finite-dimensional situation. If dk is a finite sum of Dirac measures, which is the case when L is a matrix, then the energy $H(L)$ would be $-\infty$. But if we take out the "self interaction terms" we would have

$$H(L) = N^{-1} \sum_{i \neq j} \log |E_i - E_j| \, ,$$

where for $1 \leq i \leq N$ the complex numbers E_i are the eigenvalues of L. This is exactly the energy of N vortices with constant vorticity N^{-1} located at E_i. The *motion of such vortices* is given by the differential equations

$$\frac{d}{dt} \overline{E_k} = \frac{1}{2\pi i N} \sum_{j \neq k} \frac{1}{E_k - E_j}$$

which is a Hamiltonian system for the variables $q_i = \operatorname{Re}(E_i)$ and $p_i = \operatorname{Im}(E_i)$. It can be written as a differential equation for the matrix

$$\frac{d}{dt} \overline{L} = \frac{1}{2\pi i} \frac{\partial H}{\partial L} \tag{2}$$

271

being a Hamiltonian system for $q = \text{Re}(L)$ and $p = \text{Im}(L)$

$$\dot{q} = \frac{\partial \tilde{H}}{\partial p} \ , \dot{p} = -\frac{\partial \tilde{H}}{\partial q}$$

with $\tilde{H} = \frac{1}{2\pi i} H$.

We consider now a general random Jacobi operator L which is moving according to the differential equation (2). We have now no longer to care about the self interaction problem which was urgent in the finite case, but there is another problem, namely that the energy is not necessarily Fréchet differentiable. Because $H(L) = \text{tr}(w_L(L))$, we have the formula

$$\frac{\partial H(L)}{\partial L} = 2w'_L(L) \ .$$

So, we need only to know that the Floquet exponent $w(E)$ is twice differentiable and remains differentiable in order to assure the existence of the flow.

The following Mathematica program makes a film from the evolution of arbitrarily many vortices.

```
RKStep[f_List, y_List,y0_List, dt_]:=Block[{k1,k2,k3,k4 },
  k1=dt N[ f /. Thread[y -> y0]      ];k2=dt N[ f /. Thread[y -> y0+k1/2]];
  k3=dt N[ f /. Thread[y -> y0+k2/2]];k4=dt N[ f /. Thread[y -> y0+k3  ]];
  y0+(k1+2*k2+2*k3+k4)/6];
RKLastPoint[f_List,y_List,y0_List,{t1_, dt_}:=Block[{yy},
  yy=y0;Do[yy=RKStep[f,y,yy,N[dt]],{i,1,Round[N[t1/dt]]}];yy ]
Variab[n_]:=Table[z[i],{i,n}];
Diffeq[n_]:=Table[N[1/(2 Pi I)]*
  Sum[(z[i]-z[Mod[j-1,n]+1])/(Abs[z[i]-z[Mod[j-1,n]+1]]^2+0.0001),
  {j,i+1,i+n-1}],{i,n}];
VortexFlow[cc0_,t1_,dt_]:=Block[{l=Length[cc0]},
  cc=Variab[l];hh=Diffeq[l]; RKLastPoint[hh,cc,cc0,{t1,dt}]  ];
Init[n_]:=Block[{s=2*Pi/n}, N[Table[Exp[I*s*jj],{jj,n}]] ];
Pict[z_]:=ListPlot[Table[{Re[z[[kk]]],Im[z[[kk]]]},{kk,Length[z]}],
  PlotRange->{{-3,3},{-3,3}},DisplayFunction->Identity,Axes->False];
Film[NumbPart_,NumbPict_,TimeInt_]:=Block[{c=Init[NumbPart],Movie={}},
  Do[Movie=Append[Movie,Pict[c]];
  c=VortexFlow[c,TimeInt,0.01],{m,NumbPict}]; Movie];
Display["!psfix -land -stretch > vortex.ps",
  Show[GraphicsArray[Partition[Film[31,16,1.5],4]],
  DisplayFunction->$DisplayFunction,Frame->True,
  PlotLabel->FontForm["A vortex flow with 31 particles",{"Helvetica",12}]]]
```

The program produced the following film of vortex motion:

A vortex flow with 31 particles
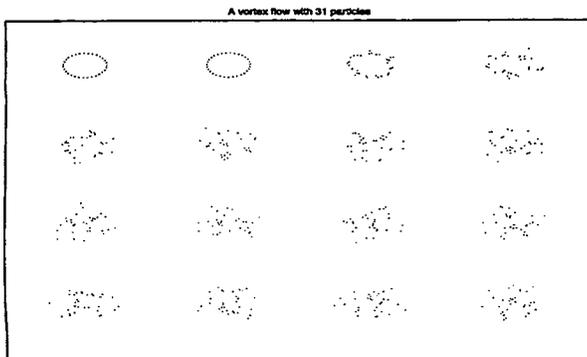
5 Questions

We add some questions.

• We would like to understand better what happens with a cyclic dynamical system in the limit when passing from the finite case to a general random limit determined by an ergodic dynamical system (X, T, m). Can one hope to understand the finite case better through the infinite system or are there new features in the infinite case? Especially we would like to know what happens in the limit if the finite cyclic case is integrable or if the global existence of the flow in the finite cyclic case implies the same in the infinite dimensional case. Is there a relation between the structure of the periodic orbits in the finite case and the random infinite dimensional case? How does non-integrability, positive topological or metric entropy in the finite case manifest itself in the infinite case?

• Can one extract from the evolution of the random system some information about the dynamical system (X, T, m)? Can new invariants for (X, T, m) be found through the study of the random differential equation?

• For which potentials $U \in C_c^\infty(\mathsf{R}, \mathsf{R})$ does there exist a function space and a manifold in this space such that the flow $\ddot{q}(x) = \int_X \nabla U(|q(x) - q(y)|) \, dm(y)$ exists for all times on this manifold? For which potentials is the flow integrable?

• We have not yet an example, where the flow of random Jacobi operators describes by its spectrum the the motion of a vortex distribution. We need a linear space of Jacobi operators for which the density of states is smooth. What further conditions have to be satisfied in order that the flow exists for all times?

References

[Are 83] H.Aref. *Integrable, chaotic and turbulent vortex motion in two dimensional flows.*
Ann. Rev. Fluid. Mech. 15:345-389, 1983

[Deim 77] K.Deimling. *Ordinary Differential Equations in Banach Spaces.*
Lecture Notes in Mathematics,No. 596,Springer 1977

[Die 68] J.Dieudonné. *Foundations of Modern Analysis, Tome I.*
Academic Press,New York and London 1968

[Kni 3] O.Knill. *Isospectral Deformations of Random Jacobi Operators.*
Commun. Math. Phys., 151:403-426, 1993

[Lan 68] O.Lanford III. *The classical mechanics of one-dimensional systems of infinitly many particles.*
Commun. Math. Phys., 9:176-191, 1968

[Lan 75] O.Lanford III. *Time evolution of large classical systems.*
Dynamical Systems: Theory and Applications, Springer Lecture notes of Physics, 38:1-111, 1975

[Sha 90] A.B.Shabat, R.I.Yamilov. *Symmetries of nonlinear chains.*
Leningrad Math. J., 2:377-400, 1990

[Sim 84] B.Simon. *Fifteen Problems in Mathematical Physics.*
Perspectives in mathematics, Anniversary of Oberwolfach 1984
Birkhäuser, Basel, 1984

[Str 89] J.-M.Strelcyn. *The Coexistence Problem for conservative dynamical systems:A review.*
Colloquium Mathematicum,LXII:331-345, 1991

Embedding of abstract dynamical systems in monotone twist maps

Contents

Abstract

Embedding an abstract dynamical system in a monotone twist map $S_\gamma : \mathsf{T}^2 \to \mathsf{T}^2$

$$(q,p) \mapsto (q + p + \gamma \cdot V'(q), p + \gamma \cdot V'(q))$$

is a variational problem.

Let Σ be the set non-degenerate critical points of $V \in C^2(\mathsf{T}, \mathsf{R})$. Using the anti-integrable limit of Aubry and Abramovici [Aub 92a],[Aub 90b], we show that there exists a constant $\gamma_0 > 0$ such that every ergodic abstract dynamical system (X, T, m) with metric entropy $h_m(T) \leq \log(|\Sigma|)$ and $|\gamma| \geq \gamma_0$ can be embedded in the twist map S_γ. For such γ, the topological entropy of S_γ is at least $\log(|\Sigma|)$.

Using a generalized Morse index, the integrated density of states of the Hessian at a critical point, we prove the existence of uncountably many different embeddings of an aperiodic dynamical system (X, T, m) if $h_m(T) \leq \log(|\Sigma| - 1)$ and $|\gamma| \geq \gamma_0$.

1 Introduction

An abstract dynamical system (X, T, m) is *embedded* in a topological dynamical system (Y, S), if there exists an S-invariant probability measure μ on Y, such that (X, T, m) and (Y, S, μ) are isomorphic as abstract dynamical systems. Interesting topological dynamical systems are monotone twist maps which are discrete Hamiltonian systems.

A general question is to decide whether a given abstract dynamical system (X, T, m), an automorphism T of a standard probability space (X, m), can be embedded in a given monotone twist map like the *generalized Standard map*

$$S_\gamma : \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} q + p + \gamma \cdot V'(q) \\ p + \gamma \cdot V'(q) \end{pmatrix} ,$$

with $V \in C^2(\mathsf{T}^1, \mathsf{R})$ and real parameter $\gamma \in \mathsf{R}$. Such a twist map (Y, S_γ) is a diffeomorphism on the two dimensional cylinder $Y = \mathsf{T} \times \mathsf{R}$ leaving invariant the Lebesgue measure. Because $S_\gamma(q, p + 1) = (q', p' + 1)$, the map S_γ acts also on the torus T^2. For each abstract dynamical system (X, T, m), the embedding question is a variational problem for a *Percival functional*

$$q \mapsto \mathcal{L}(q) = \int_X l(q(x), q(Tx)) \, dm(x)$$

on the Banach manifold $L^\infty(X, \mathsf{T}^1)$, where

$$l(q, q') = -\frac{(q' - q)^2}{2} - \gamma \cdot V(q)$$

276

is the *generating function*. Given a critical point q satisfying the *Euler equations*

$$\delta\mathcal{L}(q) = q(T) - 2q + q(T^{-1}) - \gamma \cdot V'(q) = 0 \, ,$$

a factor of the dynamical system is embedded in the twist map. The homomorphism between (X, T, m) and the embedded system (Y, S, μ) is given by the map

$$\phi : X \quad \to \mathsf{T} \times \mathsf{R},$$
$$x \quad \mapsto (q(x), p(x)) = (q(x), q(Tx) - q(x) - \gamma \cdot V'(q(x))) \ .$$

The measure μ is defined by $\mu(Z) = m(\phi^{-1}(Z))$. If q is not constant, then the factor is non trivial. If ϕ is injective on a subset of X with full measure, the system (X, T, m) itself is embedded. The Fréchet derivative of the operator

$$\delta\mathcal{L} : L^\infty(X) \to L^\infty(X), \ q \mapsto q(T) - 2q + q(T^{-1}) - \gamma \cdot V'(q)$$

is a linear Jacobi operator on $L^\infty(X)$ given by

$$Lu = u(T) - 2u + u(T^{-1}) - \gamma \cdot V''(q)u \ .$$

If this operator is invertible, then the embedded system is structurally stable in the sense that a small change in the generating function doesn't destroy the embedding.

Examples of known embeddings are :

• For every S-invariant probability measure m on the cylinder $\mathsf{T} \times \mathsf{R}$, one has an abstract dynamical system $(X, T, m) = (\text{supp}(m), S, m)$, where T is the restriction of S to X. There exists the critical point $q(x) = \pi_1(x)$ of $\mathcal{L}$, where π_1 is the projection on the first coordinate on the cylinder $\mathsf{T} \times \mathsf{R}$.

• Well studied is the question of embedding a finite ergodic dynamical system because this is equivalent to finding *periodic orbits*. For the existence of periodic orbits in twist maps see for example [Bro 75], [Kat 82], [Hal 88], [Ang 88].

• Also the embedding of irrational rotations of the circle is well investigated. Smooth functions q correspond to *KAM tori* (see [Mos 73], [Cel 88], [Her 83a], [Sal 89] and references therein), discontinuous critical points q belong to invariant Cantor sets like *Aubry-Mather sets* (also nicknamed Cantor-Aubry-Mather (CAM) sets). There exist several proofs of their existence [Mat 82], [Kat 82], [Den 76], [Mos 87], [Gol 92]; see also [Ban 88], [Mos 86].

• *Bernoulli shifts* can be embedded, if the topological entropy is positive [Kat 80] or if there exists a *homoclinic point* of the twist map. See [Fon 90] for existence results or [Zeh 73], [Gen 90] for genericity results. Angenent [Ang 92], [Ang 90] shows in

some cases that the topological entropy is positive.

The above mentioned results are in general difficult to prove. A simple approach to the embedding problem (which doesn't cover most of the above mentioned results) is due to Aubry and Abramovici [Aub 92a],[Aub 90b], who introduced the so called *anti-integrable limit* $l_0(q, q') = V(q)$ of the variational problem. The idea works if there is a nonempty set Σ of non degenerate critical points. Every $q \in L^\infty(X, \Sigma)$ is a critical point of $\mathcal{L}_0$. The anti-integrable limit doesn't correspond to a twist map any more. The variational problem, however, still makes sense. Under the condition that there are finitely many non degenerate critical points of V, a critical point of $\mathcal{L}$ has an invertible Hessian and the implicit function theorem allows the continuation of the critical points to situations which correspond to twist maps. It follows that each abstract dynamical system has a nontrivial factor embedded in a twist map.
In contrary to [Aub 90b], [Mac 92], we don't prove the existence of embedded *orbits* but the existence of embedded *abstract dynamical systems*. The main result is that we can embed every ergodic abstract dynamical system of finite entropy in a monotone twist map. More precisely, every ergodic dynamical system of metric entropy $\leq \log(|\Sigma|)$ can be embedded in the twist map S_γ if $|\gamma|$ is large enough. This implies immediately, that the topological entropy of the map S_γ is $\geq \log(|\Sigma|)$. As an example, we know then, that the topological entropy of the Standard map is $\geq \log(2)$ provided $|\gamma|$ is large enough. The proof of the embedding result Theorem 4.1 uses Krieger's theorem, which states that a system with finite entropy has a finite partition as a generator.
In Section 5 we will also assign to each embedding an *index* which is just the integrated density of states (at the energy $E = 0$) of the Hessian $L = \delta^2\mathcal{L}(q)$ at the critical point q. This index is a *generalized Morse index*, because in the case of a finite dynamical system with N points, the integrated density of states $k = k(0)$ of L is related with the Morse index K, the dimension of the stable manifold of the critical point, by the relation $k = K/N$. Because two critical points with different index belong to different embeddings, this leads in general to uncountably many different embeddings of the same dynamical system namely to an embedding for each Morse index in an interval.

2 Monotone twist maps

Given a function $l \in C^2(\mathbb{R}^2)$. We write l_i for the derivative to the i'th variable in l. Assume l satisfies the following twist and periodicity conditions

$$l_{12}(q, q') \geq \delta > 0 \, ,$$
$$l(q, q') = l(q + 2\pi, q' + 2\pi)$$

for all $(q, q') \in \mathbb{R}^2$. (The second condition is sometimes also called "no-flux condition"). With $p(q, q') = l_1(q, q')$, $p'(q, q') = -l_2(q, q')$, the real variable q' and thus

also p' can be expressed as a function of q and p. The mapping $S : \mathsf{T} \times \mathsf{R} \to \mathsf{T} \times \mathsf{R}$

$$S : (q, p) \mapsto (q', p')$$

is a *monotone twist map* with a *generating function l*. The Lebesgue measure ν is invariant on the cylinder $\mathsf{T} \times \mathsf{R}$, where $\mathsf{T} = \mathsf{R}/(2\pi\mathsf{Z})$.

Take any *abstract ergodic dynamical system* (X, T, m), where T is a measure preserving invertible map on the Lebesgue space (X, m). A critical point $q : X \to \mathsf{T}^1$ of the *Percival functional* (see [Per 80], [Mat 82], [Laz 84])

$$\mathcal{L}(q) = \int_X l(q, q(T)) \, \mathrm{dm}$$

on the Banach manifold $L^\infty(X, \mathsf{T})$ exists, if and only if

$$\delta\mathcal{L}(q) = l_1(q, q(T)) + l_2(q(T^{-1}), q) = 0 \; .$$

This *Euler equation* can be written with

$$
\begin{aligned}
p(x) &= l_1(q(x), q(Tx)) \\
p'(x) &= -l_2(q(x), q(Tx))
\end{aligned}
$$

as

$$p = p'(T^{-1}) \; .$$

A *factor* $(\tilde{X}, \tilde{T}, \tilde{m})$ of a dynamical system (X, T, m) is a homomorphic image of (X, T, m). In other words, there exists a measurable not necessarily invertible map $\phi : X \to \tilde{X}$ with $\phi T = \tilde{T} \phi$. Every dynamical system has the factor (X, T, m) and the trivial factor $|\tilde{X}| = 1$. Factors different from $|\tilde{X}| = 1$ are called *nontrivial*.

A dynamical system (X, T, m) is *embedded* in a topological dynamical system (Y, S), if there exists a S-invariant Borel probability measure μ on Y, such that (S, T, m) and (Y, S, μ) are isomorphic as abstract dynamical systems.

Lemma 2.1 *If there exists $q \in L^\infty(X, \mathsf{T}^1)$ satisfying*

$$\delta\mathcal{L}(q) = l_1(q, q(T)) + l_2(q(T^{-1}), q) = 0 \; ,$$

then a factor of the given dynamical system can be embedded in the twist map. If q is not constant, this factor is nontrivial.

Proof. Let (X, T, m) be an abstract dynamical system and let q be a critical point of the functional $\mathcal{L}$ which belongs to the monotone twist map $(\mathsf{T} \times \mathsf{R}, S)$. Define the map

$$
\begin{aligned}
\phi : X & \to \mathsf{T} \times \mathsf{R} \; , \\
x & \mapsto (q(x), l_1(q(x), q(Tx))) \; .
\end{aligned}
$$

As q is a critical point, we get, using the Euler equation $-l_2(q(T^{-1}), q) = l_1(q, q(T))$,

$$
\begin{aligned}
S\phi(x) &= S(q(x), p(x)) = ((q'(x), p'(x)) = (q(Tx), -l_2(q(x), q(Tx))) \\
&= (q(Tx), l_1(q(Tx), q(T^2x))) = (q(Tx), p(Tx)) = \phi(Tx)
\end{aligned}
$$

or shortly $S\phi = \phi T$. The image Y of ϕ is a measurable set and the dynamics of S induced on Y is a factor of T if we take the S-invariant measure $\mu(Z) = m(\phi^{-1}(Z))$ on Y. If q is not constant, then ϕ is not constant and the factor (Y, S, μ) is nontrivial. $\square$

Given a critical point q of the Percival functional $\mathcal{L}$, the second variation (the Hessian) of $\mathcal{L}$ is a bounded operator on $L^\infty(X)$ given by

$$
L(q) = \delta^2 \mathcal{L}(q) = a\tau + (a\tau)^* + b,
$$

with the multiplication operators $a, b \in L^\infty(X)$ given by

$$
\begin{aligned}
a(x) &= l_{12}(q(x), q(Tx)), \\
b(x) &= l_{11}(q(x), q(Tx)) + l_{22}(q(T^{-1}x), q(x)),
\end{aligned}
$$

where τ, τ^* are the shifts $\tau(f) = f(T)$ and $\tau^*(f) = f(T^{-1})$. The operator acts also in the same way as a bounded selfadjoint operator on the Hilbert space $L^2(X)$.

3 The existence of critical points for the Percival functional

What abstract dynamical systems (X, T, m) can be embedded in a given monotone twist map? This question is equivalent to the problem of finding a critical point q of the Percival functional $\mathcal{L}$ on $L^\infty(X, T^1)$, so that $\phi : x \mapsto (q(x), l_1(q(x), q(Tx)))$ is injective on a set of full measure in X.

Assume, the generating function l has the form

$$
l_\gamma(q, q') = -\frac{(q - q')^2}{2} - \gamma \cdot V(q),
$$

where $V \in C^2(T, \mathbb{R})$ has a non empty set Σ of non degenerate critical points. It defines the twist map

$$
S : \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} q + p + \gamma \cdot V'(q) \\ p + \gamma \cdot V'(q) \end{pmatrix}
$$

which we call *generalized Standard map*. For $\gamma = 0$, the map is integrable. Denote by $\mathcal{L}_\gamma$ the Percival functional belonging to the generating function l_γ.

Proposition 3.1 *Given any abstract dynamical system (X, T, m). For $|\gamma|$ big enough, there exists a nontrivial factor of (X, T, m) embedded as a subsystem in the twist map generated by l_γ.*

Proof. Following [Aub 90b], we take for $\epsilon := \frac{1}{\gamma}$ the equivalent generating function

$$l_\epsilon(q, q') = -\epsilon \cdot \frac{(q - q')^2}{2} - V(q)$$

with corresponding functional $\mathcal{L}_\epsilon$ and Hessian

$$L_\epsilon = \epsilon \cdot (\tau - 2 + \tau^*) - V''(q_\epsilon)$$

at a critical point q_ϵ and prove the claim for ϵ small enough.

The case $\epsilon = 0$ is called the *anti-integrable limit*. It does not generate a twist map any more but the variational problem for the Percival functional $\mathcal{L}(q)$ still makes sense.

Any function $q \in L^\infty(X, \Sigma) \subset L^\infty(X, \mathsf{T}^1)$ is a critical point of $\mathcal{L}_0$. The second variation L_0 at such a critical point q_0 is the multiplication operator

$$L_0 = -V''(q_0) \ .$$

Applying the implicit function theorem, we know that for ϵ small enough, there still exists a critical point q_ϵ of $\mathcal{L}_\epsilon$. If q_0 was a non-constant critical point, then for ϵ small enough, also q_ϵ is not constant and gives a nontrivial critical point. Applying Lemma 2.1 leads to an embedding of a nontrivial factor of (X, T, m). $\qquad\square$

We obtain immediately from Proposition 3.1

• Embeddings of Bernoulli shifts, because a nontrivial factor of a Bernoulli shift is a Bernoulli shift [Orn 70]).

• Mixing systems, because a nontrivial factor of a mixing system is mixing ([Cor 82] p 231).

• Periodic orbits with prime period P, because a cyclic permutation (X, T, m) of $|X| = P$ elements has no nontrivial factor except the system itself. One obtains at least

$$(|\Sigma|^P - |\Sigma|)/P$$

($\in \mathsf{N}$ by Fermat) different periodic orbits of period P, because there are $|\Sigma|^P - |\Sigma|$ different non constant functions in $L^\infty(X, \Sigma)$ and P of them belong to the same orbit.

• Irrational rotations, because every factor of the dynamical system $(\mathsf{T}, x \mapsto x + \alpha, dx)$ has the form $(\mathsf{T}, x \mapsto x + n \cdot \alpha, dx)$ for some $n \in \mathsf{N}$. The reason is that

every group translation has discrete spectrum and the spectrum of a factor is then a subgroup of the spectrum. The fact that a system with discrete spectrum can be reconstructed uniquely by the spectrum, leads to all the factors of the irrational rotation.

(Compare also the results in [Mac 92], where it is shown, how Cantori can be embedded in the more general case of symplectic twist maps.)

Remark. The anti-integrable limit in discrete Hamiltonian systems like twist maps can also be defined in the continuous analogue where the Lagrangian is

$$\int_a^b m \cdot \frac{\dot{x}(t)^2}{2} + V(x(t)) \, dt \, .$$

For example, the continuous analogue of the Standard map is the *pendulum* with $V(x) = \sin(x)$. The anti-integrable limit is defined to be the case, when *the mass m vanishes*. In such a case, there are critical points $x(t) = \sigma$, where σ is a critical point of the potential V. We see that because time is continuous, there are only constant critical points in the anti- integrable limit and these points keep on being non-interesting also for positive mass. In the discrete case the discreteness of time allows to jump in a time step from a critical point σ_1 of the potential V to another critical point σ_2 of V. These critical points of the functional persist, when the mass is switched on and give interesting nontrivial solutions of the system.

4 Ergodic dynamical systems with finite entropy can be embedded in a monotone twist map

We will show now that every ergodic dynamical system (X, T, m) with finite metric entropy can be embedded in a monotone twist map S on T^2. There is some obstruction in that the metric entropy $h_m(T)$ of T can't be bigger than the topological entropy $h(S)$ of S because of the *variational principle* [Wal 82]

$$h(S) = \sup_{\mu \in M_S} h_\mu(S) \, ,$$

where M_S is the set of S-invariant probability measures on T^2. It turns out that a restriction in the entropy is the only one:

Theorem 4.1 *Given an abstract ergodic dynamical system (X, T, m) with metric entropy $h_m(T) < \infty$. Denote by Σ the set of nondegenerate critical points of $V \in C^2(\mathsf{T})$. If*

$$|\Sigma| \geq e^{h_m(T)} \, ,$$

there exists $\gamma_0 > 0$ such that for any $|\gamma| > \gamma_0$, the system (X, T, m) can be be embedded in the twist map $S_\gamma : \mathsf{T}^2 \to \mathsf{T}^2$

$$\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} q + p + \gamma \cdot V'(q) \\ p + \gamma \cdot V'(q) \end{pmatrix} \ .$$

Furthermore, there exists $\gamma_1 > 0$ such that for $\gamma > \gamma_1$, the metric entropy of S_γ is bounded below by $\log(|\Sigma|)$.

Proof. We first deal with the periodic case $|X| < \infty$.

If $X = \{x_0, x_1, \ldots, x_{N-1}\}$ is finite and $Tx_i = x_{i+1 \,(\mathrm{mod}\, N)}$, we choose a sequence $(\sigma_0, \sigma_1, \ldots, \sigma_{N-1})$ with $\sigma_i \in \Sigma$ such that

$$\sigma_{i+D \,(\mathrm{mod}\, N)} = \sigma_i, \ \forall i = 0, \ldots N - 1$$

is excluded for all nontrivial integer factors $D \mid N$ of N. Take in the anti-integrable limit the critical point

$$q_0(x_i) = \sigma_i \ .$$

For small ϵ, the embedded periodic orbits must have period N.

We assume now that the system (X, T, m) is aperiodic. We are looking for critical points q of $\mathcal{L}$ such that $\phi : X \to \mathsf{T} \times \mathsf{R}$,

$$x \mapsto (q(x), p(x)) = (q(x), l_1(q(x), q(Tx)))$$

is injective on a set of full measure in X.

According to *Krieger's theorem* (see [Wal 82] p.97), every ergodic aperiodic abstract dynamical system (X, T, m) of finite entropy has a finite partition $(Y_1, \ldots Y_n)$ with $n \geq e^{h_m(T)}$ which is a generator for the dynamical system. The system (X, T, m) is then isomorphic to $(\tilde{X}, \tilde{T}, \tilde{m})$, where $\tilde{X} = \{1, 2, \ldots, n\}^{\mathsf{Z}}$ and $\tilde{T}$ is the shift transformation leaving invariant some ergodic measure $\tilde{m}$. If one defines for $x \in X$ the sequence

$$\tilde{x}_n = j, \text{ for } T^n(x) \in Y_j \ ,$$

the conjugation of (X, T, m) and $(\tilde{X}, \tilde{T}, \tilde{m})$ is given by the measurable map

$$\psi : x \mapsto \{\tilde{x}_n\}_{n \in \mathsf{Z}} \ .$$

We work now with the shift dynamical system $(\tilde{X}, \tilde{T}, \tilde{m})$ denoted again by (X, T, m) and forget about the the the old equivalent system. Choose n different points $\sigma_1, \ldots, \sigma_n \in \Sigma$. Take in the anti-integrable limit the critical point q_0 defined by

$$q_0(x) = x_0 \ .$$

Knowing the function q_0 and the shift transformation (X, T, m) allows the reconstruction of the dynamical system (X, T, m):

283

Claim, For small enough $\epsilon > 0$, the map $\phi_\epsilon : X \to \mathsf{T} \times \mathsf{R}$,

$$x \mapsto (q_\epsilon(x), p_\epsilon(x)) = (q_\epsilon(x), l_1(q_\epsilon(x), q_\epsilon(Tx)))$$

is injective on a set of measure 1.

Proof. Because $\epsilon \mapsto q_\epsilon \in L^\infty(X, \mathsf{T}^1)$ is continuous for ϵ small enough, there exists $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$ and the usual metric d on T^1,

$$d(q_\epsilon(x), q_\epsilon(y)) \geq d(q_0(x), q_0(y))/2 .$$

Given $x \neq y \in X$, there exists $n \in \mathsf{Z}$ such that $x_n \neq y_n$ and so $q_0(T^n x) \neq q_0(T^n y)$ and also $q_\epsilon(T^n x) \neq q_\epsilon(T^n y)$ or

$$S^n(\phi_\epsilon(x)) = (q_\epsilon(T^n x), p_\epsilon(T^n x)) \neq (q_\epsilon(T^n y), p_\epsilon(T^n y)) .$$

Because the twist map S is invertible, it follows that $\phi_\epsilon(x) \neq \phi_\epsilon(y)$. This finishes the proof that ϕ_ϵ is injective.

We have shown therefore that the system $(\phi_\epsilon(X), S, m)$ which is embedded in the twist map, is isomorphic to (X, T, m) if ϵ is small enough.

There exists $\gamma_1 > 0$ such that we can embed the Bernoulli shift with entropy $\log(|\Sigma|)$. From the variational principle, we know that the topological entropy is at least $\log(|\Sigma|)$, if $|\gamma| > \gamma_1$. $\qquad\qquad\Box$

For the Standard map, where $V(x) = -\cos(x)$, the set of nondegenerate critical points of V satisfies $|\Sigma| = 2$ and we get

Corollary 4.2 *Any ergodic abstract dynamical system* (X, T, m) *with entropy* $h_m(T) \leq \log(2)$ *can be embedded in some Standard map* $S_\gamma : \mathsf{T}^2 \to \mathsf{T}^2$,

$$(q, p) \mapsto (q + p + \gamma \cdot \sin(q), p + \gamma \cdot \sin(q)) .$$

There exists $\gamma_0 > 0$ *such that for* $|\gamma| > \gamma_0$, *the topological entropy of* S_γ *is at least* $\log(2)$.

Theorem 4.1 tells that monotone twist maps are very rich from the ergodic theoretical point of view. The classification problem for abstract dynamical system is already present in monotone twist maps and because this classification problem is believed to be intractable, there is also no hope to classify in general all ergodic invariant measures for a monotone twist map up to isomorphism.

Remark. There exists a one parameter family of real tori

$$V_\gamma = \{(z, w, u, v) \in \mathsf{C}^4 \mid |z| = |u| = \frac{\gamma}{2}, |v| = |w| = 1, z = \overline{u}, w = \overline{v}\}$$

in $\mathbb{C}^4$ such that the analytic map $U : \mathbb{C}^4 \to \mathbb{C}^4$

$$U : \begin{pmatrix} z \\ w \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} zwe^{z-u} \\ we^{z-u} \\ uve^{u-z} \\ ve^{u-z} \end{pmatrix}$$

induces on V_γ the Standard map S_γ [Kni 5]. It follows that every ergodic abstract dynamical system (X, T, m) with metric entropy $\leq \log(2)$ can be found in the analytic map U.

5 A generalized Morse index at the critical points

We call two embeddings of abstract dynamical systems in a monotone twist map S *different*, if the measures of the two systems are different as invariant measures of S. Even if two different critical points q_1, q_2 of the variational problem have a positive distance, these critical points may correspond to the same embedded system. If q is not constant, then for example q and $q(T)$ have positive distance but are belonging to the same embedded dynamical system.

We will define now a *generalized Morse index*, which allows to distinguish different critical points belonging to different embeddings.

5.1 The Hessian at a critical point

The Fréchet derivative L_ϵ of $\delta\mathcal{L}$ at a critical point $\delta\mathcal{L}_\epsilon(q) = 0$ is a Jacobi operator. It acts on $L^\infty(X)$ as

$$u \mapsto \epsilon(u(T) - 2u + u(T^{-1})) - V''(q)u .$$

The operator is in the same way as a selfadjoint operator on $L^2(X)$ and it can be rewritten as

$$L = \epsilon(\tau - 2 + \tau^*) - V''(q) ,$$

where $\tau : u \mapsto u(T)$ and $\tau^* : u \mapsto u(T^{-1})$ are the unitary shifts belonging to T and T^{-1} and $V''(q)$ is a multiplication operator.

5.2 A von Neumann algebra with a trace

It is advantageous to look at L not as an operator acting on $L^\infty(X)$ or $L^2(X)$ but as an element of a von Neumann algebra $\mathcal{X}$ having a trace [Kni 3]. This algebra is the *crossed product* of $L^\infty(X)$ with the dynamical system (X, T, m) defined as follows: Consider the set of sequences $K_n \in L^\infty(X)$, where $K_n \neq 0$ only for finitely many $n \in \mathbb{Z}$. This is an algebra with the multiplication

$$(KM)_n(x) = \sum_{k+m=n} K_k(x) M_m(T^k x)$$

and involution
$$(K^*)_n(x) = \overline{K_{-n}(T^n x)} .$$
$\mathcal{X}$ is the completion of this algebra with respect to the norm
$$|||K||| = |\ ||K(x)||\ |_\infty ,$$
where $K(x)$ is the bounded operator in $l^2(\mathbf{Z})$ given by the infinite matrix
$$[K(x)]_{mn} = K_{n-m}(T^m x) .$$
There exists a *trace*
$$\mathrm{tr}(K) = \int_X K_0\ \mathrm{dm} .$$
The elements $K \in \mathcal{X}$ can be written in the form
$$K = \sum_n K_n \tau^n.$$
The multiplication in $\mathcal{X}$ is the multiplication of power series with the additional rule $\tau^k K_n = K_n(T^k)\tau^k$ for shifting the τ's to the right and the requirement that $\tau^* = \tau^{-1}$. If τ is taken to be the shift operator $f \mapsto f(T)$ in $L^2(X)$ with K_n as the multiplication operator, we get a representation of $\mathcal{X}$ in $\mathcal{B}(L^2(X))$ by
$$Kf = \sum_n K_n f(T^n) .$$

5.3 Jacobi operators and the density of states

Selfadjoint operators of the form $L = a\tau + (a\tau)^* + b$ with $a, b \in L^\infty(X,\mathbf{R})$ are called *Jacobi operators*. They form a real Banach space in $\mathcal{X}$. The functional calculus for a normal element K in the C^* algebra $\mathcal{X}$ defines $f(K)$ for a function $f \in C(\sigma(K))$ where $\sigma(K)$ is the spectrum of K. The mapping
$$f \mapsto \mathrm{tr}(f(K))$$
is a bounded linear functional on $C(\sigma(K))$, and by Riesz representation theorem, there exists a measure dk on $\sigma(K)$ with
$$\mathrm{tr}(f(K)) = \int_{\sigma(K)} f(E)\ dk(E) .$$
This measure dk is called the *density of states* of K. For selfadjoint elements $K \in \mathcal{X}$, the density of states dk has its support on $\mathbf{R}$. The *integrated density of states* is for $E \in \mathbf{R}$ defined by
$$k(E) = \int_{-\infty}^E dk(E') = \mathrm{tr}(\frac{1}{\pi}\arg(L - E)) .$$
To a Jacobi operator L is attached the *Floquet exponent*
$$w(E) := -\mathrm{tr}(\log(L - E)) ,$$
which is defined for $\mathrm{Im}(E) > 0$. The branch of the logarithm is chosen so that $\log(1) = 0$.

5.4 Rotation number and Lyapunov exponent

For the *transfer cocycle*

$$A_E(x) := a^{-1}(T^{-1}x) \begin{pmatrix} E - b(x) & -a^2(T^{-1}x) \\ 1 & 0 \end{pmatrix}$$

of $L = a\tau + (a\tau)^* + b$, the *Lyapunov exponent* is defined by

$$\lambda(A_E) = \lim_{n \to \infty} n^{-1} \int_X \log \|A_E^n(x)\| \, dm(x) \, ,$$

where $A_E^n(x) = A_E(T^{n-1}x) \dots A_E(Tx)A_E(x)$ and the *rotation number* is given by $\rho(A_E) = \pi k(E)$. The rotation number can be defined for real E by the cocycle A_E alone [Del 83].

Remark. Rotation numbers have been found by Herman [Her 83] in the case of continuous cocycles homotopic to the identity and by Ruelle [Rue 85] in the case of measurable cocycles with values in the universal covering of $SL(2, \mathbb{R})$. For cocycles arising from discrete Schrödinger operators, one obtains a rotation number by counting the average number of sign-changes of a sequence $u = u_n$ satisfying $Lu = Eu$. The relation between such a rotation number and the integrated density of states is shown in [Del 83]. A rotation number for $sl(2, \mathbb{R})$ cocycles over flows is defined in [Joh 82]. For finite $|X|$ the relation between the rotation number and the Morse index is given in [Mat 84].

The *Thouless-formula* (see [Cyc 87]) relates the Floquet exponent $w(E)$ with the Lyapunov exponent and the rotation number of the cocycle A_E:

$$-\lambda(A_E) + i\rho(A_E) = w(E) \, .$$

This shows that the Floquet exponent is defined also for $\text{Im}(E) = 0$. The Floquet exponent $w(E)$ as well as the rotation number and the Lyapunov exponent are averaged quantities of the embedded dynamical system.

5.5 A generalized Morse index

We return now to the Jacobi operator which is the Hessian at a critical point q_ϵ for $\epsilon > 0$. For simplicity, we take again the old equivalent functional $\mathcal{L}_\epsilon/\epsilon$ and the corresponding Hessian

$$L_\epsilon = \tau - 2 + \tau^* - \frac{1}{\epsilon} \cdot V''(q_\epsilon) \, .$$

The *transfer cocycle*

$$x \mapsto A_E(x) = \begin{pmatrix} E + \frac{1}{\epsilon} \cdot V''(q(x)) + 2 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{C})$$

of L_ϵ is for $E = 0$ conjugated by the matrix

$$B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

to the Jacobean cocycle

$$x \mapsto dS_\gamma(x) = BA_E(x)B^{-1} = \begin{pmatrix} 1 + \frac{1}{\epsilon}V''(q(x)) & 1 \\ +\frac{1}{\epsilon}V''(q(x)) & 1 \end{pmatrix} \in SL(2, \mathbb{R})$$

of the twist map $S_\gamma : \mathsf{T} \times \mathsf{R} \mapsto \mathsf{T} \times \mathsf{R}$ with $\gamma = 1/\epsilon$.

Given a critical point q_ϵ of the functional $\mathcal{L}_\epsilon$, the *generalized Morse index* is the integrated density of states

$$k(E) = \operatorname{tr}(\frac{1}{\pi}\arg(L - E))$$

at $E = 0$ of the Hessian L_ϵ at the critical point q_ϵ. We chose $\arg(E) \in \{0, \pi\}$ for real E in a gap of the spectrum. The generalized Morse index is then a real number in $[0, 1]$.

The integrated density of states is in the anti-integrable limit $\epsilon = 0$ and for E in a gap of the spectrum of

$$L_0 = -V''(q_0)$$

given by

$$k(E) = \operatorname{tr}(\frac{1}{\pi}\arg(L_0 - E)) = m(\{x \in X \mid -V''(q_0(x)) < E\}) .$$

The Lyapunov exponent goes to ∞ in the anti-integrable limit. Notice that for E in a gap of the spectrum of the Hessian at q_0, the integrated density of states is constant for all q in a neighborhood of q_0, because an open gap in the spectrum stays open for L in a neighborhood of L_0.

6 Critical points near the anti-integrable limit correspond to hyperbolic sets

The embedded dynamical system (X, T, m) can always be chosen to be a closed subset $\overline{Y}$ of the cylinder:

Lemma 6.1 *To each critical point q of an abstract dynamical system (X, T, m), there is a continuous critical point $\overline{q} \in C(\overline{X}, \mathsf{R}^1)$ embedding a topological dynamical system $(\overline{X}, \overline{T}, \overline{m})$ isomorphic to (X, T, m).*

288

It follows that the image of $\overrightarrow{\phi} = (\overline{q}, l_1(\overline{q}, \overline{q}(T)))$ is a compact set in $\mathsf{T} \times \mathsf{R}$.

Proof. A successful embedding of (X, T, m) gives a S invariant Borel measure μ on $Y = \mathsf{T} \times \mathsf{R}$ defined by $\mu(Z) = m(\phi^{-1}(Z))$ for all measurable sets $Z \subset \mathsf{T} \times \mathsf{R}$. Such a Borel measure is automatically regular and its support is a compact subset of $\mathsf{T} \times \mathsf{R}$. The abstract dynamical system (Y, S, μ) is isomorphic to (X, T, m). Start with $(\overline{X}, \overline{T}, \overline{m}) := (Y, S, \mu)$ as the abstract dynamical system. The function $\overline{q}$: $\mathsf{T} \times \mathsf{R} \to \mathsf{T}^1$,

$$\overline{q}((q,p)) = q$$

for $(q, p) \in \mathsf{T} \times \mathsf{R}$ is a critical point of this variational problem and the corresponding map $\overline{\phi}$ has a closed image. □

Proposition 6.2 (Aubry-Mackay-Baesens) *Given a critical point q_ϵ of $\mathcal{L}_\epsilon$ with Hessian $L_\epsilon = \delta^2 \mathcal{L}(q)$. If 0 is not in the spectrum of L, the embedded factor of the dynamical system (X, T, m) is a* hyperbolic set.

See [Aub 92c] for a proof. One has to show that the Jacobean cocycle dS_γ with $\gamma = 1/\epsilon$ is uniformly hyperbolic. This is equivalent to show that the conjugated transfer cocycle A_0 of L_ϵ is uniformly hyperbolic which means that there exist one-dimensional vector spaces $x \mapsto W^{(1)}(x), W^{(2)}(x) \subset \mathsf{R}^2$ satisfying

$$A_0(x) W^{(i)}(x) = W^{(i)}(Tx)$$

and real constants $\Gamma > 0, \alpha < 1$ such that for all $x \in X$, $n \in \mathsf{N}$ and unit vectors $w^{(i)}(x) \in W^{(i)}(x)$

$$\begin{aligned} |A_0^{-n}(x) w^{(1)}(x)| &\leq \Gamma \alpha^n |w^{(1)}(x)| , \\ |A_0^n(x) w^{(2)}(x)| &\leq \Gamma \alpha^n |w^{(2)}(x)| . \end{aligned}$$

An other proof different from [Aub 92c] can also be done also by constructing a strict invariant cone bundle of the cocycle and using a theorem of Ruelle [Rue 79a].

Remarks.
a) The existence of a non-atomic invariant measure with non-zero Lyapunov exponents implies again that the topological entropy for the twist map is positive [Kat 80] for $|\gamma|$ large enough.
b) Hyperbolic Mather sets have been constructed by Goroff [?] for the Standard map with parameter $\gamma > 2\sqrt{1 + \pi^2}$.
c) For results about hyperbolic sets like the shadowing property see [Lan 85]. In [Aub 92c] there is an argument why the constructed hyperbolic sets have zero Lebesgue measure. This argument needs a review, because it relies on a result of Ruelle and Bowen [Bow 75] which assumes that the hyperbolic set is locally maximal. In general, the constructed hyperbolic set Y may fail to be *locally maximal.*

(Local maximality is defined as follows. There exists an open set U containing the hyperbolic set Y such that

$$Y = \bigcap_{n \in \mathbb{Z}} T^n(U) .$$

)

7 Multiplicity of critical points. Simultaneous embedding

We have seen that for $|\gamma|$ big enough and prime $P \in \mathbb{N}$, the twist map generated by l_γ has at least $(|\Sigma|^P - |\Sigma|)/P$ different hyperbolic periodic orbits of period P. For general $N \in \mathbb{N}$, the number of periodic orbits can be estimated also. In the next proposition, μ denotes the *Möbius function* in number theory defined as

$$\mu(n) = \begin{cases} 1 & n = 1 , \\ (-1)^r & n \text{ is the product of } r \text{ different prime numbers} , \\ 0 & n \text{ is divisible by a prime square} . \end{cases}$$

The notation $D \mid N$ means, that D is a divisor of N.

Proposition 7.1 *a) For $|\gamma|$ large enough, there are at least*

$$\rho(N) = \frac{1}{N} \cdot \sum_{D \mid N} \mu(D) \cdot |\Sigma|^{N/D}$$

periodic orbits of period N in the twist map S_γ.
b) The classical Morse index of such a hyperbolic periodic orbit with period N is the cardinality of the set

$$\{x \in X \mid -V''(q(x)) < 0\} .$$

It can take any integer value K with $0 < K < N$.

Proof.
a) Because

$$\sum_{D \mid N} D \cdot \rho(D) = |\Sigma|^N$$

the function ρ is the Möbius transform of the function $\tilde{\rho}(n) = |\Sigma|^n$. The Möbius inversion formula (see for example [Hua 82], p.108) gives

$$N \cdot \rho(N) = \sum_{D \mid N} \mu(D) \cdot \tilde{\rho}(\frac{N}{D}) = \sum_{D \mid N} \mu(D) \cdot |\Sigma|^{N/D} .$$

290

b) The rotation number ρ takes its gap-values in the set

$$\{\pi \frac{K}{N} \mid K = 0, 1, \ldots, N\}$$

and for $\rho = \pi \frac{K}{N}$ the integer

$$K = |\{x \in X \mid -V''(q(x)) < 0\}|$$

is the classical Morse index of the periodic orbit, i.e. the dimension of the stable manifold of the variational functional $\mathcal{L}$ (see [Mat 68]). □

Remark. If $|\Sigma| > 2$, one obtains also *minimal* and *maximal* periodic orbits for $|\gamma|$ large enough, because then, there are non-constant critical points with index 0 and index N in the anti-integrable limit.

Of course, if we find two critical points q_1, q_2 which have different Morse indices, the critical points must be different. We obtain

Proposition 7.2 *Let (X, T, m) be an ergodic aperiodic abstract dynamical system with metric entropy $h_m(T)$. If $|\Sigma| \geq e^{h_m(T)} + 1$, then there are uncountably many different embeddings of the system (X, T, m) in the twist map (T^2, S_γ) if $|\gamma|$ is large enough.*

Proof. Take a generating partition $(Y_1, Y_2, \ldots, Y_n)$ with $n \in \mathsf{N}$ satisfying

$$e^{h_m(T)} \leq n < e^{h_m(T)} + 1$$

and take a critical point q_0 in the anti-integrable limit defined by $q_0(x) = \sigma_i \in \Sigma$ if $x \in Y_i$. The assumption $|\Sigma| \geq e^{h_m(T)} + 1$ allows that q_0 takes different values σ_i on different sets Y_i. The generalized Morse index of the critical point q_0 (and any small enough perturbation of q_0) is

$$r = \sum_{-V''(q(\sigma_i)) < 0} m(Y_i) .$$

Since we have at least one more critical point then necessary, we can split Y_n into two disjoint measurable sets $Y_n = Y_n^{(1)} \cup Y_n^{(2)}$ and define a new function $\tilde{q}_0 \in L^\infty(X, \Sigma)$ satisfying

$$\tilde{q}_0(x) = q_0(x)$$

for $x \in X \setminus Y_n^{(2)}$ and which assumes $n + 1$ values. This new critical point $\tilde{q}_\epsilon$ defines for small ϵ again an embedding of the system (X, T, m) and the new partition $(Y_1, Y_2, \ldots, Y_n^{(1)}, Y_n^{(2)})$ is again a generator for the dynamical system (X, T, m). Assume $\sigma \in \Sigma$ is the additional value taken on Y_{n+1} and that $x \mapsto V''(\tilde{q}(x))$ takes positive values on $Y_n^{(1)}$ and negative values on $Y_n^{(2)}$. The index of the new embedding is $r + m(Y_n^{(1)}) - m(Y_n^{(2)})$ and this value can range in an interval

$$[r - m(Y_n), r + m(Y_n)]$$

depending on the measure of $Y_n^{(1)}$ and we get for each value in this interval a different embedding. □

For any (not necessarily ergodic) aperiodic dynamical system (X, T, m), we can make an *ergodic decomposition* (see [Den 76]).
There is a family of ergodic abstract dynamical systems (X_r, T_r, m_r) indexed by a parameter r in a Lebesgue probability space (R, ρ). The probability spaces (X_r, m_r) can be identified with $([0, 1], \lambda)$, where λ is the Lebesgue measure. Also R can be taken to be the unit interval $[0, 1]$ but the measure ρ can have atoms (points with positive measure).
The aperiodic ergodic systems (X_r, T_r, m_r) are called the *ergodic fibers* of (X, T, m). The system (X, T, m) is isomorphic to the dynamical system given by the transformation $(x, r) \mapsto (T_r x, r)$ on $[0, 1]^2$ leaving invariant the product measure $\lambda \times \rho$.

Proposition 7.3 *Assume $V \in C^3(\mathsf{T}, \mathsf{R})$. There exists a constant γ_V such that for $\gamma > \gamma_V$, every ergodic dynamical system (X, T, m) with $h_m(T) \leq \log(|\Sigma|)$ can be embedded into the twist map $S_{\gamma, V}$.*

Proof. We want to find first a constant $\epsilon_V > 0$ such that for all $\epsilon < \epsilon_V$, the operator

$$L_\epsilon = \epsilon \cdot (\tau - 2 + \tau^*) - V''(q_\epsilon)$$

stays always invertible. Write

$$\psi(\epsilon, q) = \delta \mathcal{L}_\epsilon(q) = \epsilon \cdot (q(T) - 2q + q(T^{-1})) + V'(q) .$$

The solution q_ϵ of $\psi(\epsilon, q_\epsilon) = 0$ satisfies

$$\frac{d}{d\epsilon} q_\epsilon = -L_\epsilon^{-1}(q(T) - 2q + q(T^{-1})) .$$

With

$$\frac{d}{d\epsilon} L_\epsilon = (\tau + \tau^* - 2) + V'''(q_\epsilon) \frac{d}{d\epsilon} q_\epsilon ,$$

we get

$$\frac{d}{d\epsilon} L_\epsilon^{-1} = - \left(\tau + \tau^* - 2 - V'''(q_\epsilon) L_\epsilon^{-1}(q(T) - 2q + q(T^{-1})) \right) L_\epsilon^{-2} .$$

We obtain a bound

$$\|\frac{d}{d\epsilon} L_\epsilon^{-1}\| \leq A\|L_\epsilon^{-1}\|^3 + B\|L_\epsilon^{-1}\|^2 + C ,$$

where A, B, C are positive constants which are only depending on V and not on the dynamical system. There exists $\epsilon_V > 0$ such that the initial value problem

$$\frac{d}{d\epsilon} y = A \cdot y^3 + B \cdot y^2 + C, \quad y(0) = \|L_0^{-1}\|$$

292

has a positive solution $y(\epsilon)$ for $\epsilon < \epsilon_V$. This solution is then a majorant satisfying $\|L_\epsilon^{-1}\| \leq y(\epsilon)$ (see [Lak 81] Theorem 4.1.1). This assures, that for $\epsilon < \epsilon_V$, the operator L_ϵ stays invertible.

All ergodic dynamical systems (X, T, m) satisfying the entropy bound

$$h_m(T_r) \leq \log(|\Sigma|) - 1)$$

can be embedded simultaneously because for each of these systems, there is then a critical point of $\mathcal{L}_{\epsilon,T}$. The map ϕ_ϵ stays injective because the invertibility of the operator L_ϵ for $\epsilon < \epsilon_V$ implies again, that the embedded system is hyperbolic and so structurally stable. We have therefore a true embedding and not only embedded a factor of the system. The faithful embedding can thus be continued until the operator L_ϵ stops being invertible. □

Having uncountably many different embeddings of the same system, the embedding result can also be generalized *in some sense* to not ergodic dynamical systems.

Corollary 7.4 *Let (X, T, m) be an aperiodic dynamical system such that for every ergodic fiber T_r, the entropy satisfies $h_m(T_r) \leq \log(|\Sigma|) - 1$. Assume $V \in C^3(\mathsf{T}, \mathsf{R})$. Then for $|\gamma| > \gamma_V$, every ergodic fiber of (X, T, m) can be embedded in the twist map (T^2, S), such that two different fibers have different embeddings.*

Proof. The multiplicity result in Proposition 5.2 tells that every ergodic dynamical system can be embedded with an index r in some interval. We can assume that the parameter interval is $[0, 1]$. We embed simultaneously all the fibers (X_r, T_r, m_r) with corresponding critical point $q_r : X_r \to \mathsf{T}$ having (normalized) index $r \in [0, 1]$. Two different fibers have different embeddings because the generalized Morse indices are different. □

Remark. We don't yet know, if we can embed every abstract (not necessarily ergodic) dynamical system. In general, we can't exclude that the supports of the S-invariant measures μ_r corresponding to the embedded fibers (X_r, T_r, m_r) are not disjoint. The critical point q (constructed in the proof of Corollary 5.3) belonging to the dynamical system (X, T, m) could fail to be injective on a set of positive measure.

8 Generalization of the results for symplectic twist maps

The results for monotone twist maps can be extended to higher dimensional *symplectic twist maps*. For such maps there is less knowledge then in the one dimensional case. For example, a higher dimensional Mather theory Is still missing. For periodic

orbits see [Gol 92a].

The generating function for such a higher dimensional symplectic twist map is a C^2 function $l : \mathsf{R}^N \times \mathsf{R}^N \to \mathsf{R}$. It should satisfy the twist condition in that the $N \times N$ matrix

$$\delta_q \delta_{q'} l$$

is positive definite and it is also required that

$$l(q + n, q' + n) = l(q, q'), \ \forall \, n \in 2\pi \mathsf{Z}^N \ .$$

Again, we assume for simplicity that

$$l(q, q') = -\frac{< q - q', q - q' >}{2} - \gamma \cdot V(q) \ ,$$

where $V \in C^2(\mathsf{T}^N, \mathsf{R})$ with $\mathsf{T}^N = \mathsf{R}^N/(2\pi \mathsf{Z}^N)$. Given an abstract dynamical system (X, T, m), the variational functional

$$\mathcal{L}(q) = \int_X l(q(x), q(Tx)) \, \mathrm{dm}(x)$$

is defined on the Banach manifold $L^\infty(X, \mathsf{T}^N)$. A nontrivial critical point q corresponds to a factor of (X, T, m) embedded in the symplectic map

$$S : \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} q + p + \gamma \cdot V'(q) \\ p + \gamma \cdot V'(q) \end{pmatrix} \ ,$$

where $p = l_1(q(x), q(Tx))$ and $p' = -l_2(q(T^{-1}x), q(x))$ and $V' = \nabla V$ is the gradient. A well known example of such a symplectic map is the *Fröschle map* with $V(q) = \gamma_1 \cos(x_1) + \gamma_2 \cos(x_2) + \mu \cos(x_1 + x_2)$. (see for example [Koo 86].)

The second variation is now a so called *random Jacobi operator on the strip* (see [Kot 88])

$$L = \tau - 2 + \tau^* - \gamma \cdot V''(q) \ ,$$

where $V''(q(x))$ is the Hessian of V at a point $q(x) \in \mathsf{T}^N$. Assume that the set of non degenerate critical points Σ is not empty. Then, the Hessian L of a critical point $q \in L^\infty(X, \mathsf{T})$ is invertible in a neighborhood of the anti-integrable limit and as before, there exist nontrivial factors embedded in the twist map for γ big enough. The operator can be seen as an element in the crossed product $\mathcal{X}$ of the algebra $L^\infty(X, M(N, \mathsf{R}))$ with the dynamical system (X, T, m). This algebra has the trace

$$\mathrm{tr}(K) = \int_X \mathrm{trace}(K_0(x)) \, \mathrm{dm}(x) \ ,$$

where $\mathrm{trace}(K_0)$ is the usual trace for the matrix $K_0(x)$ in the finite dimensional matrix algebra $M(N, \mathsf{R})$. As before, there exists a density of states dk of the operator L and a Thouless formula [Kot 88]

$$-\lambda(A_E) + \mathrm{i} k(E) = w(E) \ ,$$

294

where
$$w(E) := -\mathrm{tr}(\log(L - E))$$

is the Floquet exponent of L and $\lambda(A_E)$ is the sum of the first N Lyapunov exponents of the $SL(2N, \mathbb{C})$-matrix cocycle

$$x \mapsto A_E(x) = \begin{pmatrix} E + V''(q(x)) + 2 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2N, \mathbb{C})$$

and $k(E) = \int_{-\infty}^{E} dk(E')$ is the integrated density of states for $E \in \mathbb{R}$.

As before, every embedded dynamical system has an index, namely the value of the integrated density of states $k(E)$ at $E = 0$ and exactly in the same way, one gets also the multiplicity results like for example the existence of

$$\rho(N) = \frac{1}{N} \cdot \sum_{D|N} \mu(D) \cdot |\Sigma|^{N/D}$$

periodic orbits of period N. Only for the embedding of periodic orbits, there is a slightly different statement for the possible indices of the critical points. If Σ is the finite set of non degenerate critical points of V and $-V''(\sigma)$ has k_σ^+ positive and k_σ^- negative eigenvalues for $\sigma \in \Sigma$, then only indices

$$\sum_{\sigma \in I} k_\sigma^-, \ I \subset \Sigma$$

can occur.

9 Discussion and some questions

We call a topological dynamical system *universal*, if every abstract ergodic dynamical system can be embedded into it and *universal* of order α, if every ergodic dynamical system can be embedded into it. Our result shows, that twist maps with potential V are universal of order $\alpha = \log|\Sigma|$ if Σ is the set of nondegenerate critical points of V. This means, that such a twist map can be used to simulate every dynamical system of metric entropy $\leq \log(|\Sigma|)$.

Let us mention some results which hold for other mathematical structures and which are from similar generality then the result discussed here. The general scheme is the following: Given a categorie of objects. The aim is to find special elements in that categorie, such that a large class of the categorie is "embedded" in this "universal" element.

• (Turing) There exists a universal Turing machine, in which every Turing machine can be embedded: the universal Turing machine can simulate all the possible machines.

- (Banach-Masur) Every separable metric space is embedded isometrically in the Banach space $C[0,1]$. Spaces having this property are called universal spaces. (see [Liu 61])

- (Higman) There is a finitely presented group G which contains every finitely presented group as a subgroup. Such a group G is called a universal finitely presented group. (See [Man 77])

- (Whitney) Every smooth, connected, closed manifold of dimension n can be smoothly embedded in $\mathbb{R}^{2n+1}$.

A critical remark. In some sense the embedding result is not so surprising. Because every dynamical system with entropy $\leq \log(|\Sigma|)$ can be embedded into a Bernoulli shift with a state space $\Sigma|$, it suffices, to embed a Bernoullishift inside a topological dynamical system. So, it is enough to find a Horseshoe inside the dynamical system. Often, one can prove the existence of a homoclinic point, which gives, that an iterate of the dynamical system has a horse-shoe. The topological entropy for a dynamical system with a horse-shoe is positive but can be arbitrarily small. In any case, the anti-integrable limit gives a quantitative bound on the topological entropy which has not yet been obtained by other methods. The use of the index gives uncountably many different embeddings for aperiodic dynamical systems. It would be probably difficult to construct uncountably many different homoclinic points for a dynamical system. Necessary for this would be, that there are uncountably many periodic orbits. So, even if qualitatively similar results could be obtainable with the methods known, the anti-integrable limit gives in an easy way quantitatively much stronger results.

To the end, we add some questions:

- Can every (not necessarily ergodic) dynamical system with finite metric entropy for each ergodic fiber be embedded in a monotone twist map?

- What happens with an embedded system if the parameter γ decreases and the Hessian L becomes no more invertible? Does it bifurcate to other systems or disappear?

- What spectra do occur for the Hessians near the anti-integrable limit? Is it point spectrum when (X, T, m) is a Bernoulli shift? Is it even possible that in the aperiodic case point spectrum (Localisation) persists?

- Can one find twist maps in applications like Poincaré sections for Hamiltonian systems or for billiards, for which the results near the anti-integrable limit apply?

- Is the topological entropy of the standard map bounded above by $\log(2)$ for all γ?

• Take as the dynamical system (X, T, m) the standard map $(\mathsf{T}^2, S_\gamma, dxdy)$ and embed it into itself. The modulus of the real part of the Floquet exponent belonging to the Hessian is the metric entropy of S_γ for which nothing is known. How does the imaginary part, the index, behave for $\gamma \to \infty$?

10 Appendix: Twist maps as Hamiltonian systems

A function $f \in L^\infty(X)$ has the *discrete derivative*

$$\nabla f = f(T) - f$$

over the dynamical system (X, T, m).

Given a generating function l, one can define a Hamiltonian function

$$\mathcal{H}(q, p) = \int_X p(T) \nabla q \, dm - \mathcal{L}(q, p) \,,$$

where $p = \nabla q = q(T) - q$. The twist condition $l_{12}(q, q') \geq r$ which plays the role of the Legendre condition, assures that this *discrete Legendre transformation*

$$\mathcal{L} \mapsto \mathcal{H}$$

is possible. The twist map can be written as the Hamiltonian system

$$\begin{aligned}
\nabla q &= \mathcal{H}_{p(T)} \,, \\
\nabla p &= -\mathcal{H}_q \,.
\end{aligned}$$

Example. The generating function

$$l_\gamma(q, q') = -\frac{(q' - q)^2}{2} - \gamma \cdot V(q)$$

generates a *generalized standard map*. We get

$$\begin{aligned}
y(q, q') &= q' - q - \gamma V'(q) \,, \\
y'(q, q') &= q' - q \,.
\end{aligned}$$

The Euler equations for the variational problem

$$\mathcal{L}(q) = \int -\frac{(q(T) - q)^2}{2} - \gamma V(q) \, dm(q)$$

are

$$q(T) - 2q + q(T^{-1}) = -\gamma V'(q).$$

We get the Hamiltonian

$$\mathcal{H}(q,p) = \int_X \frac{p^2}{2} + \gamma V(q) \, \mathrm{dm}$$

and the Hamilton equations

$$\nabla q \;=\; q(T) - q = \mathcal{H}_{p(T)} = p(T) \,,$$
$$\nabla p \;=\; p(T) - p = -\mathcal{H}_q = -V'(q) \,,$$

which can be rewritten as

$$q(T) \;=\; q + p(T) = q + p + V'(q) \,,$$
$$p(T) \;=\; p + V'(q) \,.$$

Given more generally a Hamiltonian $H(q,p) = \int_X h(q,p) \, \mathrm{dm}$, where $h(x,y)$ is a smooth function on the torus. The system

$$q(T) \;=\; q + \mathcal{H}_{p(T)} \,,$$
$$p(T) \;=\; p - \mathcal{H}_q \,,$$

is a discrete Hamiltonian system. It is defined for any abstract dynamical system (X,T,m), if this system can be embedded in the map

$$(x,y) \mapsto (x',y') = (x + h_2(x,y'), y - h_1(x,y))$$

on the torus.

The advantage to write the system in a function space is that any subsystem of a twist map like a periodic orbit or invariant circle can be written as a Hamiltonian system.

11 Appendix: Embedded systems are hyperbolic sets

We give here another proof of the fact, that the embedded systems near the anti-integrable limit are hyperbolic.

Proposition 11.1 *Given an abstract dynamical system (X,T,m) which is embedded in the monotone twist map (Y,S). If the Hessian L of the embedding is invertible the embedded system is a hyperbolic set.*

Proof. If 0 is outside the spectrum of L, the integrated density of states of L is locally constant around $E = 0$ and the Lyapunov exponent $\lambda(A_E)$ is positive for small $|E|$. The last statement follows from the fact that the map $E \mapsto \lambda(A_E)$ is harmonic near 0 and $\lambda(A_E) \geq 0$ and the maximum principle for harmonic functions. This implies with *Oseledecs theorem* the existence of two measurable mappings

$$W_E^{(i)} : X \mapsto \mathsf{P}^1 \ ,$$

where P^1 denotes the projective space consisting of all one dimensional subspaces of R^2. These direction fields $W_E^{\pm}$ are *coinvariant*:

$$A_E(x) W_E^{\pm}(x) = W_E^{\pm}(Tx) \ .$$

We can write them also in projective coordinates with so called *Titchmarsh-Weyl functions*

$$m_E^{\pm}(x) = a(x) \frac{u_E^+(Tx)}{u_E^+(x)} \ ,$$

where $u_E^{\pm}(x)$ are unit vectors in $W_E^{\pm}(x)$. The invariance of $W_E^{\pm}$ is in these projective coordinates given by the *discrete Ricatti equation*

$$m_E^{\pm}(T) = E - b(T) - \frac{a^2(x)}{m_E^{\pm}(x)} \ .$$

The Green function

$$G_E(x) = [(L(x) - E)^{-1}]_{00}$$

satisfies

$$G_E(x) = \frac{1}{m_E^+(x) - m_E^-(x)}$$

and because for E in a gap the resolvent $L(x) - E$ is bounded, there exists $\epsilon > 0$ such that $|m_E^+(x) - m_E^-(x)| \geq \epsilon$. This means that the angle between the stable and unstable seperaratrices is bounded away from 0. We get also $\delta > 0$ such that

$$\{ x \in X \mid |m_0^{\pm}(x)| \leq \delta, |m_0^{\pm}(x)| \geq \delta^{-1} \}$$

has measure zero. The reason is the constance of the integrated density of states $k(E)$ which can be expressed also by the formula

$$k(E) = m(\{ x \in X \mid m^+(x) < 0 \}) \ .$$

Define for $\delta \geq 0$

$$r(\delta) = m(\{ x \in X \mid m^{\pm}(x) \in [-\delta, \delta] \}) \ .$$

Because of

$$k(E + \delta) - k(E) \geq r(\delta)/2$$

for δ small enough, there must be $r(\delta) = 0$ for δ small enough. One gets now for A_0 and for ϵ small enough the strict invariant cone bundle

$$x \mapsto [m_{E+\epsilon}^{l(x)}(x), m_{E-\epsilon}^{u(x)}]$$

for the transfer cocycle $x \mapsto A_0(x)$, where $l(x), u(x) \in \{+, -\}$ are defined as $m^{l(x)} < m^{u(x)}$. A theorem of Ruelle [Rue 85] shows then, that A_0 is uniformly hyperbolic: $\exists \Gamma > 0, \alpha < 1$ such that for almost all $x \in X$ and all $n \in \mathbb{N}$

$$
\begin{aligned}
|A_0^{-n}(x)w^{(1)}(x)| &\leq \Gamma \alpha^n |w^{(1)}(x)| , \\
|A_0^n(x)w^{(2)}(x)| &\leq \Gamma \alpha^n |w^{(2)}(x)| .
\end{aligned}
$$

Because $A_0(x)$ is conjugated to the Jacobian $dT(x)$ of the twist map at a point $\phi(x) \in \mathbb{R} \times \mathbb{T}$, the invariant embedded subset is a hyperbolic set. $\qquad \square$

12 Appendix: Topological and metric entropy

We add convenient definitions for the topological and metric entropies.

Let (X, T) be a topological dynamical system and denote with d the metric on X. Define a sequence of metrics

$$d_n(x, y) = \max_{0 \leq i \leq n-1} d(T^i x, T^i y) .$$

Following Bowen, the *topological entropy* $h(T)$ can be defined as

$$h(T) = \lim_{\epsilon \to 0} \overline{\lim_{n \to \infty}} \, \frac{\log(N(n, \epsilon))}{n} ,$$

where $N(n, \epsilon)$ is the minimal number of ϵ balls in the metric d_n covering the space X.

Let (X, T, m) now be an abstract dynamical system. We can assume that it is topological also because every abstract dynamical system is isomorphic to a topological dynamical system. Let again d denote the metric on X and let d_n be the sequence of metrics defined above.

Following Katok [Kat 80], the *metric entropy* $h_m(T)$ can be defined as

$$h_m(T) = \lim_{\epsilon \to 0} \overline{\lim_{n \to \infty}} \, \frac{\log(N(n, \epsilon, \delta))}{n} ,$$

where $N(n, \epsilon, \delta)$ is the minimal number of ϵ balls in the metric d_n covering a set of measure $\geq 1 - \delta$ and $1 > \delta > 0$ is arbitrary.

Because trivially $N(n, \epsilon, \delta) \leq N(n, \epsilon))$ one has immediately

$$h_m(T) \leq h(T) \ .$$

The *Variational principle* says

$$\sup_m h_m(T) = h(T) \ .$$

Measures which give the topological entropy are called *equilibrium measures*.

References

[Ang 88] S.Angenent. *The Periodic Orbits of an Area Preserving Twist Map.*
Commun. Math. Phys. **115**:353-374, 1988

[Ang 92] S. Angenent. *A remark on the topological entropy and invariant circles of an area preserving twist map.*
In "Twist mappings and their Applications", IMA Volumes in Mathematics, Vol. 44., Eds. R.Mc Gehee, K.Meyer 1992

[Ang 90] S.Angenent. *Monotone recurrence relations their Birkhoff orbits and topological entropy.*
Erg. Th. Dyn. Sys., 10:15-41, 1990

[Aub 92a] S.Aubry. *The concept of anti-integrability: definition, theorems and applications to the standard map.*
In "Twist mappings and their Applications", IMA Volumes in Mathematics, Vol. 44., Eds. R.Mc Gehee, K.Meyer, 1992

[Aub 90b] S.Aubry , G.Abramovici. *Chaotic Trajectories in the Standard map. The concept of anti-integrability.*
Physica D, 43:199-219, 1990

[Aub 92c] S.Aubry, R.S.Mackay, C.Baesens C. *Equivalence of uniform hyperbolicity for symplectic twist maps and phonon gap for Frenkel-Kontorova models.*
Physica D, 56:123-134, 1992

[Ban 88] V.Bangert. *Mather Sets for Twist Maps and Geodesics on Tori*
Dynamics Reported, 1:1-55, 1988

[Bow 75] R.Bowen, D.Ruelle. *The Ergodic Theory of Axiom A Flows.*
Inventiones Math., 29:181-202, 1975

[Bro 75] M.Brown, W.D. Neuman. *Proof of the Poincaré-Birkhoff fixed point theorem.*
Michigan Math. J., 24:21-31, 1975

[Cel 88] A.Celletti, L.Chierchia. *Construction of analytic KAM surfaces and effective stability bounds.*
Commun. Math. Phys., 118:119-161, 1988

[Cor 82] I.P.Cornfeld, S.V.Fomin, Ya,G.Sinai. *Ergodic Theory.*
Springer, New York, 1982

[Cyc 87] H.L.Cycon,R.G.Froese,W.Kirsch,B.Simon. *Schroedinger Operators.*
Texts and Monographs in Physics, Springer, 1987

[Del 83] F.Delyon, B.Souillard. *The Rotation Number for Finite Difference Operators and its Properties.*
Comm. Math. Phys., 89:415-426, 1983

[Den 76] M.Denker,C.Grillenberg,K.Sigmund. *Ergodic Theory on Compact Spaces.*
Lecture Notes in Math., No. 527, Springer 1976

[Fon 90] E.Fontich. *Transversal Homoclinic Points of a Class of Conservative Diffeomorphisms.*
J.Diff.Equ., 87:1-27, 1990

[Gen 90] C. Genecand. *Transversal homoclinic points near elliptic fixed points of area preserving diffeomorphism of the plane.*
Diss. ETHZ, 1990

[Gol 92] C. Golé. *A new proof of Aubry-Mather's theorem.*
Math. Z., 210:441-448, 1992

[Gol 92a] C.Golé. *Monotone maps of $T^n \times R^n$ and their periodic orbits.*
In "Twist mappings and their Applications", IMA Volumes in Mathematics, Vol. 44., Eds. R.Mc Gehee, K.Meyer 1992

[Gor 85] D.Goroff. *Hyperbolic sets for twist maps.*
Ergod. Th. & Dynam. Sys., 5:337-339, 1985

[Hal 88] G. Hall. *A remark on the multiplicity of monotone periodic orbits.*
Ergod.Th. Dyn. Sys., 8:109-118, 1988

[Her 83] M.Herman. *Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2.*
Comm.Math.Helv., 58:453-502, 1983

[Her 83a] M.Herman *Sur les courbes invariantes par les difféomorphismes de l'anneau.*
Astérisque, 103-104, Société mathématique de france 1983

[Hua 82] L.K. Hua. *Introduction to Number theory.*
Springer, Berlin, 1982

[Joh 82] R.Johnson,J.Moser. *The Rotation Number for Almost Periodic Potentials.*
Comm. Math. Phys., 84:403-438, 1982

[Kat 80] A. Katok. *Lyapunov Exponents, entropy, and periodic orbits for diffeomorphisms.*
Publ.Math. IHES, 51:137-172, 1980

[Kat 82] A. Katok. *Some remarks on Birkhoff and Mather twist map theorems.*
Ergod. Th. & Dynam. Sys., 2:185-194, 1982

[Kni 3] O.Knill. *Isospectral Deformations of Random Jacobi Operators.*
Commun. Math. Phys., 151:403-426, 1993

[Kni 5] O.Knill. *An analytic map containing the standard map family.*
Submitted to Nonlinearity

[Koo 86] H.Kook,J.D.Meiss. *Periodic orbits for reversible, symplectic mappings.*
Physica D, 35:65-86, 1986

[Kot 88] S.Kotani, B.Simon. *Stochastic Schroedinger Operators and Jacobi Matrices on the Strip.*
Comm. Math. Phys, 119:403-429, 1988

[Lak 81] V.Lakshmikantham, S.Leela. *Nonlinear Differential equations in abstract spaces.*
Pergamon Press, Oxford, 1981

[Lan 85] O.Lanford III. *Introduction to hyperbolic sets.*
in "Regular and Chaotic Motion in Dynamical systems"
Eds. G.Velo, A.S.Wightman, Nato ASI Ser. B, 118:73-102, 1985

[Laz 84] V.F.Lazutkin, D.Ya. Terman. *Percival Variational Principle for Invariant Measures and Commensurate-Incommensurate Phase Transitions in One Dimensional Chains.*
Commun. Math. Phys, 94:511-522, 1984

[Liu 61] L.A. Liusternik, V.J.Sobolev. *Elements of Functional analysis.*
Frederick Ungar Publishing Company, New York, 1961

[Man 77] Yu. I. Manin. *A course in mathematical Logic.*
Graduated texts in Mathematics 53, New York, Springer, 1977

[Mat 68] J.Mather. *Characterization of Anosov Diffeomorphisms.*
Indag. Math., 30:479-483, 1968

[Mat 82] J.Mather. *Existence of quasi-periodic orbits for twist homeomorphism of the annulus.*
Topology, 21:457-467, 1982

[Mat 84] J.Mather. *Amount of rotation about a point and the Morse Index.*
Comm. Math. Phys., 94:141-153, 1984

[Mac 92] R.S. Mackay, J.D. Meiss. *Cantori for symplectic maps near the anti-integrable limit.*
Nonlinearity, 5:149-160, 1992

[Mos 73] J. Moser. *Stable and random Motion in dynamical systems.*
Princeton University Press, Princeton, 1973

[Mos 86] J. Moser. *Recent Developments in the theory of Hamiltonian systems.*
SIAM Review, 28:459-485, 1986

[Mos 87] J. Moser. *On the construction of invariant curves and Mather sets via a regularized variational principle.*
in P.H.Rabinowitz et al (eds.) "Periodic Solutions of Hamiltonian Systems and Related Topics":221-234, 1987

[Orn 70] D.S.Ornstein. *Factors of Bernoulli shifts are Bernoulli shifts.*
Adv. Math., 5:349-364, 1970

[Per 80] I.C. Percival. *Variational principle for invariant tori and cantori.*
Amer. Instit. of Physics Conf. Proc., 57:310-320, 1980

[Rue 85] D.Ruelle. *Rotation numbers for diffeomorphisms and flows.*
Ann. de l'Inst. Henri Poincare, Physique theoretique, 42:109-115, 1985

[Rue 79a] D.Ruelle. *Analycity Properties of the Characteristic Exponents of Random Matrix Products.*
Advances in Mathematics, 32:68-80, 1979

[Sal 89] D. Salamon, E. Zehnder. *KAM theory in configuration space.*
Comm. Math. Helv., 64: 84-132, 1989

[Wal 82] P.Walters. *An introduction to ergodic theory.*
Graduate texts in mathematics 79, New York, Springer, 1982

[Zeh 73] E. Zehnder. *Homoclinic points near elliptic fixed points.*
Comm. Pure Appol. Math., 26:131-182, 1973

An analytic map containing the standard map family

Contents

.

Abstract

The analytic map $U : \mathbb{C}^4 \to \mathbb{C}^4$,

$$(z, w, u, v) \mapsto (zwe^{z-u}, we^{z-u}, uve^{u-z}, ve^{u-z})$$

has a one parameter family of two-dimensional real tori S_γ invariant, on which U is the standard map family T_γ.

We provide a rough qualitative picture of the dynamics of U and give some arguments supporting the conjecture that the metric entropy of the standard map T_γ is bounded below by $\log(\gamma/2)$.

1 The positive metric entropy conjecture

For the *Chiricov or standard mapping* on the torus $\mathsf{T}^2 = \mathsf{R}^2/(2\pi\mathsf{Z})^2$

$$T_\gamma : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y + \gamma\sin(x) \\ y + \gamma\sin(x) \end{pmatrix}$$

with real parameter γ, the *metric entropy* is measured to be greater or equal then

$$\log\left|\frac{\gamma}{2}\right| .$$

It is an open problem whether this estimate holds true.

1.1 A numerical test

The plot in Figure 1 shows a graph of the entropy of the Standard map calculated numerically with a Mathematica [Wol 91] program given by

```
T[{x_Real,y_Real},g_Real]:=N[Mod[{x+y+g*Sin[x],y+g*Sin[x]},2 Pi]];
A[{x_Real,y_Real},g_Real]:=N[{{1+g*Cos[x],1},{g*Cos[x],1}}];
Lya[{x_,y_},g_Real,n_Integer]:=Module[{t=0,B=IdentityMatrix[2],p={x,y}},
  Do[B=A[p,g].B;p=T[p,g]; t=t+Re[Log[B[[1,1]]]];B=B/B[[1,1]],{i,n}];t/n];
Entropy[g_]:=Module[{n=1000,m=10,R:=N[2*Pi*Random[]]},Sum[Lya[{R,R},g,n],{m}]/m];
```

determines numerically the integrated Lyapunov exponent

$$\lambda(dT_\gamma) = \lim_{n \to \infty} \int_{\mathsf{T}^2} \frac{1}{n} \log \|dT_\gamma^n(x,y)\| dx dy ,$$

(which we will denote shortly by *Lyapunov exponent*), by calculating

$$\lambda_{n,m}(dT_\gamma) = \frac{1}{m}\sum_{j=1}^{m}\frac{1}{n}\log\|dT_\gamma^n(x_j,y_j)\|$$

for m random points $(x_j, y_j) \in \mathsf{T}^2$. For $m, n \to \infty$, the values $\lambda_{n,m}(dT_\gamma)$ converge by Oseledec's theorem (see [Rue 79]) and *Pesin's formula* (see [Man 81]) to the metric entropy of the Standard map T_γ.
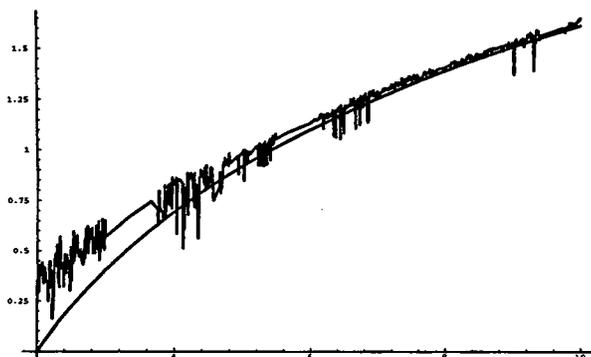
The figure shows the numerical calculation of the metric entropy $\mu(\gamma)$ of the Standard map T_γ for parameters between $\gamma = 2$ and $\gamma = 10$ in comparison with the conjectured lower bound $\log(\frac{\gamma}{2})$. The plot was produced with

```
Plot[{Entropy[s],Log[s/2]},{s,2,10}]
```

which takes some time due to the fine interpolation. We recommend just to compare by hand some values, instead of plotting the graph. Increasing the values of m, n should give more reliable results.

The measurements of our runs fit well with measurements done with other methods, see for example [Par 86]. However, one has to be careful with the interpretations of Lyapunov exponent measurements. The convergence of ergodic averages can be slow even for real valued cocycles (see [Pet 83], p.94): Given any sequence of positive real numbers b_n with $\sum_n b_n = \infty$ then there is a measurable bounded function f such

that

$$\sum_n b_n \left(\frac{1}{n} \sum_{k=0}^{n} f(T^k(x,y)) - \int_{T^2} f(x,y) \right) = \infty$$

for almost all (x,y).

An other danger is that the Lyapunov exponent is in general a *discontinuous function* of the cocycle if one is away from the uniform hyperbolic domain [Kni 91]. Positive Lyapunov exponents could be a numerical artifact. We give now three arguments supporting the quantitative and qualitative believe in positive metric entropy.

1.2 A first argument: The Lyapunov exponent of the cocycle dT_γ over the dynamical system T_0.

There is an explanation, why the number $\log|\frac{7}{2}|$ is a good choice for a lower bound of the metric entropy:
If one takes the Jacobian

$$dT_\gamma(x,y) = \begin{pmatrix} 1 + \gamma\cos(x) & 1 \\ \gamma\cos(x) & 1 \end{pmatrix}$$

of the mapping T_γ as a cocycle over the dynamical system T_0 instead of T_γ, one gets indeed, with a method of M.Herman, a lower bound $\log|\frac{7}{2}|$ for the Lyapunov exponent $\lambda(dT_\gamma)$ of this cocycle.

Proof. The argument uses complex analysis. The standard map for $\gamma = 0$ is integrable and can be written as a holomorphic map

$$(z,w) \mapsto (zw, w)$$

on the torus $\{(z,w) \in \mathbb{C}^2 \mid |z| = |w| = 1\}$. This torus is lying inside the polydisc

$$\{(z,w) \in \mathbb{C}^2 \mid |z|, |w| < r\},$$

where $r > 1$. The cocycle dT_γ is in these coordinates

$$A_\gamma(z,w) = \begin{pmatrix} 1 + \frac{7}{2}(z + z^{-1}) & 1 \\ \frac{7}{2}(z + z^{-1}) & 1 \end{pmatrix}.$$

For $|z| = 1$, the Lyapunov exponent

$$\begin{aligned}
\lambda(A_\gamma) &= \lim_{n\to\infty} \frac{1}{n} \int_{|z|=1, |w|=1} \log\|A_\gamma(U^{n-1}(z,w))\ldots A_\gamma(U(z,w))A_\gamma(z,w)\| \, dzdw \\
&= \lim_{n\to\infty} \frac{1}{n} \int_{|z|=1, |w|=1} \log\|A_\gamma^n(z,w)\| \, dzdw
\end{aligned}$$

of the cocycle A_γ is the same as the integrated Lyapunov exponent of the analytic parameterized cocycle

$$B_\gamma(z,w) = zA_\gamma(z,w) = \begin{pmatrix} z + \frac{\gamma}{2}z^2 + \frac{\gamma}{2} & z \\ \frac{\gamma}{2}z^2 + 1 & z \end{pmatrix}.$$

Because $(z,w) \mapsto \frac{1}{n}\log||B_\gamma^n(z,w)||$ is plurisubharmonic, the Lyapunov exponent can be estimated from below as

$$
\begin{aligned}
\lambda(A_\gamma) &= \lambda(B_\gamma) = \lim_{n\to\infty} \int_{|z|=1,|w|=1} \frac{1}{n}\log||B_\gamma^n(z,w)||\ dz\ dw \\
&\geq \lim_{n\to\infty} \frac{1}{n}\log||B_\gamma^n(0,0)|| = \log(\frac{\gamma}{2}).
\end{aligned}
$$

$\square$

1.3 A second argument: A huge class of cocycles has positive Lyapunov exponents

There is another argument why it is reasonable to believe that the entropy in the standard map is measured to be positive for γ big enough. Assume, we determine the Lyapunov exponent and make random mistakes in the calculation of the cocycle in that each Jacobian dT is multiplied with a random rotation $R(\phi(x,y)) \in SO(2,\mathbb{R})$, where $\phi(x,y)$ is a random angle measurably depending on $(x,y) \in \mathbb{T}^2$. This corresponds to an error in the calculation of the angle of the images of tangent vectors. There is an astonishing result of Herman [Her 83] which implies that an arbitrary error can be shifted in a deterministic way to get positive Lyapunov exponents:

Given any error-function $\phi(x,y)$ (also $\phi(x,y) = 0$ is allowed), the Lebesgue measure of the set of values $\beta \in [0,2\pi)$ giving zero Lyapunov exponents to the cocycle

$$R(\phi(x,y) + \beta)dT_\gamma$$

is smaller then

$$\frac{8}{\log(1 + \gamma^2/2)}.$$

Proof. The estimate can be derived from a result of Herman ([Her 83], p.498) and the hint, that the Jacobian cocycle

$$dT_\gamma(x,y) = \begin{pmatrix} 1 + \gamma \cdot \cos(x,y) & 1 \\ \gamma \cdot \cos(x,y) & 1 \end{pmatrix}$$

satisfies

$$dT_\gamma(x,y) = R(\frac{\pi}{4}) \circ \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 0 \\ 1 + 2 \cdot \gamma \cdot \cos(x) & 2 \end{pmatrix} = R(\frac{\pi}{4}) \circ D_\gamma(x,y) .$$

Herman's result is a subharmonic estimate of the Lyapunov exponents: For any cocycle $R(\beta)A(x,y) = R(\phi(x,y)+\beta) \circ D(x,y)$ with

$$D(x,y) = \begin{pmatrix} c(x,y) & 0 \\ b(x,y) & c^{-1}(x,y) \end{pmatrix} ,$$

where the functions b, c, c^{-1} are in $L^\infty(\mathsf{T}^2, \mathsf{R})$, one has

$$\int_{|w|=1} \lambda(B(w)) \, dw \geq \int_{\mathsf{T}^2} \log \sqrt{\frac{1}{4}((c+c^{-1})^2 + b^2)} \, dx dy .$$

The proof of this estimates goes as follows. Define $w = e^{i\beta}$ and the complex cocycle

$$B(w)(x,y) = w \cdot e^{i\phi(x,y)} \cdot D(x,y) .$$

Because $|w \cdot e^{i\phi(x,y)}| = 1$, we have $\lambda(R(\beta)A) = \lambda(B(w))$. With $G = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$, one can write

$$B(w,x) = (G + w^2 \cdot e^{2i\phi(x,y)} \overline{G}) \circ D(x,y) .$$

The Lyapunov exponent $w \mapsto \lambda(B(w))$ is subharmonic leading to $\int_{|w|=1} \lambda(B(w)) \, dw \geq \lambda(B(0))$. We calculate with $L = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}$

$$\int_{|w|=1} \lambda(B(w)) \, dw \geq \lambda(B(0)) = \lambda(GD) = \lambda(L^{-1}GDL)$$

$$= \int_{\mathsf{T}^2} \log \sqrt{\frac{1}{4}((c+c^{-1})^2 + b^2)} \, dx dy .$$

This implies with

$$\lambda(R(\beta)A) \leq \int_{\mathsf{T}^2} \log \sqrt{(c+c^{-1})^2 + b^2} \, dx \, dy$$

that the Lebesgue measure of values β with $\lambda(R(\beta) \circ A) = 0$ is smaller then

$$1/ \left(\int_{\mathsf{T}^2} \log \sqrt{(c+c^{-1})^2 + b^2} \, dx dy \right) .$$

Putting in the specific values $c(x,y) = 1/\sqrt{2}$ and $b(x,y) = 1/\sqrt{2} + \sqrt{2}\gamma \cdot \cos(x)$ from the cocycle dT_γ gives the desired estimate. $\qquad\square$

1.4 A third argument: Positive Lyapunov exponents for a dense set of bounded measurable cocycles

One can change a given cocycle over an aperiodic dynamical system on a set of arbitrary small measure, in an arbitrary small way, to get positive Lyapunov exponents:

Given $\epsilon > 0$, there is a set $Y \subset \mathsf{T}^2$ of Lebesgue measure $\leq \epsilon$ and a cocycle $C(x,y) \in SL(2,\mathsf{R})$ with $\|C(x,y) - 1\| \leq \epsilon$ and $C(x,y) \neq 1$ only on Y, such that the cocycle

$$B = dT_\gamma C$$

has positive Lyapunov exponents $\int \frac{1}{n} \log \|B(x,y)\| dxdy$.

The proof [Kni 92] uses Herman's result treated in the last subsection.
The density of cocycles with positive Lyapunov exponents explain in a qualitative way the obtained results in the numerical experiments. More convincing would be a result showing that the set of cocycles with positive Lyapunov exponents is residual or that this set even contains an open dense set. These questions are open.
We outline the proof which can be found in [Kni 92]. We first give the four ingredients used for the proof.

• **Rohlin's lemma**: If (X, T, m) is aperiodic, there exists a $(n, \epsilon)-$ Rohlin set Y such that $Y, T(Y), \ldots, T^{n-1}(Y)$ are pairwise disjoint and $m(Y_{rest}) \leq \epsilon$.

• **Oseledec's theorem**: For $A \in \mathcal{P}$, there exists a stable direction field $W \in L^\infty(X, \mathbf{P}^1)$ satisfying the co-invariance $AW = W(T)$ and $\lambda(A) = -\int_X \log|Aw|\, \mathrm{dm}$, where $w(x) \in W(x)$ is a unit vector.

• **Result of M.Herman**: Call ν the Lebesgue measure on $SO(2, \mathsf{R})$,

$$\nu(\{\phi \mid \lambda(AR(\phi)) = 0\}) \leq \lambda(A)^{-1}\,.$$

For $A \notin L^\infty(X, SO(2, \mathsf{R}))$, there $\exists$

$$\beta \in \mathsf{R},\ AR(\beta) \in \mathcal{P}\,.$$

• **Abramov type result**: Let (X, T, m) be ergodic and $Z \subset X$ be of positive measure. Let (Z, T_Z, m_Z) be the induced system, A_Y the derived cocycle. $\lambda(A_Y) = m(Z)^{-1} \cdot \lambda(A)$.

The main steps in the proof of the density result are:
1) **Ergodic decomposition**: Assume (X, T, m) to be ergodic without loss of generality.

2) **Rohlin**: Build $Z = Y \cup Y_{rest}$, where Y is a (n, ϵ) Rohlin-set. We use the aperiodicity here.

3) **First perturbation:** Perturb A a first time so that A_Z is not in $L^\infty(X, SO(2, \mathbf{R}))$.

4) **Herman:** There exists β_0 with $A_Z R(\beta) \in \mathcal{P}$ and also $AR(1_Z\beta_0) \in \mathcal{P}$.

5) **Oseledec:** There exists $W \in L^\infty(X, \mathbf{P}^1)$ with $AR(1_Z\beta_0)W = W(T)$. We use ergodicity assumption here.

6) **Second perturbation:** Given $\mu > 1$, one estimates for

$$E = R(w(T))\mathrm{Diag}(\mu^{-1}, \mu)R(w(T))AR(1_Z\beta_0) ,$$

the Lyapunov exponent $\lambda(E) \geq \log(\mu) > 0$.

7) **Abramov formula:** $\lambda(E_Z) \geq \log(\mu) \cdot n/2$. With this, we achieve a big Lyapunov exponent for the derived cocycle.

8) **Herman:** $\exists \beta_1 \leq 4\pi/(n\log(\mu))$ with $\lambda(E_Z R(\beta_1 - \beta_0)) > 0$.

9) **Third perturbation:** $B = ER(1_Z(\beta_1 - \beta_0)) \in \mathcal{P}$ because $B_Z = E_Z R(\beta_1 - \beta_0)$.

10) **Choice of parameters:** Give $\epsilon > 0$. Choosing $\mu - 1 > 0$ small and $n \in \mathsf{N}$ large depending on ϵ, μ, achieves that $B \in \mathcal{P}$ is in a ϵ neighborhood of A.

2 Analytic extension of the standard map family

For the first heuristic argument, it was crucial that the integrable dynamical system T_0 could be embedded into an analytic system and that the Jacobian cocycle could be extended to an analytic cocycle.

Is it possible to embed in a similar way the standard map into a holomorphic map of $\mathbf{C}^n$? There is an obvious embedding into the holomorphic map in $\mathbf{C}^2$ by just extending x, y to C (see for example [Gel 92]). A handicap is that one has to take the real parts $Re(x), Re(y)$ modulo 2π, in order to get the standard map. Of course, one could skip this identification. But this has some disadvantages like for example that periodic orbits in the quotient space are no more periodic in the lift.

We propose an embedding of the Standard map family T_γ in *one single* analytic map

$$U : \begin{pmatrix} z \\ w \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} zwe^{z-u} \\ we^{z-u} \\ uve^{u-z} \\ ve^{u-z} \end{pmatrix} .$$

The standard map is induced on a one parameter family of invariant real tori

$$S_\gamma = \{(z, w, u, v) \in \mathbb{C}^4 \mid |z| = |u| = \frac{\gamma}{2}, |v| = |w| = 1, z = \overline{u}, w = \overline{v}\} \ .$$

Notice that the parameter γ doesn't appear in the map U. We didn't succeed in estimating the Lyapunov exponents by estimating with a subharmonicity argument the Lyapunov exponent of an analytic cocycle over U. Such an estimate (if at all possible !) would probably need additional ideas. The problem is, that the torus S_γ is not a so-called *distinguished boundary* of a polydisc in contrary to the invariant tori

$$\overline{S}_\gamma = \{(z, w, u, v) \in \mathbb{C}^4 \mid |z| = |u| = \frac{\gamma}{2}, |w| = |v| = 1, z = u, w = v\} \ ,$$

where U induces the integrable map T_0 and on which the Lyapunov exponent could be estimated from below by $\log(\frac{\gamma}{2})$ as we saw before.

Nevertheless, we think, that the map U is interesting itself. It allows to study the whole family of standard maps with *one analytic map* and it could be, that bifurcation phenomena and critical phenomena will be analyzable better like this. Why not try to understand more about a holomorphic map which contains a family of discrete Hamiltonian systems as subsystems? On the other hand, one has already a lot of information about U inherited from results about T_γ. We think of results like existence of periodic orbits, existence and nonexistence of invariant tori, Aubry-Mather sets, homoclinic points, special chaotic orbits, universality in bifurcation scenarios of periodic points or renormalisation schemes in the break-up of invariant curves. And who knows, if in future, mathematics will have results about iteration of multidimensional analytic maps, which will give back results about twist maps.

In the next section, we want to begin a modest qualitative study of U_γ. We give an integral, symmetries, a list of all the fixed points with their stability properties and other invariant sets like tori and vector spaces.

3   Some properties of the analytic map U

An integral. An integral of U is the determinant

$$I(z, w, u, v) = \det(DU) = wv$$

of the Jacobean

$$DU(z, w, u, v) = \begin{pmatrix} (w + wz)e^{z-u} & ze^{z-u} & -zwe^{z-u} & 0 \\ we^{z-u} & e^{z-u} & -we^{z-u} & 0 \\ -uve^{u-z} & 0 & (v + uv)e^{u-z} & ue^{u-z} \\ -ve^{u-z} & 0 & ve^{u-z} & e^{u-z} \end{pmatrix} \ .$$

and on $I^{-1}(1)$ there is another integral $J = zu$ and the map on an invariant set $wv = 1, uz = J, z \neq 0$ the map is given by

$$\tilde{U} : \begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} zwe^{z-J/z} \\ we^{z-J/z} \end{pmatrix},$$

which is product of the two integrable maps

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z \\ we^{z-J/z}, \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} zw \\ w \end{pmatrix}.$$

For $J = \frac{\gamma^2}{4}$, the map $\tilde{U}$ leaves invariant the torus

$$\tilde{S}_\gamma = \{(z,w) \in \mathbb{C}^2 \mid |z| = \frac{\gamma}{2}, |w| = 1\}$$

and induces there the standard map T_γ.

Invariant vector spaces. There is the obvious direct sum of invariant real vector spaces

$$\mathbb{R}^4, i\mathbb{R}^4 \subset \mathbb{C}^4.$$

There is also the two dimensional invariant complex plane

$$E = \{(0,w,0,v) \mid w,v \in \mathbb{C}\},$$

consisting of fixed points. Also the complex 2 dimensional plane

$$\{z = u, w = v\}$$

which contains the tori $\overline{S}_\gamma$ is invariant. Finally, for $n \in \mathbb{Z}$, the 1-dimensional complex lines

$$\begin{aligned} G_n &= \{(0,w,2n\pi i,1) \mid w \in \mathbb{C}\}, \\ H_n &= \{(2n\pi i,1,0,v) \mid v \in \mathbb{C}\}, \end{aligned}$$

consisting on fixed points are invariant.

Maps commuting with U. There is an involution

$$\begin{aligned} z &\mapsto & u &\mapsto z, \\ v &\mapsto & w &\mapsto v, \end{aligned}$$

commuting with U.

Reversibility on $I^{-1}(\pm 1)$. On the two invariant sets $I^{-1}(\pm 1)$, the map U is conjugate to its inverse U^{-1}

$$U = VU^{-1}V,$$

314

where V is the involution

$$V : \begin{pmatrix} z \\ w \\ u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ we^{z-u} \\ z \\ ve^{u-z} \end{pmatrix}.$$

Because $I = \det(DU)$ is invariant under conjugation, we have no reversibility on $I^{-1}(\mu)$ if $\mu \neq 1, -1$.

Invariant two dimensional real tori. The map U_γ leaves invariant the tori

$$S_\gamma = \{(z, w, u, v) \in \mathbf{C}^4 \mid |z| = |u| = \frac{\gamma}{2}, |w| = |v| = 1, z = \overline{u}, w = \overline{v}\},$$
$$\overline{S}_\gamma = \{(z, w, u, v) \in \mathbf{C}^4 \mid |z| = |u| = \frac{\gamma}{2}, |w| = |v| = 1, z = u, w = v\}.$$

Write $z = \frac{\gamma}{2}e^{2\pi i x}, w = e^{2\pi i y}$ on S_γ and $\overline{S}_\gamma$. The analytic map U restricted to S_γ is the standard map

$$T_\gamma : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y + \gamma\sin(x) \\ y + \gamma\sin(x) \end{pmatrix} .$$

The map U restricted to $\overline{S}_\gamma$ is the integrable map

$$T_\gamma : (x, y) \mapsto (x + y, y).$$

For $\gamma = 0$, both tori $S_0, \overline{S}_0$ collapse to 1-dimensional tori

$$S_0 = \{(0, 0, v, \overline{v}) \mid |v| = 1\},$$
$$\overline{S}_0 = \{(0, 0, v, v) \mid |v| = 1\}.$$

The fixed point at the origin. The point $O = (0, 0, 0, 0)$ is a fixed point of U. The Jacobean $DU(O) = Diag(0, 1, 0, 1)$ at this point shows that there is a two dimensional complex stable manifold

$$W^+ = \{(z, w, u, v) \mid U^n \to 0\}$$

passing through O. The center manifold to the two eigenvalues 1 is the two dimensional invariant line

$$E = \{(0, w, 0, v) \mid w \in \mathbf{C}, v \in \mathbf{C}\}$$

containing O. It consists of fixed points.

Fixed point lines. The 1-dimensional complex line

$$F_n = \{z - u = 2n\pi i, w = 1, v = 1\}$$

and each point of such a line is a fixed point of U. The spectrum of DU at a point $P = (u, 1, u + 2n\pi i, 1) \in F_n$ is

$$\{1, 1, \lambda_n(u), \lambda_n^{-1}(u)\},$$

with

$$\lambda_n(u) = 1 + u + n\pi i + \sqrt{(u + n\pi i)(2 + u)}\ .$$

We see, that except for positive real $(u + n\pi i)(2 + u)$, there is an eigenvalue inside and an eigenvalue outside the unit ball in $\mathbb{C}$. It follows that in this case, there is a complex 1-dimensional stable W_u^+ and a complex 1-dimensional unstable manifold W_u^- passing through P. This is especially true for $u = \gamma \in \mathbb{R}, n = 0$, where the stable and unstable manifolds cut the tori S_γ.

For $n \in \mathbb{Z}$, the 1-dimensional complex lines

$$\begin{aligned} G_n &= \{(0, w, 2n\pi i, 1) \mid w \in \mathbb{C}\}, \\ H_n &= \{(2n\pi i, 1, 0, v) \mid v \in \mathbb{C}\} \end{aligned}$$

consist on fixed points. The spectrum of DU at a point $(0, w, 2n\pi i, 1) \in G_n$ is

$$\left\{w, 1, 1 + n\pi i - \sqrt{2n\pi i - n^2\pi^2}, 1 + n\pi i + \sqrt{2n\pi i - n^2\pi^2}\right\},$$

the spectrum of DU at a point $(2n\pi i, 1, 0, v) \in H_n$ is

$$\left\{v, 1, 1 + n\pi i - \sqrt{2n\pi i - n^2\pi^2}, 1 + n\pi i + \sqrt{2n\pi i - n^2\pi^2}\right\}.$$

Fixed point plane. The two dimensional fixed point plane

$$E = \{(0, w, 0, v \mid w, v \in \mathbb{C}\}$$

is the center manifold passing through the origin O. The spectrum of DU at a fixed point $(0, w, 0, v) \in E$ is

$$\{1, 1, v, w\}.$$

The eigenspace to the eigenvalues 1 is of course E. The eigenspace to the eigenvalue v is spanned by $(0, -w, v - 1, v)$ and the eigenspace to the eigenvalue w is spanned by $(1 - w, -w, 0, v)$.

In summary, we have found all fixed points of U:

Every fixed point of U is contained in the set

$$\{O\} \cup E \cup \bigcup_{n \in \mathbb{Z}} F_n \cup G_n \cup H_n\ .$$

Proof. If $z \cdot w \cdot u \cdot v \neq 0$ then $w = 1, v = 1$ and $z - u = 2n\pi i$. Thus $(z, w, u, v) \in F_n$. If $z \cdot w \cdot u \cdot v = 0$, we get the other sets by combinatorial reasoning, using that $w = 0$ implies $z = 0$ and $v = 0$ implies $u = 0$. $\qquad\square$

Partial uniform hyperbolicity Because the determinant $I = \det DU = wv$ is an integral, we know that for $|I| > 1$

$$\det DU^n = I^n \to \infty, \; n \to \infty$$
$$\det DU^n = I^n \to 0, \; n \to -\infty .$$

This means that the map is dissipative in the domain $I < 1$ and expansive in the domain $I > 1$. For any periodic orbit in $I > 1$ there is an unstable manifold of positive dimension and for any periodic orbit in $I < 1$ there is a stable manifold of positive dimension. Given any U invariant probability measure in $I > 1$, there is a positive Lyapunov exponent. Analogous, for any U invariant probability measure in $I < 1$, there is a negative Lyapunov exponent.

4 Questions

We repeat and add some questions.

• Is the at the beginning mentioned bound for the metric entropy in the standard mapping true? The numerical calculations are convincing.

• What is the dynamics of U on the invariant manifolds of the various fixed points? What are the Julia sets of the map induced on the one dimensional invariant manifolds?

• Can one draw a qualitative picture of the stable and unstable *complex* manifolds of a periodic hyperbolic orbit on an invariant torus S_γ, where U induces the standard map?

References

[Gel 92] V.Gelfreich, V.Lazutkin, C.Simo, M.Tabanov. *Fern-like structures in the wild set of the standard and semistandard maps in* C^2. International J. of Bif. and Chaos,2: 353-370, 1992

[Her 83] M.R.Herman. *Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2.* Comment.Math.Helv., 58:453-502, 1983

[Kni 91] O.Knill. *The upper Lyapunov exponent of SL(2,R) cocycles: Discontinuity and the problem of positivity.* Lecture Notes in Math., No. 1486, Springer 1991

[Kni 92] O.Knill. *Positive Lyapunov exponents for a dense set of bounded measurable SL(2,R) cocycles.*
Erg. Th. Dyn. Syst., 12:319-331, 1992

[Man 81] R.Mané. *A proof of Pesin's formula.*
Erg. Th. Dyn. Syst., 1: 95-102 (1981)

[Par 86] M.Paramio,J.Sesma. *A continued fraction Procedure to determine invariant directions in twist mappings .*
Phys. Lett. A,117:333-336,1986

[Pet 83] K.Peterson. *Ergodic theory.*
Cambridge Studies in advanced mathematics 2, Cambridge University press, 1983

[Rue 79] D.Ruelle. *Ergodic theory of differentiable dynamical systems.*
Publ.Math. de l'IHES, 50:27-58, 1979

[Wol 91] Wolfram Research. *Mathematica Version 2.0.*
Illinois, 1991

Cohomology of dynamical systems

Contents

Abstract

We discuss some cohomological constructions for dynamical systems.

A group dynamical system is a pair $(\mathcal{R}, \mathcal{G})$, where $\mathcal{R}$ is a group acting on $\mathcal{G}$ by group automorphisms. An algebra dynamical system is a triple $(\mathcal{R}, \mathcal{M}, tr)$, where $\mathcal{R}$ is a group acting on the C^* algebra $\mathcal{M}$ by automorphisms leaving invariant a trace tr.

(1) For a group dynamical system $(\mathcal{R}, \mathcal{G})$ there is the Eilenberg-McLane cohomology.

(2) For a group dynamical system $(\mathbb{Z}, \mathcal{G})$ we define a sequence of Halmos homology and cohomology groups.

(3) For an algebra dynamical system $(\mathbb{Z}^d, \mathcal{M}, tr)$, there is a discrete version of de Rham's cohomology and an abstract version of Stokes theorem holds.

(4) For a group dynamical system $(\mathbb{Z}^d, \mathcal{G})$, there is a de Rham cohomology for groups.

1 Introduction

There are different cohomologies which are useful in ergodic theory. They all lead to algebraic invariants for ergodic dynamical systems. We are dealing with two categories of dynamical systems. A *group dynamical system* is a pair $(\mathcal{R}, \mathcal{G})$ of Abelian groups, where $\mathcal{R}$ is acting by group automorphisms on $\mathcal{G}$. An *algebra dynamical system* $(\mathcal{R}, \mathcal{M}, tr)$ is an Abelian group $\mathcal{R}$ acting by automorphisms on the C^* algebra $\mathcal{M}$. We will always assume $\mathcal{R} = \mathbb{Z}^d$ or the *cyclic case* $\mathcal{R} = \mathbb{Z}$.

• *The cohomology of groups for a group dynamical system.* We illustrate this cohomology by an example. Let (X, T, m) be an abstract dynamical system and $\mathcal{G} = L^\infty(X, \mathbb{T})$ be the group of measurable circle-valued functions on X. The transformation T induces a group automorphisms $f \mapsto f(T)$ giving the group dynamical system $(\mathcal{R}, \mathcal{G})$. The group $\mathcal{G}$ has a subgroup $\mathcal{C} = \{g(T)g^{-1}\}$ of so-called *coboundaries*. The group $\mathcal{H}^1(T, \mathcal{G})$ is the first *Eilenberg-McLane cohomology group* of the group dynamical system $(\mathcal{R}, \mathcal{G})$. Assume T and T^2 are ergodic. The constant function $f(x) = -1$ is not in $\mathcal{C}$ because $g(T) = -g$ would imply $g(T^2) = g$ and $g = const$ which contradicts $g(T) = g^{-1}$ unless $g = 1$. This non-trivialityœ of the cohomology is in some sense a *global constraint* coming from the requirement that g has to be measurable. If we would give up the obstruction that g has to be measurable, we could easily build functions g satisfying $g(T)g^{-1} = -1$. In other words, if we would use algebra instead of ergodic theory and take any bijective map T of a set X and form the group $\mathcal{G} = \mathbb{T}^X$ of all maps from X to the circle $\mathbb{T}$, then $\mathcal{H}^1(\mathcal{R}, \mathcal{G})$ would be trivial. The measure theoretic structure of the dynamical system acts as a boundary condition which makes the algebraic topology more interesting.

• *Halmos homology and cohomology groups for a cyclic group dynamical system.*
An automorphism T of an arbitrary abelian group $\mathcal{G}$ gives a discrete differentiation
$f \mapsto df = f(T)f^{-1}$ on the group $\mathcal{G}$. If $\mathcal{E}$ is the trivial group $\{1\}$ in $\mathcal{G}$, then $d^{-n}\mathcal{E}$ is
a subgroup of $\mathcal{G}$ and one can think of them as "polynomials" of degree $\leq (n-1)$
because n times "differentiation" makes them vanishing. Halmos defined the groups

$$\mathcal{H}_n(\mathcal{R}, \mathcal{G}) = d^{-n}\mathcal{E}/d^{-(n-1)}\mathcal{E} ,$$

which he called *generalized eigenvalues* in the case $\mathcal{G} = L^\infty(X)$, where the group au-
tomorphism is coming from a dynamical system (X, T, m). Then, $\mathcal{H}_1(T, \mathbb{R}) = d^{-1}\mathcal{E}$
is the space of functions f left-invariant under T and the first *Halmos homol-*
ogy group $\mathcal{H}_1$ measures how many ergodic components T has. The next group
$\mathcal{H}_2(T, \mathbb{R}) = d^{-2}\mathcal{E}/d^{-1}\mathcal{E}$ is the vector space of nontrivial eigenfunctions to eigen-
values different from 1. This second Halmos homology group measures the point
spectrum of the unitary Koopman- operator belonging to T and Halmos called the
other groups $\mathcal{H}_n$ *generalized eigenvalues.*
There is a dual construction. Start with $\mathcal{G}$ and form the groups

$$\mathcal{H}^n = d^{n-1}\mathcal{G}/d^n\mathcal{G} ,$$

which we call the *Halmos cohomology groups.* The first of these groups is identical
with the first Eilenberg MacLane cohomology group defined above.

• *de Rham cohomology groups for an algebra dynamical system.* We illustrate the sit-
uation with an example. Let $T_1, T_2, \ldots T_d$ be an ergodic $\mathbb{Z}^d$ action on the probability
space (X, m) and take $\mathcal{M} = L^\infty(X)$. Look at the *vector field*

$$v = (v_1, v_2, \ldots, v_d) \in L^\infty(X, \mathbb{R})^d .$$

The *rotation* of this vector field is

$$(dv)_{ij} = (v_i(T_j) - v_i - (v_j(T_i) - v_j)) \in L^\infty(X, \mathbb{R}) ,$$

where i, j runs over all pairs $i < j$. For $f \in L^\infty(X)$ is defined the *gradient*

$$(f(T_1) - f, f(T_2) - f, \ldots f(T_d) - f) ,$$

which is a vector field. One sees immediately, that the rotation of a gradient is
vanishing. A cohomology problem of de Rham type is to determine the vector space
of vector fields with vanishing rotation modulo the vector space of gradients.
For any constant vector field $v = (c_1, c_2, \ldots, c_d)$ with $c_i \in \mathbb{R}$, the rotation is zero but
except in the case when all c_i are vanishing, the vector fields v are not gradients
because there is no measurable function f on X, such that $c_i = f(T_i) - f$ for $c_i \neq 0$.
(To see this, just integrate the equation $c_i = f(T_i) - f$.) So, the first cohomology
group contains at least the group $\mathbb{R}^d$.

• *de Rham cohomology groups for a group dynamical system.* We consider a group dynamical system $(\mathcal{R}, \mathcal{G})$ with $\mathcal{R} = \mathbf{Z}^d$. We think of $A = (A_1, A_2, \ldots A_d)$ as a *one-form* or *connection* or *gauge-field* depending on topological, geometrical or physical preferences. Each A_i is attached at one of the transformations T_i generating $\mathcal{R}$. The *curvature* of a gauge field is

$$F_{ij} = dA = A_i A_j (T_i) A_i (T_j)^{-1} A_j^{-1}$$

and each F_{ij} is attached at a *plaquette* spanned by two transformations T_i, T_j. An element B in $\mathcal{G}$ can also act as a *Gauge transformation* on gauge fields

$$A = (A_1, A_2, \ldots, A_d) \mapsto AdB = (BA_1 B(T_1^{-1}), BA_2 B(T_2^{-1}), \ldots BA_d B(T_d^{-1})) .$$

Fields $A = dB$, which are cohomologuous to 1 are *gradients* of a function B. They satisfy automatically $dA = 1$, which means physically that they are *zero curvature fields*. Geometrically one knows then that parallel transport around a closed curve is the identity and space is *flat.* The problem is to find the *moduli space of zero curvature fields*, how many zero curvature fields exist modulo Gauge transformations. This can also be considered as the second cohomology group of a de Rahm cohomology which is defined in the group dynamical system $(\mathcal{R}, \mathcal{G})$.

After these examples, we turn to the history of cohomologies for dynamical systems. The general framework for cohomology has been worked out by Eilenberg and McLane [Eil 47]. It is called *cohomology of groups.* The "cohomology of dynamical systems" was invented in the beautiful paper [Kir 67] of Kirillov in 1967. We owe this reference to J.P. Conze. Kirillov asked a lot of interesting questions for this cohomology and used also $H^1(T, \mathbf{Z}_2)$ what we call the *cohomology of sets.* Cohomology of dynamical systems also appeared in the paper [Liv 72] without mentioning Kirillov's work. But already before, people were using cohomology in dynamical systems without using this name. The first appearance we could locate is in von Neumann's paper [Neu 32] (p.641) who was investigating the spectra of ergodic flows. (In Hilbert's list of problems [Hil 02], the cohomological equation $f(T) - f = g$ appeared in a side remark in problem 5 already. Hilbert gave it as an example of an analytic functional equation which has only a continuous non-differentiable solution). We mention also [Anz 51] who used the name "equivalent" instead of cohomologuous. In the book of Halmos [Hal 56], the notion of "generalized eigenvalues" is introduced. This is a cohomology construction which Halmos used it to distinguish dynamical systems. The same notation is found in [Akc 65]. Also influenced by Halmos was [Ste 71], who continued the work aiming to find invariants for dynamical systems. The cohomology of dynamical systems as a special case of group cohomology was worked out in [Moo 78] (and references therein) in the context of ergodic equivalence relations. Some ideas have been followed further and we refer to the book [Sch 89] (chapter 2).

Cohomolocal equations appear at different other places in other function spaces. They are important when studying *structural stability* of dynamical systems. When conjugating two dynamical systems T, $\tilde{T} = T + f$ with a conjugating map $\phi = Id + h$, the unknown function h has to satisfy the functional equation

$$T(x) + h(T(x)) = T(x + h(x)) + f(x + h(x)) .$$

In the case $T : T^1 \to T^1, x \mapsto x + \alpha$, this equation can be written as

$$h(x + \alpha) - h(x) = f(x + h(x))$$

which has the linearisation the so called *homological equation*

$$h(x + \alpha) - h(x) = f(x) .$$

Solving this equation in a space of analytic periodic functions is important in Arnold's proof [Arn 88] of the theorem, that an analytic diffeomorphism of the circle $\tilde{T}$ with Diophantine rotation number can be conjugated to the rigid rotation T if the two diffeomorphisms $T, \tilde{T}$ are close enough to each other. (Arnold conjectured, that this is true independent of the closeness of T and $\tilde{T}$. This conjecture was later proved by Herman.)

2 Cohomology of groups and dynamical systems

Let $\mathcal{R}$ be a group acting continuously on an abelian topological group $\mathcal{G}$. We write the group operations in $\mathcal{R}$ multiplicatively and the group operation in $\mathcal{G}$ additively and the action of $\mathcal{R}$ on $\mathcal{G}$ as $g \mapsto rg$. Denote by C^n the set of functions from $\mathcal{R}^n \to \mathcal{G}$ which are called $n-$ dimensional *cochains*. They form in a natural way a group. The *coboundary operator* $d^{(n)} : C^n \to C^{n+1}$ is defined as

$$
\begin{aligned}
d^{(n)} f(r_1, \ldots r_{n+1}) &= r_1 f(r_2, \ldots r_{n+1}) \\
&+ \sum_{i=1}^{n} (-1)^i f(r_1, \ldots, r_i r_{i+1}, \ldots, r_{n+1}) + \\
&+ (-1)^{n+1} f(r_1, \ldots, r_n) .
\end{aligned}
$$

The elements in $\mathcal{Z}^n = \ker(d^{(n)})$ are called $n-$ dimensional *cocycles* and $\mathcal{B}^n = \operatorname{im}(d^{(n-1)})$ is the set of *coboundaries*. They are both subgroups of C^n. It is of advantage to write the cochains also in a homogeneous way using one variable more:

$$F(r_0, r_1, \ldots, r_n) := r_0 f(r_0^{-1} r_1, r_1^{-1} r_2, \ldots, r_{n-1}^{-1} r_n) .$$

The function F satisfies the homogeneity condition $F(rr_0, \ldots rr_n) = rF(r_0, \ldots, r_n)$ and the old function can be recovered with the formula

$$f(r_1, \ldots r_n) = F(1, r_1, r_1 r_2, \ldots, r_1 r_2 \ldots r_n) .$$

In such homogeneous variables, the coboundary operator is

$$d^{(n)}F(r_0,\dots,r_{n+1}) = \sum_{i=0}^{n+1}(-1)^i F(r_0,\dots,\hat{r}_i,\dots,r_{n+1}) \ .$$

($\hat{r}_i$ means that the variable r_i has been taken away). If the coboundary operator is written in this form we see immediately that

$$d^{(n+1)}d^{(n)} = 0 \ .$$

This implies that $\mathcal{B}^n$ is a subgroup of $\mathcal{Z}^n$ and one can form the *cohomology group* $\mathcal{H}^n = \mathcal{Z}^n/\mathcal{B}^n$. This is the *group cohomology* of Eilenberg-McLane [Eil 47].

Examples.
• For $n = 0$ we get

$$\begin{aligned}
\mathcal{Z}^0 &= \{f \in \mathcal{G} | rf - f\}, \\
\mathcal{B}^0 &= 0
\end{aligned}$$

and so

$$\mathcal{H}^0 = \{g \in \mathcal{G} \mid rg = g, \forall r \in G\}$$

is the subgroup of $\mathcal{G}$ consisting of elements which are left invariant under all $r \in R$.

• For $n = 1$,

$$\begin{aligned}
\mathcal{Z}^1 &= \{f : \mathcal{R} \to \mathcal{G} \mid f(r_1 r_2) = f(r_1) + r_1 f(r_2)\} \ , \\
\mathcal{B}^1 &= \{f : \mathcal{R} \to \mathcal{G} \mid f(r) = rg - g \mid g \in \mathcal{G}\} \ .
\end{aligned}$$

In the case when $\mathcal{R}$ is abelian then an element $f \in \mathcal{Z}$ is also called simply a "cocycle". It follows then that

$$f(r_1) + r_1 f(r_2) = f(r_2) + r_2 f(r_1) \ .$$

In the case when $\mathcal{R} = \mathbf{Z}^d$ has the generators $e_1,\dots,e_d$, one can define $A_i = f(e_i)$ and the above cocycle condition is equivalent to the zero curvature condition

$$A_i + e_i A_j = A_j + e_j A_i \ .$$

• For $n = 2$, one gets

$$\begin{aligned}
\mathcal{Z}^2 &= \{f(r_1,r_2) = r_1 f(r_2,r_3) + f(r_1,r_2 r_3) - f(r_1 r_2,r_3)\} \ , \\
\mathcal{B}^2 &= \{f(r_1,r_2) = r_1 g(r_2) + g(r_1) - g(r_1 r_2)\} \ .
\end{aligned}$$

A special case of the cohomology of Eilenberg McLane is the *cohomology of abstract dynamical systems.*

If the group $\mathcal{R}$ acts by automorphisms on a probability space (X, m), we call $(X, \mathcal{R}, m)$ a dynamical system. We are mainly interested in the case $\mathcal{R} = \mathbb{Z}$ and write then (X, T, m) with $< T >= \mathbb{Z}$. Let G be a commutative locally compact topological group and define $\mathcal{G} = L^\infty(X, G)$. The group $\mathcal{R}$ is acting in a natural way on $\mathcal{G}$. The action of an element $T \in \mathcal{R}$ on $\mathcal{G}$ is $A(x) \mapsto A(T(x))$. The above group cohomology of $(\mathcal{R}, \mathcal{G})$ gives a cohomology of dynamical systems. The invention of this cohomology is due to Kirillov [Kir 67] who made also the simple remark that if two dynamical systems $(X_1, \mathcal{R}_1, m_1)$ and $(X_2, \mathcal{R}_2, m_2)$ are conjugate, they have isomorphic cohomology groups.

We are mainly interested in the *first cohomology group* because of the following theorem

Theorem 2.1 (Eilenberg-McLane, Feldmann-Moore) *If $\mathcal{R}$ is free or hyperfinite, then the groups $\mathcal{H}^n(\mathcal{R}, \mathcal{G})$ are trivial for $n \geq 2$.*

The zeroth cohomology group $\mathcal{H}^0$ is also not so interesting. If the action of $\mathcal{R}$ is ergodic, then this group is isomorphic to G. The first cohomology group is the quotient $\mathcal{H}^1 = \mathcal{G}/\mathcal{C}$, where

$$\mathcal{C} = \{ A \in \mathcal{G} \mid \exists B \in \mathcal{G}, \exists T \in \mathcal{R}, \ A = B(T) - B \} \ .$$

This is *the group of interest* and we will turn to this group in the next section.

3 The cohomology group $\mathcal{H}^1(T, G)$

In this section we deal with the first cohomology group in the case $\mathcal{G} = L^\infty(X, G)$, where $T : \mathcal{G} \to \mathcal{G}$ comes from an abstract dynamical system (X, T, m) and $G = SL(2, \mathbb{R})$. With non-abelian groups G, this leads only to *cohomology classes* and not to cohomology groups. We also want to treat the case, when G is an abelian subgroup of $SL(2, \mathbb{R})$. Especially $G = \mathbb{Z}_2$ or $G = SO(2, \mathbb{R})$ are interesting and the cohomology question for these groups sheds also light on the case $SL(2, \mathbb{R})$.

3.1 Circle-valued cocycles

The case when the group G is $SO(2, \mathbb{R})$ and $\mathcal{R} = \mathbb{Z} =< T >$ has been investigated extensively already (see for example [Bag 88]). Especially the case when the dynamical system is an ergodic group translation on the torus has attracted much attention especially in the context of ergodic skew products which bear the name *Anzai products*. In this case $G = SO(2, \mathbb{R})$, the elements in $\mathcal{G}$ are called *circle-valued cocycles*.

They have already been studied by von Neumann in 1932 ([Neu 32] p. 641) in connection with ergodic flows obtained by suspension. The dynamical system is the irrational rotation $x \mapsto x + \alpha$ on the circle. The problem can be formulated as a spectral problem for an operator A given as

$$A e^{ig(x)} = e^{if(x)} e^{ig(x+\alpha)}$$

which is also called a *weighted composition operator*. Finding an eigenvalue $e^{2\pi i\theta}$ with eigenfunction g for A is equivalent to have discrete spectrum of A because $ge^{2\pi ikx}$ is then a complete orthonormal system with eigenvalues $e^{2\pi i(k\alpha+\theta)}$. The existence of the eigenfunction g to the eigenvalue $e^{i\theta}$ is equivalent to the solvability of the *cohomological equation*

$$g(x+\alpha) - g(x) = f(x) - \theta \pmod{2\pi}$$

for the 2π periodic L^2 function g. Taking Fourier series $f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{2\pi ikx}$, this equation has the formal solution

$$g(x) = \sum_{k\in\mathbb{Z}} \frac{\hat{f}_k}{(1 - e^{2\pi ik\alpha})} \cdot e^{2\pi ikx} \; .$$

If this Fourier series converges, one has discrete spectrum for A.

It is not known how to determine the cohomology group

$$\mathcal{H}(T,G) = \mathcal{G}(T,G)/\mathcal{C}(T,G) \; .$$

In the case when $|X|$ is finite, and the dynamical system is ergodic, it is easy to show that $\mathcal{H}(T, SO(2,\mathbb{R})) = SO(2,\mathbb{R})$. In general, when X is a finite set, on has also $\mathcal{H}(T,G) = G$ for any group G. In the aperiodic case, the question is open and it seems that the determination of the group is hopeless.

We know ([Kni 2]) for $G = SL(2,\mathbb{R})$

Proposition 3.1 *If (X,T,m) is aperiodic, the subgroup $\mathcal{C}$ is dense in $\mathcal{G}$.*

and therefore the cohomology group is not a topological group when taking the quotient topology from $\mathcal{G}$.

3.2   Cohomology of measurable sets

An interesting case is $G = \mathbb{Z}_2$, because the group $\mathcal{G}$ is then isomorphic to the group of measurable subsets of X and the coboundaries are the subsets Y which can be represented as $Y = Z(T)\Delta Z$. On $\mathcal{G}$ we can take the metric $d(Y,Z) = m(Y\Delta Z)$. We showed in [Kni 1]:

Theorem 3.2 *If (X,T,m) is aperiodic, the subgroup $\mathcal{C}$ as well as its complement are both dense in $\mathcal{G}$ with respect to the metric d.*

The problem to determine whether a measurable set $Y \subset X$ is a coboundary is called *the cohomology problem of measurable sets*. Open is also how to determine the first cohomology group $\mathcal{H}^1(T,\mathbb{Z}_2)$. The cohomology of sets appeared already in [Kir 67] and from the same reference, the following result can be deduced which shows that in general $\mathcal{H}(T,G) \neq G$ unlike in the finite case $|X| < \infty$.

Proposition 3.3 *In the case $(X, T, m) = (\mathsf{T}^1, x \mapsto x + \alpha, dx)$, where α is irrational, the first cohomology group $\mathcal{H}^1(T, \mathbb{Z}_2)$ has at least 3 elements.*

Proof. The set X itself is not a coboundary because T^2 is ergodic. Choose $n \in \mathsf{N}$ and decide X into n intervals $Y_1, \ldots Y_n$ of the same length. Assume one of these intervals Y_k is a coboundary. Then all of them are coboundaries and also

$$X = \bigcup_{k=1}^{n} Y_k$$

is a coboundary. This is a contradiction and the assumption that Y_k was a coboundary is wrong. For $n = 2$, the sets X, Y_1, Y_2 are pairwise not cohomologuous. $\square$

This idea can be generalized. A dynamical system is called *reversible* if there exists an involution S such that $STS = T^{-1}$.

Proposition 3.4 *Given a reversible aperiodic ergodic dynamical system (X, T, m) with $STS = T^{-1}$ such that T^2 is ergodic. If there exists a set Y with $m(Y) = 1/2$ such that $Y \triangle S(Y) = X$ then the cardinality of $\mathcal{H}^1(T, \mathbb{Z}_2)$ is bigger then 2.*

Proof. We show that all of the sets $Y, S(Y), X$ are not coboundaries and that they are pairwise not cohomologuous.
X is not a coboundary because T^2 is ergodic. The set Y is a coboundary if and only if $S(Y)$ is a coboundary because

$$Y = Z \triangle T(Z) \Rightarrow S(Y) = S(Z) \triangle ST(Z) = S(Z) \triangle T^{-1} S(Z) \, .$$

So, if Y would be a coboundary also $X = Y \triangle S(Y)$ would be a coboundary. From

$$Y \triangle S(Y) = X, Y \triangle X = Y, S(Y) \triangle X = S(Y)$$

follows that no pair of them can be cohomologuous. $\square$

The following remark we owe to J.P Conze:

Remark. It follows from a recent result by Derrien [Der 93] that there is an infinite countable group contained in $\mathcal{H}^1(T, \mathbb{Z}_2)$ when working over the dynamical system $(\mathsf{T}^1, x \mapsto x + \alpha, dx)$. If there exists a partition $X = Y_1 \cup Y_2 \cup \ldots \cup Y_n$ of T^1 where Y_i are intervals with rational length then any of the sets $Y = \cup_{i \in I} Y_i$ is not a coboundary and two different such sets are not cohomologuous.

3.3   Relation between cohomologies

Let $\mathcal{E} = L^\infty(X, \mathbb{Z}_2))$ denote the group of $\mathbb{Z}_2$-valued cocycles, with

$$\mathcal{O} = L^\infty(X, SO(2, \mathsf{R}))$$

the group of circle-valued cocycles and with

$$\mathcal{A} = L^\infty(X, SL(2, \mathbf{R}))$$

the group of $SL(2, \mathbf{R})$ cocycles. The group $\mathcal{E}$ is the center of the multiplicative group $\mathcal{A}$ and $\mathcal{O}$ is the maximal abelian subgroup of $\mathcal{A}$. We have already prove, that $A \in \mathcal{E}$ is not a coboundary in $\mathcal{E}$, then A is neither a coboundary in $\mathcal{O}$ nor a coboundary in $\mathcal{A}$. This implies that if we find cocycles, which are not coboundaries in $\mathcal{E}$ then we have found cocycles in $\mathcal{A}$ which are not coboundaries in $\mathcal{A}$. Here is again the result.

Lemma 3.5 *(a) $A \in \mathcal{E}$ is a coboundary in $\mathcal{E}$ if and only if it is a coboundary in $\mathcal{O}$.*
(b) If $A_1, A_2 \in \mathcal{O}$ and $\exists C \in \mathcal{A}$ such that $A_1 = C(T)A_2C^{-1}$, then there exists $C \in \mathcal{O}$ with $A_1 = C(T)A_2C^{-1}$.
(c) $A \in \mathcal{O}$ is a coboundary in $\mathcal{O}$ if and only if it is a coboundary in $\mathcal{A}$.
(d) $A \in \mathcal{E}$ is coboundary in $\mathcal{E}$ if and only if it is a coboundary in $\mathcal{A}$.

This has a consequence for the cohomology groups:

Corollary 3.6 $\mathcal{H}^1(T, \mathbf{Z}_2)$ *is a subgroup of* $\mathcal{H}^1(T, SO(2, \mathbf{R}))$, *which is a subset of* $\mathcal{H}^1(T, SL(2, \mathbf{R}))$.

4   Halmos homology and cohomology groups in the cyclic case

We are dealing in this section with a cyclic group dynamical system which means that there is given an abelian group $\mathcal{G}$ with a group automorphism $T : \mathcal{G} \to \mathcal{G}$.

There is a generalization of the first cohomology group

$$\mathcal{H}^1(T, \mathcal{G}) = \mathcal{G}/\{B(T) - B \mid B \in \mathcal{G}\}$$

leading to a sequence of cohomology and homology groups which we call *Halmos cohomology groups*, because Halmos was dealing with such constructions under the name "generalized eigenvalues" in the case when $\mathcal{G} = L^\infty(X)$. Define the *discrete Lie derivative*

$$d : \mathcal{G} \to \mathcal{G}, \ A \mapsto dA = A(T) - A$$

and the sequence of groups $\mathcal{H}^0(T, \mathcal{G}) = \mathcal{G}$,

$$\mathcal{H}^n(T, \mathcal{G}) = d^{n-1}\mathcal{G}/d^n\mathcal{G}$$

called *Halmos cohomology groups* as well as the sequence of groups $\mathcal{H}_0 = \{0\} = \mathcal{E} \subset \mathcal{G}$,

$$\mathcal{H}_n = d^{-n}\mathcal{E}/d^{-(n-1)}\mathcal{E} ,$$

328

called *Halmos homology groups*, where

$$d^{-1}A = \{B \in \mathcal{G} \mid dB = A\} \ .$$

Like this, there are defined two sequences of groups

$$\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}^2, \mathcal{H}^1, \mathcal{H}^0 \ ,$$

which are invariants of the dynamical system $\mathcal{G}, T)$. In the special case when $\mathcal{G} = L^\infty, G)$ we denote the groups $\mathcal{H}_i(\mathcal{G}, T)$ also $\mathcal{H}_i(G, T)$. If two such dynamical systems (X, T_1, m) and (X, T_2, m) are conjugate, then $\mathcal{H}_i(T_1, G) = \mathcal{H}_i(T_2, G)$ and $\mathcal{H}^i(T_1, G) = \mathcal{H}^i(T_2, G)$. Also the numbers

$$
\begin{aligned}
\mu(T, G) &= \min\{n \geq 1 \mid \mathcal{H}_n = \{0\}\} \in \mathbb{N} \cup \{\infty\} \ , \\
\nu(T, G) &= \min\{n \geq 1 \mid \mathcal{H}^n = \{0\}\} \in \mathbb{N} \cup \{\infty\} \ ,
\end{aligned}
$$

are invariants of the dynamical system. Halmos [Hal 56] used the number μ [Hal 56] in the case $\mathcal{G} = L^\infty(X, \mathbb{R})$. With

$$
\begin{aligned}
d^{-1}\mathcal{E} &= \{A(T) = A\} \ , \\
d^{-2}\mathcal{E} &= \{A(T) - A = B \in d^{-1}\mathcal{E}\} = \{A(T^2) - 2A(T) + A = 0\} \ , \\
d^{-3}\mathcal{E} &= \{A(T) - A = B \in d^{-2}\mathcal{E}\} = \{A(T^3) - 3A(T^2) + 3A(T) - A = 0\} \ ,
\end{aligned}
$$

we get the homology groups

$$
\begin{aligned}
\mathcal{H}_0 &= d^{-1}\mathcal{E}/\mathcal{E} = \{A(T) = A\} \ , \\
\mathcal{H}_1 &= d^{-2}\mathcal{E}/d^{-1}\mathcal{E} = \{A(T^2) - 2A(T) + A = 0\}/\{A(T) - A = 0\} \ .
\end{aligned}
$$

On the other side, we have

$$
\begin{aligned}
d\mathcal{G} &= \{A = B(T) - B\} \ , \\
d^2\mathcal{G} &= \{A = B(T^2) - 2B(T) + B\} \ , \\
d^3\mathcal{G} &= \{A = B(T^3) - 3B(T) - 3B(T^2) + B\}
\end{aligned}
$$

and the cohomology groups

$$
\begin{aligned}
\mathcal{H}^0 &= \{A(T) = A\} \ , \\
\mathcal{H}^1 &= \mathcal{G}/\{A = B(T) - B\} \ , \\
\mathcal{H}^2 &= \{A = B(T) - B\}/\{A = B(T^2) - B(T) + B\} \ .
\end{aligned}
$$

There is the obvious problem to determine the groups $\mathcal{H}^n, \mathcal{H}_n$ and to find out, what they say all about the dynamical system. Also interesting would be to understand the dynamics of $d : \mathcal{G} \to \mathcal{G}$. which is already a nontrivial problem in the case when

the the group $\mathcal{G}$ is finite.

Examples:

• If we take for $\mathcal{G} = L^\infty(X, \mathbb{Z}_2)$, the algebra of measurable sets and for T an automorphism of the probability space, we define

$$T : \mathcal{G} \to \mathcal{G}, \ Y \mapsto T(Y) \ ,$$

and the determination of $\mathcal{H}^1(\mathcal{G}, T)$ is the "cohomology problem for measurable sets". More generally, with $\mathcal{G} = L^\infty(X, G)$ one gets the situation described below with $\mathcal{G} = L^\infty(X, G)$ where G is an abelian group and T is an automorphism of a probability space.

• Take any abelian group $\mathcal{G}$ and $T : g \mapsto g^{-1}$. For $\mathcal{G} = \mathbb{Z}_p$, where p is a prime, we have $\mathcal{H}^1(\mathbb{Z}_p, T) = \mathbb{Z}_2$ because half of the residue classes are quadratic residues. For $\mathcal{G} = \mathbb{Z}$ we get $\mathcal{H}^n = \mathbb{Z}_2$ and $\mathcal{H}_n = \{0\}$.

• For a finite abelian group $\mathcal{G}$, we calculated experimentally with a Cayley program that $\mathcal{H}^n(G, T) = \mathcal{H}_n(G, T)$ holds. This suggests, that in general, a duality could hold.

• Let $\mathcal{G} = C^\omega(U)$ be the additive group of all analytic functions on a domain $U \subset \mathbb{C}$. Define $d(f) = f'$ and $\mathcal{E} = \{0\} \subset \mathcal{G}$. $d^{-n}\mathcal{E}$ is the group of polynomials having degree $\leq n$ and $\mathcal{H}_n(T, \mathcal{G}) = \mathbb{R}$. On the other hand $\mathcal{H}^n(T, \mathcal{G})$ is the trivial group because any analytic function is also a derivative.

5 de Rham cohomology for C^* dynamical systems

In the case $\mathcal{R} = \mathbb{Z}^n$, $\mathcal{G} = L^\infty(X, G)$, only the first Eilenberg- McLane cohomology group is interesting. More interesting groups are obtained by mimicking the de Rham construction.

Let $\mathcal{M}$ be any C^* algebra with trace tr and $\mathcal{R} = <T_1, \ldots T_n>$ be a representation of $\mathbb{Z}^n$ as automorphisms of $\mathcal{M}$ which leave invariant the trace. The triple $(\mathcal{M}, \mathcal{R}, \mathrm{tr})$ is called a C^* dynamical system with time $\mathcal{R} = \mathbb{Z}^n$.

For example, one can take $\mathcal{M} = L^\infty(X, G)$, where $G = M(N, \mathbb{R})$ is the algebra of matrices on $\mathbb{R}^N$ and take $T_1, \ldots T_n$ as ergodic commuting measure preserving transformations on the probability space (X, m).

The *crossed product* $\mathcal{X} = \mathcal{M} \otimes \mathcal{R}$ with multiplication

$$f \cdot g = \sum_I f_I \tau_I \cdot \sum_J g_J \tau_J = \sum_{I,J} f_I(T^I) g_J \tau_{IJ} = \sum_{IJ=K} (fg)_K \tau_K \ ,$$

is a graded Banach algebra having the involution

$$(K^*)_I(x) = \overline{K_{-I}}(T^I x) \ .$$

An other algebra is obtained if the multiplication of the τ_i is not symmetric but anti-symmetric. Denote by Ω the algebra generated by $\tau_i, \tau_i^*, \ i = 1, \ldots, d$ which satisfies

$$
\begin{aligned}
\{\tau_i, \tau_j\} &= \{\tau_i^*, \tau_j^*\} = 0 \ , \\
\tau_i \wedge \tau_j^* &= \tau_i^* \wedge \tau_j = \delta_{ij} \ .
\end{aligned}
$$

A basis in Ω is given by the elements

$$\tau_{i_1} \wedge \ldots \wedge \tau_{i_p}, \ \tau_{i_1}^* \wedge \ldots \wedge \tau_{i_p}^*, 1 \le p \le n$$

with $i_1 < i_2 < \ldots < i_k$. The algebra has dimension $n(n-1)-1$. The tensor product $\Lambda = \mathcal{M} \otimes \Omega$ the space of *difference forms* or the *skew-crossed product* of the algebra $\mathcal{M}$ with the dynamical system $\mathcal{R}$. It is a graded algebra

$$\Lambda = \oplus_{p=-d}^d \Lambda_p$$

with exterior multiplication

$$f \wedge g = \sum_I f_I \tau_I \sum_J f_J \tau_J = \sum_{I,J} f_I g_J(T^I) \tau_I \wedge \tau_J \ .$$

If f is a p-form and g is a q-form, then fg is a $p+q$ form. The space of $0-$ forms Λ_0 can be identified with $\mathcal{M}$. The algebra Λ has an involution

$$\sum_I f_I \tau_I \mapsto \sum_I f_I^*(T_I^{-1}) \tau_I^*$$

and a trace

$$\mathrm{tr}(\sum_I f_I \tau_I) = \mathrm{tr}(f_0)$$

giving the scalar product

$$< f, g > = \mathrm{tr}(f \wedge g^*) = \mathrm{tr}(\sum_I f_I g_I) \ .$$

Define the 1-form $\tau = \sum_{i=1}^d \tau_i$ and

$$d : \Lambda_p \to \Lambda_{p+1}, \ f = \sum_I f_I \tau_I \mapsto df = \sum_I [\tau, f_I] \tau_I \ .$$

The operator d is called a *coboundary operator*. The adjoint of τ is the -1 form $\tau^* = \sum_{i=1}^{d} \tau_i^*$. Dual to d is the operator d^* defined by

$$d^* \sum_I g_I \tau_I = [\tau^*, g_I] \tau_I .$$

Examples. For $f \in \Lambda^0$ we get the *gradient*

$$df = \sum_{i=1}^{p} (f(T_i) - f) \tau_i .$$

For a 1-form $f = \sum_i f_i \tau_i$, we get the *rotation*

$$\begin{aligned}
df &= \sum_{i,j} (f_i(T_j) - f_i) \tau_j \wedge \tau_i \\
&= \sum_{i<j} (f_i(T_j) - f_i) - (f_j(T_i) - f_j) \tau_{ij} .
\end{aligned}$$

For a $(d-1)$-form $f = \sum_{I_i} f_{I_i} \tau_{I_i}$, with $\tau_{I_i} = \tau_1 \wedge \tau_2 \wedge \ldots \wedge \hat{\tau}^i \wedge \ldots \wedge \tau_d$, one obtains the *divergence*

$$df = \sum_{i=1}^{d} (f_{I_i}(T_i) - f_{I_i}) \tau_{1,2,\ldots,d} .$$

Lemma 5.1
$$d^2 f = (d^*)^2 f = 0$$

Proof. Given $f = \sum f_I \tau_I$, we calculate

$$d^2 f = \sum_{I,i,j} (f_I(T_i T_j) - f_I(T_i) - f_I(T_j) + f_I) \tau_i \wedge \tau_j \wedge \tau_I .$$

Interchanging a specific pair i, j on the right hand side changes the sign but not the value of the sum. This implies $d^2 f = 0$. Analogous we get $(d^*)^2 f = 0$. $\quad\Box$

It follows that the image of $d_k = d$ on Λ^k is contained in the kernel of d_{k+1} on Λ^{k+1}. When $\mathcal{M}$ is abelian this allows the definition of the *de Rham cohomology groups*

$$H^k(\mathcal{M}, T) = \mathrm{Ker}(d_{k+1})/\mathrm{Im}(d_k) .$$

In the non-abelian case, $H^k(\mathcal{M}, T)$ is only a set, which we call the *Moduli space of k-forms.*

Proposition 5.2 *Let $(\mathcal{M}, \mathcal{R}, \mathrm{tr})$ be a C^* dynamical system with time $\mathbf{Z}^n$. For each $(p-1)-$ form f and each $p-$ form g one has*

$$< df, g > = < f, d^* g > .$$

Proof. By linearity, it is enough to show it for $f = f_I \tau_I$ and $g = g_{iI} \tau_{iI}$

$$< df, g > = \mathrm{tr}((f_I(T_i) - f_I)g_{iI}^*) = \mathrm{tr}(f_I(g_{iI}^*(T_i^{-1}) - g_{iI}^*)) = < f, d^* g > .$$

□

This can be seen as a version of *theorem of Green-Stokes-Gauss*. We will illustrate later with examples how the classical theorem of Green-Stokes-Gauss can be obtained as a limit of the above proposition.

Hodge theory
There exists the involution called *Hodge-* operator*

$$* : \Lambda_p \to \Lambda_{n-p}, \ f = \sum_I f_I \tau_I \mapsto f_I^* \tau_{I^*} ,$$

with $(\tau_I)^* = \tau_J$, where $J = I^*$ is the unique complementary index satisfying $I \wedge J = \tau_1 \wedge \ldots \wedge \tau_d$. This gives a natural isomorphism between Λ_p and Λ_{n-p}. The operator $*$ satisfies $** = Id$.

It is natural to ask if a Hodge-theory analogous to the classical case exists. One can define the *Laplace-Beltrami* operator

$$L = dd^* + d^* d$$

which acts on Λ and leaves each space Λ_p of p-forms invariant. The Laplace-Beltrami operator is essentially the usual discrete Laplace operator:

Lemma 5.3

$$L \sum_I f_I \tau_I = -2(\sum_{i,I}(f_I(T_i) - 2f_I + f_I(T_i^{-1}))\tau_I .$$

Proof. We calculate

$$
\begin{aligned}
d^* df_I \tau_I &= \sum_{i,j}(f_I(T_i T_j^{-1}) - f_I(T_j^{-1}) - f_I(T_i) + f_I)\tau_j^* \tau_i , \\
dd^* f_I \tau_I &= \sum_{i,j}(f_I(T_i T_j^{-1}) - f_I(T_j^{-1}) - f_I(T_i) + f_I)\tau_i \tau_j^* ,
\end{aligned}
$$

Because for $i \neq j$ we have $\tau_j^* \tau_i = -\tau_i \tau_j^*$, we get

$$(d^* d + dd^*)f_I \tau_I = -2 \sum_i (f_I(T_i) - 2f_I + f_I(T_i)^{-1})\tau_I .$$

The claim follows by linearity.

□

Remark: The coboundary operator d can also be defined by taking a one form

$$\tau = \sum_i a_i \tau_i$$

333

satisfying the zero curvature equation

$$[a_i \tau_i, a_j \tau_j] = 0 .$$

In the same way one has then $d^2 = 0$. The Laplace-Beltrami operator satisfies under the condition $[a_i \tau_i, (a_j \tau_j)^*] = 0$

$$(d^* d + d d^*) f_I \tau_I = - \sum_i 2 a_i f_I(T_i) a_i^* - b_i f_I - f_I b_i + 2 a_i^*(T_i^{-1}) f_I(T_i^{-1}) a_i \tau_I ,$$

where $b_i = a_i^*(T_i^{-1}) a_i(T_i^{-1}) + a_i a_i^*$. Define on Λ the operator

$$D = d + d^*$$

and the operator P which satisfies

$$P = (-1)^p$$

on Λ_p. With $d^2 = (d^*)^2 = 0$ one checks the simplest version of *super-symmetry*

$$D^2 = L, P^2 = 1, \{D, P\} = 0 .$$

The kernel of $L = L_k$ acting on Λ_k is called the space of *random harmonic k-forms*
Examples
• $d = 1$.
(i) Let f be a 0-form and $g = g\tau$ be a 1-form. With

$$df = (f(T) - f)\tau, \ d^*(g\tau) = g(T^{-1}) - g$$

we get

$$
\begin{aligned}
< df, g > \ &= \ \operatorname{tr}(df g^*) = \operatorname{tr}((f(T) - f)\tau \tau^* g^*) \\
&= \ \operatorname{tr}(f(g(T^{-1}) - g)^*) = \operatorname{tr}(f(d^* g)^*) \ = < f, d^* g > \ .
\end{aligned}
$$

In the case $\mathcal{M} = L^\infty(X)$ with $\operatorname{tr}(f) = \int_X f \, dm$, this can be read as a *discrete partial integration*.

• $d = 2$.
(i) Given a 0-form f and a 1-form $g = g_1 \tau_1 + g_2 \tau_2$ we calculate

$$
\begin{aligned}
df &= (f(T_1) - f)\tau_1 + (f(T_2) - f)\tau_2 , \\
d^* g &= [\tau^*, g_1]\tau_1 + [\tau^*, g_2]\tau_2 = g_1(T_1^{-1}) - g_1 + g_2(T_2^{-1}) - g_2
\end{aligned}
$$

and get

$$
\begin{aligned}
< df, g > \ &= \ \operatorname{tr}(f(T_1) - f)g_1^* + (f(T_2) - f)g_2^*), \\
< f, d^* g > \ &= \ \operatorname{tr}(f(g_1^*(T_1^{-1}) - g_1^* + g_2^*(T_2^{-1}) - g_2^*)) .
\end{aligned}
$$

Comparison of the right-hand sides gives $< df, g >=< f, d^*g >$.

(ii) Take a 1-form $f = f_1\tau_1 + f_2\tau_2$ and a 2-form $g = g_{12}\,\tau_1 \wedge \tau_2$. With

$$df = (f_2(T_1) - f_2) - (f_1(T_2) - f_1)\tau_1 \wedge \tau_2 ,$$
$$d^*g = (g_{12}(T_1^{-1}) - g_{12})\tau_2 - (g_{12}(T_2^{-1}) - g_{12})\tau_1 ,$$

we get

$$< df, g > = \text{tr}((f_2(T_1) - f_2 - f_1(T_2) + f_1)g_{12}^*\,dm(x) ,$$
$$< f, d^*g > = \text{tr}(-f_1(g_{12}^*(T_2^{-1}) - g_{12}^*) + f_2(g_{12}^*(T_1^{-1}) - g_{12}^*)) .$$

Again both right hand sides coincide.

The classical de Rham theory as a limit
We will see in examples that in some sense the classical Gauss-Green-Stokes theorem can be obtained in the limit. For this we take on the d dimensional torus T^d the measurable Z^d action defined by the group translations

$$T_i(x)_k = x + \epsilon e_k \; ; ,$$

where e_i forms the standard basis in R^n. The unitary operators $U_i : h \mapsto h(T_i)$ give then the Z^d action on $\mathcal{M} = L^\infty(X)$ with the trace $\text{tr}(h) = \int_X h\,dm$.
• $d = 1$.
Let I be an interval $[a, b] \subset \mathsf{T}$ with $a \neq b$ and $g = 1_I \in L^\infty(X)$. Let $f : \mathsf{T}^1 \to \mathsf{R}$ be differentiable. We get

$$\frac{1}{\epsilon} < df, g > = \int_I \frac{f(x + \epsilon) - f(x)}{\epsilon}\,dx$$
$$\frac{1}{\epsilon} < f, d^*g > = \frac{1}{\epsilon}(\int_a^{a+\epsilon} f(x)\,dx - \int_b^{b+\epsilon} f(x)\,dx) .$$

and for $\epsilon \to 0$, we get

$$\frac{1}{\epsilon} < df, g > \quad \to \quad \int_I f'(x)\,dx$$
$$\frac{1}{\epsilon} < f, d^*g > \quad \to \quad f(b) - f(a) .$$

This shows that the usual Hauptsatz of calculus is obtained in the limit when the translation parameter ϵ approaches 0 and some regularity for f and g is assumed.

• $d = 2$.
We take the dynamical system

$$(X = \mathsf{T}^2, T_1(x, y) = (x + \epsilon, y), T_2(x, y) = (x, y + \epsilon) .$$

335

(i) Given a smooth 0-form f. Given a smooth curve $\gamma : [0,1] \to \mathsf{T}^2$ with velocity vector $\dot{\gamma}$, we define the smooth 1-form

$$g = \dot{\gamma}_1 1_\Gamma \tau_1 + \dot{\gamma}_2 1_\Gamma \tau_2 \,,$$

where

$$\Gamma = \{\gamma(t) + s \cdot (-\dot{\gamma}_2, \dot{\gamma}_1) \mid (t,s) \in [0,1] \times [-\epsilon, \epsilon] \subset \mathsf{T}^2$$

is a thick curve of thickness ϵ. We calculated

$$
\begin{aligned}
< df, g > &= \operatorname{tr}(f(T_1) - f)g_1^* + (f(T_2) - f)g_2^*), \\
< f, d^* g > &= \operatorname{tr}(f(g_1^*(T_1^{-1}) - g_1^* + g_2^*(T_2^{-1}) - g_2^*)) \,.
\end{aligned}
$$

For $\epsilon \to 0$, $\frac{1}{\epsilon} < df, g >$ is approaching the line integral $\int_\gamma \nabla f \, ds$ and $\frac{1}{\epsilon} < f, d^* g >$ is approaching $f(\gamma(1)) - f(\gamma(0))$.

(ii) Given a smooth 1-form $f = f_1 \tau_1 + f_2 \tau_2$ defined by 2 smooth functions $f_i : \mathsf{T}^2 \to \mathsf{R}$. Take a region $G \subset \mathsf{T}^2$ with a smooth boundary and define the 2-form $g = 1_G \, \tau_1 \wedge \tau_2$. We calculated

$$
\begin{aligned}
< df, g > &= \operatorname{tr}((f_2(T_1) - f_2 - f_1(T_2) + f_1)g_{12}^* \, dm(x) \,, \\
< f, d^* g > &= \operatorname{tr}(-f_1(g_{12}^*(T_2^{-1}) - g_{12}^*) + f_2(g_{12}^*(T_1^{-1}) - g_{12}^*)) \,.
\end{aligned}
$$

For $\epsilon \to 0$,

$$< df, g > \to \int_G \operatorname{rot}(f) \, dm$$

and

$$< f, d^* g > \to \int_{\delta G} f \, ds$$

so that the Theorem of Green is obtained in the limit.

6 de Rham cohomology for group dynamical systems

We do a related construction of de Rham cohomology with less structure by replacing the algebra dynamical system by a group dynamical system.

The idea is to define with a dynamical system a geometry on a group. A geometry is in the non discrete case defined by differential operators on a smooth structure. For example a connection (covariant derivative) on a Riemannian manifold defines the geometry. Here, the geometry is given by d commuting automorphisms on the group $\mathcal{G}$.

The application we have in mind is the group $\mathcal{G} = L^\infty(X, G)$, where G is an abelian group and T_i come from automorphisms of the probability space (X, m)

$$T_i : A \mapsto A(T_i) \,.$$

We are trying to do a de Rham cohomology on a group instead on an algebra by taking the cohomology for algebras and to identify the additive and multiplicative structures of the algebra.

Given a group dynamical system $(\mathcal{R}, \mathcal{G})$, where $\mathcal{R} = \mathbf{Z}^d$ is generated by d automorphism $T_1, \dots T_d$. For $1 \leq k \leq d$, the group Λ_k of $k - forms$

$$f = \sum_{|I|=k} f_I \tau_I$$

is a $\begin{pmatrix} k \\ d \end{pmatrix}$ tuple of group elements in $\mathcal{G}$. Addition of two $k - forms$ is defined as

$$\sum_I f_I \tau_I + \sum_J g_J \tau_J = \sum_I f_I g_I \tau_I$$

and the inverse is defined as

$$(\sum_I f_I \tau_I)^{-1} = \sum_I (f_I)^{-1} \tau_I \, .$$

Given two index sets I, J of the same length. If an odd permutation of indices in I gives J, we require that

$$f_I = (f_J)^{-1}$$

holds. We define a *multiplication* $\Lambda_k \times \Lambda_m \to \Lambda_{k+m}$

$$\sum_I f_I \tau_I \cdot \sum_J g_J \tau_J = \sum_{I,J} f_I g_J (T^I) \tau_I \tau_J$$

With special 1-form is $\tau = \sum_i \tau_i$ we define an exterior derivative by

$$f = \sum_I f_I \tau_I \to df = \sum_{I,i} f_I (T_i) f_I^{-1} \tau_i \tau_I \, .$$

In the case when the group G is abelian one gets

$$d^2 = 0$$

in the sense that

$$d^2 A = 1$$

for every k-form A. This implies that the image of $d = d_k$ on Λ_k is contained in the kernel of d_{k+1} on Λ_{k+1}. This leads to the cohomology groups

$$\mathcal{H}^k(G, T) = \text{Ker}(d_{k+1})/\text{Im}(d_k)$$

which are invariants of the group dynamical system $(\mathcal{R}, \mathcal{G})$.

If $\mathcal{G}$ is abelian, the space $\mathcal{H}^1(\mathcal{R}, \mathcal{G})$ is the *moduli space of zero curvature fields* over the group dynamical system $(\mathcal{R}, \mathcal{G})$. This moduli space labels the equivalence classes of zero curvature Gauge fields modulo the Gauge fields which are gradients. Examples: Take $d = 3$.

(i) A 0 form f is just an element in $\mathcal{G}$. The exterior derivative defines the *gradient*

$$df = f(T_1)f^{-1}\tau_1 + f(T_2)f^{-1}\tau_2 + f(T_3)f^{-1}\tau_3 \; .$$

The zeroth cohomology group $\mathcal{H}^0$ consists of the space of group elements which are invariant under all T_i:

$$\mathcal{H}^0 = \{f \in \mathcal{G} \mid f(T_i) = f\}$$

(ii) The exterior derivative of a 1-form f is the "rotation"

$$
\begin{aligned}
df &= d(f_1\tau_1 + f_2\tau_2 + f_3\tau_3) \\
&= f_2(T_3)f_2^{-1}\tau_3 \wedge \tau_2 + f_3(T_2)f_3^{-1}\tau_2 \wedge \tau_3 \\
&\quad + f_1(T_2)f_1^{-1}\tau_2 \wedge \tau_1 + f_2(T_1)f_2^{-1}\tau_1 \wedge \tau_2 \\
&\quad + f_3(T_1)f_3^{-1}\tau_1 \wedge \tau_3 + f_1(T_3)f_1^{-1}\tau_3 \wedge \tau_1 \\
&= f_2(T_3)f_2^{-1}f_3f_3(T_2)^{-1}\tau_3 \wedge \tau_2 \\
&\quad + f_1(T_2)f_1^{-1}f_2f_2(T_1)^{-1}\tau_2 \wedge \tau_1 \\
&\quad + f_3(T_1)f_3^{-1}f_1f_1(T_3)^{-1}\tau_1 \wedge \tau_3
\end{aligned}
$$

For an abelian group G, this exterior derivative is equivalent to the *curvature*

$$\sum [f_i\tau_i, f_j\tau_j]\tau_i \wedge \tau_j$$

of the one-form A.

The first cohomology group $\mathcal{H}^1$ is the moduli space of zero curvature fields:

$$\mathcal{H}^1 = \frac{\{f_i(T_j)f_i^{-1} = f_j(T_i)f_j^{-1}\}}{f_i = g(T_i)g^{-1}} \; .$$

(iii) The exterior derivative of a 2 form is

$$
\begin{aligned}
df &= d(f_{23}\tau_2 \wedge \tau_3 + f_{31}\tau_3 \wedge \tau_1 + f_{12}\tau_1 \wedge \tau_2) \\
&= f_{23}(T_1)f_{23}^{-1}f_{31}(T_2)f_{31}^{-1}f_{12}(T_3)f_{12}^{-1}\tau_1 \wedge \tau_2 \wedge \tau_3 \; .
\end{aligned}
$$

The second cohomology group $\mathcal{H}^2$ consists of all the 2-forms (f_{23}, f_{31}, f_{12}) which have "zero divergence"

$$f_{23}(T_1)f_{23}^{-1}f_{31}(T_2)f_{31}^{-1}f_{12}(T_3)f_{12}^{-1} = 1$$

modulo the space of 2-forms which arise as curvature of a 1-form (f_1, f_2, f_3)

$$f_{ij} = f_i(T_j)f_i^{-1}f_jf_j(T_i)^{-1} \; .$$

7 Questions

We add some questions.

- A main problem is the concrete determination of the defined groups in special examples.

- Does a duality for the Halmos cohomology and homology groups hold? Is $\mathcal{H}^n(\mathcal{G}, T)$ isomorphic to $\mathcal{H}_n(\mathcal{G}, T)$?

- Does the classical Hodge theory have an analogue for the situation here? Especially: is $\ker(L_k) = \mathcal{H}^k$? Is there a *Poincare duality*

$$\mathcal{H}^q = \mathcal{H}^{d-q}$$

for the de Rham cohomology groups?

References

[Akc 65] M.A.Akcoglu,R.V.Chacon. *Generalized eigenvalues of automorphisms.*
Proc.Amer.Math.Soc., 16:676-680, 1965.

[Anz 51] H.Anzai. *Ergodic Skew Product Transformations on the Torus.*
Osaka Math. J., 3:83-99, 1951.

[Arn 88] V.Arnold. *Geometrical methods in the theory of ordinary differential equations.*
Springer Grundlehren No. 250, New York, 1988

[Bag 88] L.Baggett. *On circle valued cocycles of an ergodic measure preserving transformation.*
Israel J. Math., 61:29-38, 1988.

[Der 93] J.-M. Derrien. *Critères d'ergodicité de cocycles en escalier. Exemples.*
C.R.Acad.Sci.Paris Série I, 316:73-76, 1993

[Eil 47] S.Eilenberg,S. McLane. *Cohomology theory in abstract groups.*
Annals of Mathematics, 48:51-78, 1947.

[Fel 77] J.Feldmann, C.Moore. *Ergodic equivalence relations, cohomology and von Neumann algebras I,II.*
Transactions of the Americal Mathematican Society, 234: 289-359, 1977

[Hal 56] P.Halmos. *Lectures on Ergodic Theory.*
The mathematical society of Japan, 1956

[Hil 02]   D.Hilbert. *Mathematische Problem.*
           Gesammelte Abhandlungen III, Berlin 1935

[Kir 67]   A.A. Kirillov. *Dynamical systems, factors and representations of groups.*
           Russian Math. Surveys, 22:63-75, 1967

[Kni 2]   O.Knill. *'Positive Lyapunov exponents for a dense set of bounded measurable $SL(2, R)-$ cocycles',*
           Ergod. Th. Dynam. Sys. 12, 319-331, 1992

[Kni 1]   O.Knill. *The upper Lyapunov exponents of $SL(2, R)$ cocycles: Discontinuity and the problem of positivity.*
           Lect. Notes Math., 1486, Springer 1991

[Liv 72]   A.Livsic. *Cohomology of dynamical systems.*
           Izv. Akad. Nauk SSSR, 36:1278-1301, 1972

[Moo 78]   C.Moore,K.Schmidt. *Coboundaries and homomorphisms for non-singular actions and a problem of H.Helson.*
           Proc. London Math.Soc., 40:443-475, 1980

[Neu 32]   J. von Neumann. *Zur Operatorenmethode in der Mechanik.*
           Ann. Math. 33: 587-642, 1932

[Sch 89]   K. Schmidt. *Algebraic ideas in ergodic theory.*
           Regional conference Series in mathematics No. 76, AMS Providence 1989

[Ste 71]   A.M.Stepin. *Cohomologies of automorphism groups of a Lebesgue space.*
           Functional Anal. Appl., 5:167-168, 1971

Nonuniform and uniform hyperbolic cocycles

Contents

Abstract

We study hyperbolic properties of bounded $SL(2,\mathbb{R})$ cocycles over a dynamical system.

Nonuniform hyperbolic cocycles have by definition Lyapunov exponents which are different almost everywhere. We reprove a theorem of Wojtkowski which states that sufficient for positive Lyapunov exponents is the existence of an invariant cone bundle. We make the remark that the condition in the theorem is also necessary.

A cocycle is *uniformly hyperbolic* if a strict invariant cone bundle exists. We reprove a theorem of Ruelle which states, that in this situation, the Lyapunov exponents depend real analytically on the cocycle.

We investigate the relation between the rotation number of Ruelle for measurable matrix cocycles and the hyperbolic behavior of the cocycle. We show that a cocycle is uniformly hyperbolic if and only if the rotation number is locally constant along a so called Herman circle $\beta \mapsto AR(\beta)$ which is obtained by multiplying A with a constant rotation $R(\beta)$ with angle β.

In the end we study spectra for $SL(2,\mathbb{C})$ matrix cocycles and prove that the spectrum of a cocycle acting on $L^2(X, \mathbb{C}^2)$ is the same then the *Sacker-Sell spectrum*, which is the set of complex numbers such that $z \cdot A$ has exponential dichotomy.

1 Introduction

Many numerical experiments support the believe that unstable behavior in conservative dynamical systems on a two dimensional manifold is typical. Anosov systems are prototypes of systems showing erratic, chaotic motion. It is just this extreme form of hyperbolicity which makes them tractable and the disorder in the system brings simplicity in the mathematical description.

A weaker form of hyperbolicity is present if there is a splitting of the seperatrices which is no longer uniform. This is the case when the Lyapunov exponents are different almost everywhere. Pesin theory shows that there are present still strong stochastic properties in this case. The mathematics of nonuniform hyperbolic dynamical systems seems to be much more difficult than the uniform hyperbolic systems.

What distinguishes uniform and non-uniform hyperbolic dynamical systems? How large are both classes in the space of all dynamical systems and how looks the boundary, where a transition between uniform and non-uniformity happens? Not much seems to be known about these questions. We just mention an unpublished result

of Mané [Man 83], [Man 83a] which states that a C^1 generic diffeomorphism on a compact connected manifold is either Anosov or has zero metric entropy.

In this chapter, we want to investigate the question of uniform and non-uniform hyperbolicity from a slightly different point of view: we fix a dynamical system and look at the set $\mathcal{A}$ of all bounded measurable $SL(2,\mathbb{R})$ cocycles over this dynamical system. In reality, the cocycles are coupled with the dynamical system but we think that understanding of general uniform and non-uniform cocycles over a *fixed* dynamical system could also help to understand some differences between uniform and non-uniform dynamical systems.

There is a subset $\mathcal{P}$ of the manifold $\mathcal{A}$ of measurable $SL(2,\mathbb{R})$ cocycles, where the Lyapunov exponents are positive almost everywhere. We have shown [Kni 2] that $\mathcal{P}$ is dense in the L^∞ topology and that $\mathcal{P} \setminus \text{int}\mathcal{P}$ is not empty if the dynamical system is aperiodic [Kni 1]. $\mathcal{P}$ contains an open subset $\mathcal{S}$, where the cocycles show uniform hyperbolic behavior. Ruelle's theorem (we will give in this chapter again a proof) tells, that Lyapunov exponents depend real-analytically on $\mathcal{S}$. Together with our discontinuity result, we know that for aperiodic dynamical systems $\mathcal{P} \setminus \mathcal{S}$ is never empty.

This chapter can be viewed as an appendix to our above mentioned works [Kni 1] and [Kni 2] and is organized as follows: we formulate and reprove a theorem of Wojtkowski which is a criterion for showing positive Lyapunov exponents almost everywhere. It is the only criterion, which works over general dynamical systems. It says, that a measurable invariant cone bundle, which is strict invariant on a sweep-out-set of positive measure implies positive Lyapunov exponents almost everywhere. We prove a remark which says that this criterion is also necessary.

Next, we formulate some equivalent definitions for uniform hyperbolic cocycles. Such cocycles have been investigated by Ruelle [Rue 79], who showed that in this open set $\mathcal{S}$, the Lyapunov exponents are depending real analytically on the cocycle. Ruelle gave also explicit formulas for the Fréchet derivative. We reprove this and illustrate his formula by calculating the derivative of the Lyapunov exponent on a special curve. This result illustrates that the derivative can get big in modulus, if the stable and unstable co-invariant direction fields get close.

Ruelle also defined a rotation number for measurable $SL(2,\mathbb{R})$ cocycles [Rue 85]. This rotation number is not unique. It is defined for cocycles with values in the *universal covering* of $SL(2,\mathbb{R})$ and the rotation number depends on the chosen lift. In certain cases, one can define a rotation number which is unique modulo 2π. One of these cases is treated by Herman [Her 83], who defined a rotation number for continuous cocycles homotopic to the identity.

The Lyapunov exponent and the rotation number can be joined together to a complex function $w = -\lambda + i\rho$ defined on the space $L^\infty(X, \widetilde{SL}(2, \mathbb{R}))$ of cocycles taking values in the universal covering group of $SL(2, \mathbb{R})$. In the theory of discrete Jacobi operators, (where no universal covering is necessary because of the special form of the cocycles), the number w is the *Floquet exponent* and a result in [Dei 83] shows that there is then a link between the Lyapunov exponent and the rotation number. We believe that this should be the case also in the more general set-up of general cocycles but have no results in this direction. Instead we will prove a relation between the Lyapunov exponent and the rotation number which says that local constance of the rotation number along a circle $\beta \mapsto AR(\beta)$ (where $R(\beta)$ is a rotation about an angle β), is equivalent with $A \in \mathcal{S}$.

There are different types of spectra which can be defined for matrix cocycles. First of all, a cocycle $A\tau^*$ is an element of a C^* algebra which is the crossed product of $L^\infty(X, M(2, \mathbb{C}))$ with the dynamical system. It can also be viewed as an operator acting on $L^2(X, \mathbb{C}^2)$ acting as $u \mapsto Au(T^{-1})$ generalizing the Koopman operator in ergodic theory. For almost all $x \in X$, $A\tau^*$ defines an operator on $l^2(\mathbb{C}^2)$ defined by $(Au)_n = A(T^n x)u_{n-1}$ leading to another spectrum which we call individual spectrum. An other spectrum is the Sacker-Sell spectrum which is defined to be the set of complex numbers z such that $z \cdot A$ has exponential dichotomy on $L^2(X, \mathbb{C}^2)$. In the end of the chapter we will have a look at such spectra.

2 Invariant cone bundles and nonuniform hyperbolicity

A *dynamical system* (X, T, m) is an automorphism T of a probability space (X, m). We define the Banach manifold $\mathcal{A} = L^\infty(X, SL(2, \mathbb{R}))$ in the real Banach algebra $\mathcal{X} = L^\infty(X, M(2, \mathbb{R}))$. The elements in $\mathcal{X}$ are called *matrix cocycles*. We use the notation $A(T)$ for the cocycle $x \mapsto A(T(x))$ and write $A^n = A(T^{n-1}) \cdots A$. For an element $A \in \mathcal{X}$ the *Lyapunov exponent* is defined as

$$\lambda(A, x) = \lim_{n \to \infty} n^{-1} \log \|A^n(x)\|$$

and the integrated Lyapunov exponent $\lambda(A) = \int_X \lambda(A, x)\, dm(x)$. We define the set $\mathcal{P} \subset \mathcal{A}$ as the set of $SL(2, \mathbb{R})$-matrix cocycles, where $\lambda(A, x)$ is positive for almost all $x \in X$.

A *cone* is a closed proper subset in $\mathbb{R}^2$, such that for $a, b \in C$, also $a + b \in C$ and if $a \in C$ and $t \in \mathbb{R}^+$, then $t \cdot a \in C$. A cone C is determined by two vectors $c^{(1)}, c^{(2)}$ namely

$$C = \{v \in \mathbb{R}^2 \mid \exists t_1, t_2 \in \mathbb{R}^+, \ v = t_1 c^{(1)} + t_2 c^{(2)}\} \ .$$

A *measurable cone bundle* C is a map $x \mapsto C(x) \cup (-C(x))$, where $C(x)$ is a cone and where the two mappings

$$c^{(1)}, c^{(2)} : X \mapsto \mathbb{R}^2$$

344

determining the cone bundles are measurable.

We identify the one dimensional projective space P^1 with $\mathsf{R}/(\pi\mathsf{Z})$ and identify an element in P^1 with an angle in $[0,\pi)$. Define the mapping

$$\mathsf{R}^2 \setminus \{0\} \to P^1, \ v \mapsto \overline{v} \in [0,\pi) \,,$$

where $\overline{v}$ is the angle v makes with the first basis vector $e^{(1)} = (1,0)$ in R^2 taken modulo π. We define on P^1 the metric

$$|u^{(1)} - u^{(2)}| = |\sin(u^{(1)} - u^{(2)})| \,.$$

The cone bundle C given by the mappings $c^{(1)}$ and $c^{(2)}$ can also be described by two measurable mappings

$$\overline{c}^{(1)}, \overline{c}^{(2)} : X \to P^1$$

and we write

$$C = [\overline{c}^{(1)}, \overline{c}^{(2)}] \,.$$

The notation $A\overline{c}$ means $\overline{Ac}$. If we assume to have given an orientation on P^1 it makes sense to speak of intervals in P^1 and the cone $C(x)$ is just represented by the interval $[\overline{c}^{(1)}(x), \overline{c}^{(2)}(x)]$ in P^1.

We say a cocycle $A \in \mathcal{A}$ *admits an invariant cone bundle* if there exists a cone bundle $[\overline{c}^{(1)}, \overline{c}^{(2)}]$ such that for almost all $x \in X$, there exists $\epsilon = \epsilon(x) > 0$ with

$$[A(x)\overline{c}^{(1)}(x), A(x)\overline{c}^{(2)}(x)] \subset [\overline{c}^{(1)}(T(x)) + \epsilon(x), \overline{c}^{(2)}(T(x)) - \epsilon(x)] \,.$$

We have not changed the notation when $A(x)$ acts on P^1 instead on R^2.

We say, that A *admits a strict invariant cone bundle* if there exists $\epsilon > 0$ independent of x such that for almost all $x \in X$

$$[A(x)\overline{c}^{(1)}(x), A(x)\overline{c}^{(2)}(x)] \subset [\overline{c}^{(1)}(T(x)) + \epsilon, \overline{c}^{(2)}(T(x)) - \epsilon] \,.$$

The following theorem of Wojtkowski states that invariant cone bundles give positive Lyapunov exponents.

Theorem 2.1 (Wojtkowski) *If A admits an invariant cone bundle $[\overline{c}^{(1)}, \overline{c}^{(2)}]$ then*

$$\lambda(A) \geq \int_X \log \frac{\sqrt{\xi}+1}{\sqrt{\xi}-1} \, dm > 0$$

with

$$\xi = \frac{|\overline{c}^{(2)}(T) - A\overline{c}^{(1)}| \cdot |A\overline{c}^{(2)} - \overline{c}^{(1)}(T)|}{|\overline{c}^{(2)}(T) - \overline{c}^{(1)}(T)| \cdot |A\overline{c}^{(2)} - A\overline{c}^{(1)}|} \,.$$

We took the formulation in [Woj 86]. In another form, the theorem has been proved in [Woj 85]. Because we have changed the language a little (we have a measurable cocycle, where Wojtkowski has a piecewise continuous cocycle being the Jacobean of a piecewise C^1 mapping on a two dimensional manifold) and used another notation,

we will give again a proof.

We need preliminary lemmas for the proof. The first lemma is an Abramov type result (formulated by Wojtkowski) which relates the Lyapunov exponent of A with the Lyapunov exponent of the derived cocycle A_Z.

Lemma 2.2 *If (X, T, m) is ergodic and $Z \subset X$ has positive measure, then*

$$\lambda(A_Z) \cdot m(Z) = \lambda(A) .$$

For a proof see [Kni 1].

Define the function $F : \mathbb{R}^2 \to \mathbb{C}$ by

$$v = (v_1, v_2) \mapsto F(v) = (v_1 \cdot v_2)^{1/2} .$$

For a matrix

$$A \in SL(2, \mathbb{R})^+ = \{A \in SL(2, \mathbb{R}) \mid [A]_{ij} \geq 0\}$$

define

$$\rho(A) = \inf_{F(v)=1} F(Av) .$$

The function ρ as a kind of norm in $SL(2, \mathbb{R})$.

Lemma 2.3 (Wojtkowski) *For $A, B \in SL^+(2, \mathbb{R})$,*

$$a) \quad \|A\| \geq \rho(A)$$
$$b) \quad \rho(AB) \geq \rho(A) \cdot \rho(B) ,$$
$$c) \quad \rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (ad)^{1/2} + (bc)^{1/2} .$$
$$d) \quad A_{ij} \geq B_{ij}, \ i, j = 1, 2 \Rightarrow \rho(A) \geq \rho(B)$$

Proof.

a) Take $v = (1, 1)$. Then $\|A\| \geq \frac{|Av|}{|v|} \geq \frac{F(Av)}{F(v)} \geq \rho(A)$.

b)

$$\begin{aligned}
\rho(AB) &= \inf_{F(v)=1} F(ABv) \\
&\geq \inf_{F(Bv)=1} \frac{F(ABv)}{F(Bv)} \cdot \inf_{F(v)=1} F(Bv) \\
&= \rho(A) \cdot \rho(B) .
\end{aligned}$$

c) Direct computation: take the vector $v = (r, r^{-1})$. Then $F(v) = 1$ and

$$F(Bv) = (ar + br^{-1})^{1/2}(cr + dr^{-1})^{1/2} .$$

The infimum is attained for $r^4 = (bd)/(ac)$ which gives the result.

d) Follows from c). □

Corollary 2.4 *For* $A \in L^\infty(X, SL(2,\mathbb{R})^+)$ *one has*

$$\lambda(A) \geq \int_X \log(\rho(A))\, dm \ .$$

Proof. Using lemma 2.3 b), one has

$$
\begin{aligned}
\lambda(A) &= \lim_{n\to\infty} n^{-1} \int_X \log \|A^n\|\, dm \geq \lim_{n\to\infty} n^{-1} \int_X \log\Big(\prod_{k=0}^{n-1} \rho(A(T^k))\Big)\, dm \\
&= \lim_{n\to\infty} n^{-1} \int_X \sum_{k=0}^{n-1} \log(\rho(A(T^k)))\, dm = \int_X \log(\rho(A))\, dm \ .
\end{aligned}
$$

□

Proof of the theorem of Wojtkowski. We can assume without loss of generality that the dynamical system is ergodic because the general result is obtained by integrating over the Choquet simplex of ergodic invariant measures.

Assume $A \in \mathcal{A}$ admits the invariant cone bundle $C = [\bar{c}^{\,(1)} + \bar{c}^{\,(2)}]$. We find a sequence of measurable sets Y_l with $m(Y_l) \to 1$ such that the derived cocycles $A_l := A_{Y_l}$ over the induced dynamical systems (Y_l, T_l, m_l) have a strict invariant cone bundle C_l which is just the restriction of C onto the set Y_l. (Note however that this does not imply that A_l is uniformly hyperbolic because A_l is not bounded in general.) From Lemma 2.2 we obtain for $n \to \infty$ $\lambda(A_l) \to \lambda(A)$. We get

$$
\begin{aligned}
\xi &= \frac{|\bar{c}^{\,(2)}(T) - A\bar{c}^{\,(1)}| \cdot |A\bar{c}^{\,(2)} - \bar{c}^{\,(1)}(T)|}{|\bar{c}^{\,(2)}(T) - \bar{c}^{\,(1)}(T)| \cdot |A\bar{c}^{\,(2)} - A\bar{c}^{\,(1)}|} \\
&= \frac{\sin(\bar{c}^{\,(2)}(T) - A\bar{c}^{\,(1)}) \cdot \sin(A\bar{c}^{\,(2)} - \bar{c}^{\,(1)}(T))}{\sin(\bar{c}^{\,(2)}(T) - \bar{c}^{\,(1)}(T)) \cdot \sin(A\bar{c}^{\,(2)} - A\bar{c}^{\,(1)})} \\
&= \frac{(\cot \bar{c}^{\,(2)}(T) - \cot A\bar{c}^{\,(1)}) \cdot (\cot A\bar{c}^{\,(2)} - \cot \bar{c}^{\,(1)}(T))}{(\cot \bar{c}^{\,(2)}(T) - \cot \bar{c}^{\,(1)}(T)) \cdot (\cot A\bar{c}^{\,(2)} - \cot A\bar{c}^{\,(1)})} \\
&= [\cot A\bar{c}^{\,(1)}, \cot A\bar{c}^{\,(2)}, \cot \bar{c}^{\,(1)}(T), \cot \bar{c}^{\,(2)}(T)] \ ,
\end{aligned}
$$

where $[\mu_1, \mu_2, \mu_3, \mu_4]$ is the *cross-ratio* which is independent of the coordinate system. Take in the fiber over x the new basis $c^{\,(1)}(x), c^{\,(2)}(x)$ and assume that for each $l \in \mathbb{N}$ the coordinate transformation from the standard basis to the new basis is given by the bounded measurable mapping

$$D_l : Y_l \to SL(2,\mathbb{R}) \ .$$

347

In the new coordinates, the cocycle

$$B_l = D_l(T_l)A_l D_l^{-1}$$

is a cocycle with values in $SL^+(2,\mathbb{R})$ and one has $\lambda(A_l) = \lambda(B_l)$. The cocycle $B_l \in L^\infty(X, SL(2,\mathbb{R})^+)$ maps the cone bundle

$$D_l(C_l) = \{v = (v_1, v_2) \mid v_1 v_2 \geq 0\}$$

inside the cone bundle $D_l(A(C_l))$. Denote by

$$E_l(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \in SL^+(2,\mathbb{R}) \ .$$

a cocycle which maps the cone bundle C_l exactly on $D_l(A(C_l))$. Because such a cocycle E_l satisfies $[B_l]_{ij} \geq [E_l]_{ij}$, we have with lemma 2.3 d) $\rho(B_l) \geq \rho(E_l)$. Because the cross-ratio is invariant under coordinate transformations, we can write $\xi(x)$ as

$$\xi(x) \leq [0, \infty, \frac{c(x)}{a(x)}, \frac{d(x)}{b(x)}] = \frac{a(x)d(x)}{b(x)c(x)} > 0 \ .$$

We check

$$\sqrt{ad} + \sqrt{bc} = \frac{\sqrt{\xi}+1}{\sqrt{\xi}-1}$$

and get with lemma 2.3 d) and corollary 2.4

$$
\begin{aligned}
\lambda(A_l) &= \lambda(B_l) \geq \int_{Y_l} \log(\rho(B_l)) \, dm_l \\
&\geq \int_{Y_l} \log(\rho(E_l)) \, dm_l \\
&= \int_{Y_l} \log((ad)^{1/2} + (bc)^{1/2}) \, dm_l \\
&= \int_{Y_l} \log \frac{\sqrt{\xi}+1}{\sqrt{\xi}-1} \, dm_l > 0 \ .
\end{aligned}
$$

The claim of the theorem follows after taking the limit $l \to \infty$.

$$\lambda(A) \geq \int_X \log \frac{\sqrt{\xi}+1}{\sqrt{\xi}-1} \, dm > 0 \ .$$

$\square$

3 The converse of Wojtkowsky's theorem

The theorem of Wojtkowski can be reversed. The idea for this remark is a slight modification of the concept of the Lyapunov metric. We refer to [You 86], where we take the essence for the following result:

Theorem 3.1 *If $A \in \mathcal{P}$ then A admits an invariant cone-bundle.*

Proof. Fix $A \in \mathcal{P}$ and call $\lambda := \lambda(A)$ and for $i = 1, 2$ denote with $W^{(i)}(x)$ the two measurable co-invariant direction fields. Let $w^{(i)}(x)$ be a unit vector in $W^{(i)}(x)$ and denote with $e^{(1)}$ and $e^{(2)}$ the usual basis in $\mathbb{R}^2$.

We show first that there exists a measurable map $B : X \to GL(2, \mathbb{R})$ such that

$$B(T)AB^{-1} = \mathrm{Diag}(\mu, \nu)$$

with $\mu \geq e^{\lambda/2}$ and $\nu \leq e^{-\lambda/2}$.

Define for $x \in X$ the matrix $B(x)$ through

$$B(x)e^{(1)} = \left(\sum_{n=0}^{\infty} |A^{-n}(x)w^{(1)}(x)| \, e^{n\lambda/2}\right)^{-1} w^{(1)}(x) \, ,$$

$$B(x)e^{(2)} = \left(\sum_{n=0}^{\infty} |A^{n}(x)w^{(2)}(x)| \, e^{n\lambda/2}\right)^{-1} w^{(2)}(x) \, .$$

We show now that

$$|B(T(x))^{-1}A(x)B(x)e^{(1)}| \geq e^{\lambda/2} \, .$$

If we define for $k \in \mathbb{N}$ the number

$$s_k(x) = |A^{-1}(T^k x)w^{(1)}(T^k x)|e^{\lambda/2}$$

we can write

$$B(x)e^{(1)} = \left(\sum_{n=0}^{\infty} s_{-1}s_{-2}\ldots s_{-n}\right)^{-1} w^{(1)}(x)$$

and

$$|B(T(x))^{-1}A(x)B(x)e^{(1)}|$$

$$= |A(x)w^{(1)}(x)| \cdot \frac{\sum_{n=1}^{\infty} s_0 s_{-1} \ldots s_{-(n-1)}}{\sum_{n=0}^{\infty} s_{-1}s_{-2}\ldots s_{-n}}$$

$$= |A(x)w^{(1)}(x)| \cdot s_0 \cdot \frac{\sum_{n=1}^{\infty} s_0 s_{-1} \ldots s_{-(n-1)}}{\sum_{n=0}^{\infty} s_0 s_{-1} \ldots s_{-n}}$$

$$= |A(x)w^{(1)}(x)| \cdot s_0$$

$$= |A(x)w^{(1)}(x)| \cdot |A^{-1}(x)w^{(1)}(x)|e^{\lambda/2} \geq e^{\lambda/2} \, .$$

In the same way one checks, that

$$|B(T(x))^{-1}A(x)B(x)e^{(2)}| \leq e^{-\lambda/2} \, .$$

To construct an invariant cone bundle for the cocycle A, one just takes an invariant cone bundle $C(x)$ for the diagonal cocycle $B(T)^{-1}AB$. The image $B(C)$ is then an invariant cone bundle for A. $\qquad \square$

Remark. Unlike in the case of uniform hyperbolicity to which we will turn in the next section, the diagonalisation in the above proof is in general not possible in a bounded way.

4 Uniform hyperbolicity

We consider now the open set

$$S = \{A \in \mathcal{A} \mid \exists C \in \mathcal{A}, \exists \epsilon > 0, \ [C(T)AC^{-1}]_{ij} \geq \epsilon\} \ .$$

Ruelle ([Rue 79]) defined S as the set of cocycles leaving strictly invariant a cone bundle. We will prove the equivalence of these two definitions and also reprove the result of Ruelle which says that the Lyapunov exponent $\lambda : S \to \mathbf{R}$ is real analytic. We consider a simpler case than Ruelle because we deal only with 2×2 matrices.

The next lemma of Wojtkowski [Woj 85] gives an estimate of the Lyapunov exponent in the uniform hyperbolic case.

Lemma 4.1 (Wojtkowski) *If* $A \in SL(2, \mathbf{R})^+$, $[A]_{ij} \geq \epsilon$ *then*

$$\rho(A) \geq (1 + 2\epsilon^2)^{1/2}$$

and so

$$\lambda(A) \geq \frac{1}{2}\log(1 + 2\epsilon^2) \ .$$

Proof. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $w = (w_1, w_2)$ with $F(w) = (w_1 w_2)^{1/2} = 1$ then

$$
\begin{aligned}
F(Aw) &= (aw_1 + bw_2)^{1/2}(cw_1 + dw_2)^{1/2} \\
&\geq (ad - bc + 2bc)^{1/2} \geq (1 + 2\epsilon^2)^{1/2} \ .
\end{aligned}
$$

$\square$

Lemma 4.2 *If* $A \in \mathcal{A}$ *has a strict invariant cone bundle then also* $B = D(T)AD^{-1}$ *has a strict invariant cone bundle for every* $D \in \mathcal{A}$.

Proof. A strict invariant cone bundle $C = [\overline{c}^{\,(1)}, \overline{c}^{\,(2)}]$ satisfies

$$A(C) = [A\overline{c}^{\,(1)}, A\overline{c}^{\,(2)}] \subset [\overline{c}^{\,(1)} + \epsilon, \overline{c}^{\,(2)} - \epsilon]$$

and the cocycle $B = D(T)AD^{-1}$ has the strict invariant cone bundle $D(C)$. $\square$

Lemma 4.3 *Given two measurable mappings*

$$\overline{d}^{\,(1)}, \overline{d}^{\,(2)} : X \to \mathbf{P}$$

such that for some $\epsilon > 0$ *and almost all* $x \in X$

$$|\overline{d}^{\,(1)}(x) - \overline{d}^{\,(2)}(x)| \ \geq \epsilon \ ,$$

then there exists $B \in \mathcal{A}$ *such that for* $i = 1, 2$

$$B(x)\overline{d}^{(i)}(x) = \overline{e}^{(i)}(x) \ .$$

Proof. Let $d^{(i)}(x)$ be unit vectors in the equivalence classes $\overline{d}^{(i)}(x)$. Define the cocycle E by

$$E(x)e^{(i)}(x) = d^{(i)}(x) \ .$$

The cocycle $B(x) = E(x)/\det(E(x))$ satisfies

$$B(T)^{-1}AB\overline{e}^{(i)} = \overline{e}^{(i)} \ .$$

Because for almost all $x \in X$

$$|\overline{d}^{\ (1)}(x) - \overline{d}^{\ (2)}(x)| \geq \epsilon$$

we have

$$\det(E(x)) \geq \text{Const} \cdot \epsilon$$

and B is bounded. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Lemma 4.4 *Given $A, B \in \mathcal{A}$ with $C(T)AC^{-1} = B$. There exists $\Gamma > 1$ such that for all $n \in \mathbb{N}$*

$$\Gamma^{-1} \cdot \|B^n(x)\| \leq \|A^n(x)\| \leq \Gamma \cdot \|B^n(x)\| \ .$$

Proof. We have $C(T^n)A^nC^{-1} = B^n$. The claim follows with

$$\Gamma = \||C\|| \cdot \||C^{-1}\|| .$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Proposition 4.5 *The following statements are equivalent:*
a) $A \in \mathcal{S}$.
b) The cocycle A admits a strict invariant cone bundle.
c) If $w^{(i)} \in W^{(i)}$ then there exist $\Gamma > 0, \alpha < 1$ such that for almost all $x \in X$ and all $n \in \mathbb{N}$

$$\begin{aligned}
|A^{-n}(x)w^{\ (1)}(x)| &\leq \Gamma\alpha^n|w^{\ (1)}(x)| \ , \\
|A^n(x)w^{\ (2)}(x)| &\leq \Gamma\alpha^n|w^{\ (2)}(x)| \ .
\end{aligned}$$

d) $\exists C \in \mathcal{A}$ such that $C(T)AC^{-1} = \text{Diag}(\gamma, \gamma^{-1})$ and $\gamma(x) \leq \alpha^{1/2} < 1$.

Proof.
b) $\Rightarrow$ a): Assume A admits the strict invariant cone bundle $C = [\overline{c}^{\ (1)}, \overline{c}^{\ (2)}]$ with

$$[A\overline{c}^1, A\overline{c}^2] \subset [c^{\ (1)} + \delta, c^{\ (2)} - \delta] \ .$$

By Lemma 4.3, there exists $E \in \mathcal{A}$ with $E\overline{c}^{(i)} = \overline{e}^{(i)}$, where $e^{(i)}$ is the standard basis. The cocycle $B = E(T)AE^{-1}$ has the strict invariant cone bundle

$$D = [\overline{d}^{\,(1)}(x), \overline{d}^{\,(2)}(x)] = [0, \pi/2] \,,$$

because E maps the cone bundle C into D and $A(C)$ into

$$A(D) = [A\overline{d}^{\,(1)}(x), A\overline{d}^{\,(2)}(x)] \subset [\overline{d}^{\,(1)}(x) + \delta |||E|||^{-1}, \overline{d}^{\,(2)}(x) - \delta |||E|||^{-1}] \,.$$

We have therefore

$$[B]_{ij} \geq \tan(\delta |||E|||^{-1}) |||B|||^{-1}$$

which shows that $A \in \mathcal{S}$.

$a) \Rightarrow b)$: Assume $A \in \mathcal{S}$. There $\exists F \in \mathcal{A}$ and $\epsilon > 0$ with $[F(T)AF^{-1}]_{ij} \geq \epsilon$. Because $B = F(T)AF^{-1}$ admits the strict invariant cone bundle

$$C(x) = [\overline{c}^{\,(1)}(x), \overline{c}^{\,(2)}(x)] = [0, \pi/2]$$

satisfying

$$[B\overline{c}^{\,(1)}(x), B\overline{c}^{\,(2)}(x)] \subset [\arctan(\epsilon |||B|||^{-1}), \pi/2 - \arctan(\epsilon |||B|||^{-1})] \,,$$

the cocycle A admits the strict invariant cone bundle $F^{-1}C(x)$ (Lemma 4.2).

$a), b) \Rightarrow c)$: Assume $A \in \mathcal{S}$. There $\exists F \in \mathcal{A}$ and $\epsilon > 0$ with $[F(T)AF^{-1}]_{ij} \geq \epsilon$. From Lemma 4.1, we have $\rho(B) \geq (1 + 2\epsilon^2)^{1/2} =: \beta$ and so $||B^n|| \geq \rho(B)^n \geq \beta^n$ and therefore $\lambda(A) \geq log(\beta) > 0$. According to Oseledec's theorem, there exists a splitting $\mathbb{R}^2 = W^{(1)} \oplus W^{(2)}$ which is co-invariant $AW^{(i)}(x) = W^{(i)}(Tx)$. We assume that $w^{(1)}(x) \in C(x)$ which means that $\overline{w}^{\,(1)}(x)$ is the expanding direction. Because A has a strict invariant cone bundle, there exists $\delta > 0$ such that

$$|\overline{w}^{\,(1)}(x) - \overline{w}^{\,(2)}(x)| \geq \delta \,.$$

From Lemma 4.3 we get $E \in \mathcal{A}$ such that

$$E(T)^{-1}BE = D = \mathrm{Diag}(\gamma, \gamma^{-1}) \,.$$

We have

$$||D^n(x)|| = \gamma^n(x) = \Pi_{i=0}^{n-1}\gamma(T^i x) \,.$$

From Lemma 4.4 follows that there exists $\Gamma_1 > 0$ such that

$$\gamma^n(x) = ||D^n(x)|| \geq \Gamma_1 ||B^n(x)|| \geq \Gamma_1\beta^n \,.$$

There exists $\Gamma_2 > 0$ such that

$$
\begin{aligned}
|A^{-n}(x)w^{(1)}(x)| &\leq \Gamma_2 |D^{-n}(x)e^{(1)}(x)| = \Gamma_2\gamma^{-n}(x) \leq \Gamma_1 \cdot \Gamma_2\beta^{-n} \,, \\
|A^n(x)w^{(2)}(x)| &\leq \Gamma_2 |D^n(x)e^{(2)}(x)| = \Gamma_2(\gamma^n)(x))^{-1} \leq \Gamma_2 \cdot \Gamma_1\beta^{-n} \,.
\end{aligned}
$$

The claim follows now with $\Gamma = \Gamma_1 \cdot \Gamma_2$ and $\alpha = \beta^{-1}$.

c) $\Rightarrow$ d): Let $w^{(i)}(x)$ be a unit vector in $W^{(i)}(x)$. Define for $x \in X$ the matrix $B(x)$ through

$$B(x)e^{(1)} = (\sum_{n=0}^{\infty} |A^{-n}(x)w^{(1)}(x)|\alpha^{-n/2})^{-1}w^{(1)}(x)$$

$$B(x)e^{(2)} = (\sum_{n=0}^{\infty} |A^{n}(x)w^{(2)}(x)|\alpha^{-n/2})^{-1}w^{(2)}(x).$$

Because we have $|A^{-n}(x)w^{(1)}(x)| \leq \alpha^n \Gamma$ and $|A^n(x)w^{(2)}(x)| \leq \alpha^n \Gamma$, both sums

$$\sum_{n=0}^{\infty} |A^{-n}(x)w^{(1)}(x)|\alpha^{-n/2})^{-1}, \ \sum_{n=0}^{\infty} |A^{n}(x)w^{(2)}(x)|\alpha^{-n/2})^{-1}$$

converge to positive limits. The cocycle $C(x) = B(x)/\det(B(x))$ can be diagonalised

$$D(x) = C(T(x))^{-1}A(x)C(x) = D(\gamma(x), \gamma^{-1}(x)) = \text{Diag}(\gamma, \gamma^{-1})$$

and we calculate like in the converse of Wojtkowsky's theorem

$$|\lambda(x)| = |D(x)e^{(1)}| = |B(T(x))^{-1}A(x)B(x)e^{(1)}| \leq \alpha^{1/2}.$$

d) $\Rightarrow$ a): Assume $C(T)AC^{-1} = \text{Diag}(\gamma, \gamma^{-1})$ is diagonal with $\gamma(x) \geq \alpha^{1/2}$. Define $\epsilon := (\alpha - \alpha^{-1})/2 > 0$. With the rotation R about the angle $-\pi/4$ one gets

$$R(T(x)) \circ \text{Diag}(\gamma(x), \gamma^{-1}(x)) \circ R^{-1}(x)$$
$$= \begin{pmatrix} (\gamma(x) - \gamma(x)^{-1})/2 & (\gamma(x) + \gamma(x)^{-1})/2 \\ (\gamma(x) + \gamma(x)^{-1})/2 & (\gamma(x) - \gamma(x)^{-1})/2 \end{pmatrix}$$

and $A \in \mathcal{S}$. $\qquad \square$

5 Analyticity of the Lyapunov exponent

The space

$$\mathcal{U} = L^{\infty}(X, \mathbf{P})$$

is a real-analytic Banach manifold. We will write in this paragraph for $\bar{u}(x) \in \mathbf{P}$ simply $u(x)$. Given $u \in \mathcal{U}$ lying in a cone bundle $C = [c^{(1)}, c^{(2)}]$. We can look at the cone bundle C As a neighborhood of u in $\mathcal{U}$. The mapping

$$C \to L^{\infty}(X), v \mapsto (v - u)$$

maps C into a neighborhood of 0 in the Banach space $L^{\infty}(X)$. A collection of cone bundles C together with such mappings gives an atlas of the Banach manifold $\mathcal{U}$.

We can redefine a cone bundle to be a nonempty convex simply connected closed set in the manifold $\mathcal{U}$.

Also the group $\mathcal{A} = L^\infty(X, SL(2, \mathbb{R}))$ is a real analytic Banach manifold in the Banach algebra $L^\infty(X, M(2, \mathbb{R}))$ and $\mathcal{A}$ is acting on $\mathcal{U}$ in a natural way: define

$$\psi : \mathcal{A} \times \mathcal{U} \to \mathcal{U}, (A, u) \to Au(x) = A(T^{-1}x)u(T^{-1}x) .$$

We can say that a cone bundle C is strictly invariant if A maps C into its interior. Oseledec's theorem can be restated in saying that for $A \in \mathcal{P}$ the mapping $\phi_A(\cdot) = \psi(A, \cdot)$ has exactly two fixed points $w^{(1)}(A), w^{(2)}(A)$. A part of Proposition 4.5 can be reformulated in saying that that $A \in \mathcal{S}$ if and only if there exists a nonempty convex simply connected set $C \subset \mathcal{U}$ which is mapped into its interior by ϕ_A.

A differentiable mapping ϕ on a Banach space is called *hyperbolic* if the derivative $d\phi$ is a linear hyperbolic operator. A fixed-point P of ϕ is called *hyperbolic*, if $d\phi(P)$ is hyperbolic. A fixed point is called *stable* if it is hyperbolic and if the spectrum is inside the unit disc. It is called *unstable*, if it is hyperbolic and if the spectrum is not intersecting the unit disc.

Lemma 5.1 *For $A \in \mathcal{S}$, the two fixed points $w^{(1)}, w^{(2)}$ of $\phi_A : \mathcal{U} \to \mathcal{U}$ are hyperbolic fixed points. One is stable, the other is unstable.*

Proof. If $F(T)AF^{-1} = B$, then the mappings ϕ_A and ϕ_B are conjugated. To see this we write $\phi_A = \overline{A}\tau^*$, where

$$\tau^* : \mathcal{U} \to \mathcal{U}, u \mapsto u(T^{-1})$$

and $\overline{A}u(x) = \overline{A(x)u(x)}$. This gives $\phi_B = \overline{F}\phi_A \overline{F}^{-1}$, because

$$\phi_B = \overline{B}\tau^* = \overline{F}\phi_A\overline{F}^{-1} = \overline{FA}\tau^*\overline{F}^{-1} .$$

From this fact follows that $d\phi_B(w^{(i)}) = dFd\phi_AdF^{-1}(Fw^{(i)})$ and $d\phi_A, d\phi_B$ have the same spectrum.

We apply this now to $A \in \mathcal{S}$ which is cohomologous to a diagonal $D = \text{Diag}(\gamma^{-1}, \gamma)$ with $\gamma(x) \leq \alpha < 1$. We can calculate the derivatives $d\phi_D : L^\infty(X) \to L^\infty(X)$ at the two fixed points $w^{(1)} = 0$ and $w^{(2)} = \pi/2$ as

$$\begin{aligned}
d\phi_D(0)f(x) &= \gamma(x)^2 f(T^{-1}x) , \\
d\phi_D(\pi/2)f(x) &= \gamma(x)^{-2} f(T^{-1}x) ,
\end{aligned}$$

because the diagonal cocycle D is acting on $\mathcal{V}$ as $u \to \arctan(\gamma^2 \cdot \tan(u))$ which has around $u = 0$ the linearisation $u \to \gamma^2 \cdot u$. From the fact that $\gamma(x) \leq \alpha < 1$ and

354

$\gamma^{-1} \geq \alpha^{-1} > 1$ follows that the spectrum of $d\phi_D(\pi/2)$ is located outside the disc with radius α^{-2} and the spectrum of $d\phi_D(0)$ is located inside the disc with radius α^2. $\qquad\square$

The above lemma can be reversed. The existence of two hyperbolic fixed points for ϕ_A implies also that $A \in \mathcal{S}$. We prove this later.

Proposition 5.2 *Given $A \in \mathcal{S}$ there exists a neighborhood $\mathcal{N}$ of A and neighborhoods $\mathcal{V}^{(i)}$ of $w^{(i)}(A)$ in $\mathcal{U}$ such that the mappings $w^{(i)} : \mathcal{N} \to \mathcal{V}^{(i)}$ are real analytic.*

Proof. Given $A \in \mathcal{S}$ and assume that A has the cone bundle C strict invariant. Take a neighborhood $\mathcal{N}$ of A such that a cocycle B in $\mathcal{N}$ has also the cone bundle C strictly invariant. The set $\mathcal{V} = \text{int}(C)$ is an open neighborhood of $w^{(1)}(A)$. We have a mapping $w : \mathcal{N} \to \mathcal{V}$ which assigns to a cocycle $B \in \mathcal{N}$ the fixed point of ϕ_B in $\mathcal{V}$: $\phi_A(w(A)) = w(A)$. The mapping

$$\psi : \mathcal{N} \times \mathcal{V} \to \mathcal{V}, (A, u) \mapsto \phi_A u$$

is real analytic. Because the spectrum of ϕ_A doesn't intersect the unit circle, the linear mapping

$$d\phi_A - \text{Id} : L^\infty(X) \to L^\infty(X)$$

is invertible. By the implicit function theorem, there exists a real analytic mapping $w : \mathcal{N} \to \mathcal{V}$ such that $\phi_A w(A) = w(A)$. This implies that the fixed point $w^{(1)} \in \mathcal{V}$ depends real analytically from A. The same can be shown for the other fixed point where one has to take the cone bundle $\overline{\mathcal{U} - C}$ which is strictly invariant for A^{-1}. $\square$

We give now the proof of Ruelle for the following theorem

Theorem 5.3 (Ruelle) *The mapping $\lambda : \mathcal{S} \to \mathbb{R}$ is real analytic.*

Proof. Given $A \in \mathcal{S}$ which has the fixed points $w^{(i)}$ in $\mathcal{U}$. Denote by $v(x)$ a unit vector such that $\overline{v}(x) = w^{(1)}(x)$ and with $w(x)$ a unit vector orthogonal to $w^{(2)}(x)$. We can write

$$\begin{aligned}
\lambda(A) &= \int \log|A(x)v(x)| \, dm(x) = n^{-1} \int \log|A^n(x)v(x)| \, dm(x) \\
&= \lim_{n \to \infty} n^{-1} \int \log\langle w(T^n x), A^n(x)v(x)\rangle \, dm(x) \, .
\end{aligned}$$

The last equation was obtained because

$$\langle w(T^n x), A^n(x)v(x)\rangle = |A^n(x)v(x)| \cdot \langle w(T^n x), v(T^n x)\rangle \, ,$$

where the scalar product on the right hand side is bounded away from zero. The formula

$$\lambda(A) = \lim_{n \to \infty} n^{-1} \int \log\langle w(T^n x), A^n(x)v(x)\rangle \, dm(x)$$

is still correct if we replace v and w by functions in $\mathcal{U}$ close enough to v and w. By definition we have $\langle w(T), Av \rangle = 0$. Therefore $\langle A^* w(T), v \rangle = 0$. The element $w \in \mathcal{U}$ is a fixed point of the cocycle A^* which is a cocycle over the dynamical system (X, T^{-1}, m).

We can calculate now the Fréchet derivative $d\lambda$ For $U \in L^\infty(X, M(2, \mathbf{R}))$, we get

$$
\begin{aligned}
d\lambda(A)U &= \lim_{n\to\infty} n^{-1} d(\int \log\langle w(T^n x), A^n(x)v(x)\rangle \, dm(x))U \\
&= \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \frac{\langle w(T^n x), A^{n-k} U A^{k-1}(x)v(x)\rangle}{\langle w(T^n x), A^n v(x)\rangle} \, dm(x) \\
&= \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \frac{\langle (A^*)^{n-k}(T^n x)w(T^n x), U A^{k-1}(x)v(x)\rangle}{\langle (A^*)^{n-k}(T^n x)w(T^n x), A^k v(x)\rangle} \, dm(x) \\
&= \int \frac{\langle w(Tx), Uv(x)\rangle}{\langle w(Tx), Av(x)\rangle} \, dm(x) \, .
\end{aligned}
$$

The mapping

$$
(A, U) \mapsto \frac{\langle w(T), Uv \rangle}{\langle w(T), Av \rangle} \in L^\infty(X)
$$

is real analytic in A and U because it is linear in U. The mappings $A \to w^{(i)}$ are real analytic. So, $A \to d\lambda(A)$ is real analytic and hence also the mapping $A \mapsto \lambda(A)$. $\square$

6 Illustration of the formula of Ruelle

We want to illustrate this formula of Ruelle for the derivative of the Lyapunov exponent by calculating the directional derivative along a special curve in $\mathcal{A}$.

Denote by $R(\beta)$ the constant cocycle in $\mathcal{O} = L^\infty(X, SO(2, \mathbf{R}))$ which assigns to all $x \in X$ the matrix belonging to a rotation about the angle β. For each $A \in \mathcal{A}$ we have a circle

$$
\beta \mapsto A(\beta) = R(\beta)A \, .
$$

We call this circle *Herman circle*. A result of Herman (see [Kni 1]) implies that the Lebesgue measure of values β with $\lambda(A(\beta)) > 0$ is larger than $1 - 1/\lambda(A)$. Roughly speaking: as bigger as the Lyapunov exponent is, as longer we stay in $\mathcal{P}$ when moving on the circle $\beta \mapsto R(\beta)A$. We assume now that $A \in \mathcal{S}$ and we want to calculate

$$
\frac{d}{d\beta}\lambda(A(\beta))
$$

at the parameter $\beta = 0$. With

$$
R(\alpha)A(\beta) = A(\beta) + \alpha U(\beta) + O(\alpha^2)
$$

we obtain

$$
U(\beta) = \lim_{\alpha\to 0} \alpha^{-1}(A(\alpha + \beta) - A(\beta)) = JA(\beta) \, ,
$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. With the above formula of Ruelle we have

$$\frac{d}{d\beta}\lambda(\beta) = \int \frac{\langle w(Tx), JAv(x)\rangle}{\langle w(Tx), Av(x)\rangle} \, dm(x) = \int \frac{\langle w(x), Jv(x)\rangle}{\langle w(x), v(x)\rangle} \, dm(x) \ .$$

If we denote with $\omega(x)$ the angle between $w^{(1)}(x)$ and $w^{(2)}(x)$, we have

$$\begin{aligned} \langle w(x), Jv(x)\rangle &= \cos(\omega(x)) \ , \\ \langle w(x), v(x)\rangle &= \sin(\omega(x)) \ , \end{aligned}$$

and so

$$\frac{d}{d\beta}\lambda(\beta) = \int \cot(\omega(x)) \, dm(x) \ . \tag{1}$$

This formula tells us that the Lyapunov exponent can change drastically if the stable and unstable direction fields are close together. We get also the following corollary.

Corollary 6.1 *The Herman circle $\beta \mapsto A(\beta)$ can not lie completely inside S. We always pass a region with zero Lyapunov exponents or a region with nonuniform hyperbolicity.*

Proof. We calculate from Formula 1 the second derivative

$$\frac{d^2}{d\beta^2}\lambda(\beta) = -\int \frac{d\omega}{d\beta} \cdot \sin(\omega)^{-2} \, dm$$

which is negative because $d\omega/d\beta \geq 2$. A periodic real-analytic function can not have everywhere a negative second derivative. □

As an example, we can calculate for a trigonal cocycle

$$A(x) = \begin{pmatrix} c & b(x) \\ 0 & c^{-1} \end{pmatrix}$$

the derivative $d/d\beta\lambda(A(\beta))$ at the point $\beta = 0$.

We have $w^{(1)}(x) = 0$ and $w^{(2)}(x) = \omega(x)$. Then

$$\cot(\omega(Tx)) = \frac{c \cdot \cos(\omega(x)) + b(x) \cdot \sin(\omega(x))}{c^{-1}\sin(\omega(x))} = c^2 \cot(\omega(x)) + b(x)c$$

and after integration

$$\frac{d}{d\beta}\lambda(A(\beta)) = \int_X \cot(\omega(x)) \, dm(x) = \frac{c}{1 - c^2} \int b \, dm \ .$$

The same formula is true for

$$A(x) = \begin{pmatrix} c & 0 \\ b(x) & c^{-1} \end{pmatrix} .$$

We apply this to the Standard mapping on the torus $\mathsf{T}^2 = \mathsf{R}^2/(2\pi\mathsf{Z}^2)$:

$$T_\gamma : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y + \gamma\sin(x) \\ y + \gamma\sin(x)) \end{pmatrix}$$

which has the Jacobian cocycle

$$A(x) = \begin{pmatrix} 1 + \gamma \cdot \cos(x) & 1 \\ \gamma \cdot \cos(x) & 1 \end{pmatrix} .$$

It is an open problem, whether there exists a parameter γ such that the Lyapunov exponent of $A = dT_\gamma$ over the dynamical system $(\mathsf{T}^2, T_\gamma, m)$ is positive. One measures numerically the value

$$\log(\frac{\gamma}{2}) .$$

On the Herman circle $A(\beta) = R(\beta)A$, there is the point

$$A(-\pi/4) = \begin{pmatrix} 2^{-1/2} & 0 \\ 2^{-1/2} + 2^{1/2}\gamma \cdot \cos(x) & 2^{1/2} \end{pmatrix}$$

lying in $\mathcal{S}$. As an application of the above formula we can calculate the derivative

$$\frac{d}{d\beta}A(-\pi/4) = \sqrt{2} .$$

A side remark. We take the opportunity and mention what Herman's result implies in the case of the Standard map. Given

$$A(x) = \begin{pmatrix} c(x) & b(x) \\ 0 & c^{-1}(x) \end{pmatrix} ,$$

Herman's estimate gives

$$\int \lambda(A(\beta))d\beta \geq \delta = \int_X \log\sqrt{(\frac{c + c^{-1}}{2})^2 + b^2} \, dm .$$

This implies that the measure of the set

$$\{\beta \in \mathsf{T} \mid \lambda(A(\beta)) > 0\}$$

can be estimated from below by $\delta/(\delta + \sqrt{2})$ because the Lyapunov exponent of $\lambda(A(\beta))$ can not get bigger then $\delta + \sqrt{2}$. For $\gamma \to \infty$, the measure of this set goes

to 1.

We add an other application of formula 1. Consider the dynamical system $(X, T, m) = (T^1, x \mapsto x + \alpha, dx)$ and the cocycle

$$A(x) = R(x)\text{Diag}(c, c^{-1}),$$

where $R(x)$ is the the rotation about the angle x and c is a constant $c > 2$. The above estimate of Herman gives

$$\int \lambda(A(\beta)) d\beta \geq \log(\frac{c + c^{-1}}{2}).$$

On the other hand

$$\lambda(A) = \lambda(A(T^n)) = \lambda(R(n \cdot \alpha)A)$$

implies that the mapping $\beta \mapsto \lambda(\beta)$ is constant. Because the second derivative of $\lambda(A(\beta))$ is always negative, whenever $A \in \mathcal{S}$, we conclude that the Herman circle $A(\beta)$ lies entirely in $\mathcal{P} \setminus \mathcal{S}$.

7 Relation between Lyapunov exponents and rotation number

A rotation number for $SL(2, \mathbb{R})$ cocycles is a mapping $\rho : \mathcal{A} \to \mathbb{R}$ which measures how much a vector v rotates in average under the evolution $n \mapsto A^n(v)$. Because a canonical definition of a rotation number is in general not possible, one is forced to look at cocycles with values in the universal covering group $\widetilde{SL(2, \mathbb{R})}$ of $SL(2, \mathbb{R})$. A definition of a rotation number in this case has been given by [Rue 85]. We will review here his definition and the properties.

For $\tilde{A} \in \widetilde{SL(2, \mathbb{R})}$ we denote with A its projection onto $SL(2, \mathbb{R})$. We can make polar decomposition of $\tilde{A}$ by

$$\tilde{A} = \tilde{R}(\phi(A)) \circ |A|,$$

where $|A| = (A \circ A^*)^{1/2}$ and $\tilde{R}(\phi)$ is in the universal covering group of $SO(2, \mathbb{R})$. We can think of $\tilde{R}(\phi)$ as a rotation about $\phi \in \mathbb{R}$. For two elements $\tilde{A}, \tilde{B}$ of $\widetilde{SL(2, \mathbb{R})}$ one has

$$|\phi(\tilde{A} \circ \tilde{B}) - \phi(\tilde{A}) - \phi(\tilde{B})| < \pi \qquad (*).$$

Proof. One can write

$$\begin{aligned}
\tilde{A} \circ \tilde{B} &= \tilde{R}(\phi(\tilde{A}) + \phi(\tilde{B})) \circ \tilde{R}(-\phi(\tilde{B})) \circ |A| \circ \tilde{R}(\phi(\tilde{B})) \circ |B| \\
&= \tilde{R}(\phi(\tilde{A}) + \phi(\tilde{B})) \circ P \circ Q,
\end{aligned}$$

where

$$P = \tilde{R}(-\phi(\tilde{B})) \circ |A| \circ \tilde{R}(\phi(\tilde{B}))$$

359

and $Q = |B|$ are positive selfadjoint. The claim follows from the fact, that $|\phi(PQ)| < \pi$. We take on $SL(\tilde{2}, \mathbf{R})$ the "norm"

$$\|\tilde{A}\| = \|A\| + |\phi(A)|$$

which allows to give the distance $\|\tilde{A} - \tilde{B}\|$. Let $\tilde{\mathcal{A}}$ be the space $L^\infty(X, SL(\tilde{2}, \mathbf{R}))$ of all measurable mappings from X to $SL(\tilde{2}, \mathbf{R})$ with the topology

$$\||\tilde{A} - \tilde{B}\|| = |\,\|\tilde{A} - \tilde{B}\|\,|_\infty\ .$$

As in $\mathcal{A}$ we use the notation

$$\tilde{A}^n(x) = \tilde{A}(T^{n-1})(x) \circ \ldots \circ \tilde{A}(x)\ .$$

Theorem 7.1 (Ruelle) *The limit*

$$\rho(\tilde{A}, x) = \lim_{n \to \infty} n^{-1}\phi(\tilde{A}^n(x))$$

exists for almost all $x \in X$ and is T invariant. The rotation number

$$\rho(A) = \int_X \rho(\tilde{A}, x)\, dm$$

depends continuously on $\tilde{A}$.

Proof. Fix $m \in \mathbf{N}$ and write a number $n \in \mathcal{N}$ as $n = km + r$, where $k, m > 0$ and $0 \le r < m$. We get from $(*)$

$$(km)^{-1}|\phi(\tilde{A}^n) - \phi(\tilde{A}^r(T^{km}x)) - \sum_{i=0}^{k-1}\phi(\tilde{A}^m(T^{mi}x))| < (km)^{-1}k\pi = \pi/m\ .$$

Birkhoff's ergodic theorem implies that

$$\lim_{k \to \infty}(km)^{-1}\sum_{i=0}^{k-1}\phi(\tilde{A}^m(T^{mi}x))$$

exists for almost all $x \in X$. Because m can be chosen arbitrarily big and for $\phi \in L^\infty(X)$

$$\lim_{n \to \infty}(km)^{-1}\tilde{A}^r(T^{km}x) = 0\ ,$$

we have

$$\lim_{n \to \infty}n^{-1}\phi(\tilde{A}^n(x)) = \lim_{k \to \infty}(km)^{-1}\sum_{i=0}^{k-1}\phi(\tilde{A}^m(T^{mi}x))\ .$$

Clearly $\rho(\tilde{A}, x) = \rho(\tilde{A}, Tx)$.
We want to see now that the mapping

$$\rho : \tilde{\mathcal{A}} \to \mathbf{R}$$

360

is continuous. We know that for $n \to \infty$

$$\rho_n(\tilde{A}) = \int n^{-1} \phi(\tilde{A}^n) dm \to \rho(\tilde{A}) \ .$$

From $(*)$ we have

$$|\rho_{2n}(\tilde{A}) - \rho_n(\tilde{A}) - \rho_n(\tilde{A})| \ \leq n^{-1}\pi$$

and so

$$|\rho_{2^k n}(\tilde{A}) - \rho_{2^{k-1} n}(\tilde{A})| \leq 2^{-(k-1)} n^{-1} \pi \ .$$

Summing up gives

$$|\rho(\tilde{A}) - \rho_n(\tilde{A})| \leq 4 n^{-1} \pi \ .$$

The sequence ρ_n of continuous functions on $\tilde{\mathcal{A}}$ is uniformly convergent. Therefore ρ is continuous. $\qquad\qquad\square$

There is a connection with Herman's rotation number [Her 83] which is defined if we have a dynamical system (X, T, m) such that X is compact metric and T a homeomorphism which leaves invariant a Borel probability measure m. If X is connected, $A \in C(X, SL(2, \mathbf{R}))$ is homotopic to the identity and one can define a rotation number which is unique up to 2π. ([Her 83]). If A is homotopic to the identity, one can define a continuous lifting $\tilde{A} : X \to \widetilde{SL}(2, \mathbf{R})$. If $\tilde{A}_1$ and $\tilde{A}_2$ are two liftings of A then

$$\tilde{A}_1 \tilde{A}_2^{-1} = \tilde{R}(\phi(x)) \ ,$$

where $\phi(x)$ is a continuous mapping $X \to \mathbf{Z}$. Because X is connected, ϕ must be constant $\phi = 2\pi k$. This implies that

$$\rho(\tilde{A}_1) = \rho(\tilde{A}_2) + 2\pi k \ .$$

Define

$$\tilde{\mathcal{S}} = \{ \tilde{A} \mid A \in \mathcal{S} \} \ .$$

Lemma 7.2 *The rotation number ρ is constant on every connected set in $\tilde{\mathcal{S}}$.*

Proof. It is enough to prove that ρ is locally constant in $\tilde{\mathcal{S}}$
The projection A of $\tilde{A}$ admits a strict invariant cone bundle $C = [c^{(1)}, c^{(2)}]$ and there exists a neighborhood $\mathcal{N}$ of A which has the same cone bundle strictly invariant. Take a neighborhood $\tilde{\mathcal{N}}$ of $\tilde{A}$ such that

$$\tilde{\mathcal{N}} \subset \{ \tilde{A} \in \tilde{\mathcal{A}} \mid A \in \mathcal{N} \} \ .$$

Take $\tilde{B} \in \tilde{\mathcal{N}}$. Because

$$|\phi(\tilde{A}^n(x)) - \phi(\tilde{B}^n(x)| \leq |c^{(2)}(x)) - c^{(1)}(x)| \leq \pi \ ,$$

we get $\rho(\tilde{A}) = \rho(\tilde{B})$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We want to reverse now the above lemma showing that the set $\mathcal{S}$ is characterized by the fact that the rotation number is there locally constant along the Herman circle $\beta \mapsto A(\beta)$ at $\beta = 0$.

Given $\tilde{A} \in \tilde{\mathcal{A}}$ we look at the function

$$\beta \to \rho(\tilde{R}(\beta)\tilde{A}) \ .$$

In contrary to the rotation number ρ, the difference

$$\rho_A(\beta) = \rho(\tilde{R}(\beta)\tilde{A}) - \rho(\tilde{A})$$

is *independent of the chosen lift.*
To every $A \in \mathcal{A}$ is like this assigned a continuous circle mapping

$$\rho_A : \mathsf{T} \to \mathsf{T}$$

which is not invertible in general because we have just seen that it is locally constant when $A(\beta) \in \mathcal{S}$.

Proposition 7.3 $A \in \mathcal{S}$ *if and only if there is an open interval I containing 0 such that $\rho_A(\beta) = 0$ for $\beta \in I$.*

We need some preparation for the proof. In the same way as A acts on P, a lift $\tilde{\mathcal{A}}$ of A acts on the covering $\tilde{\mathsf{P}}$ of P which is isomorphic to R. We write $\tilde{u}$ for an element in $\tilde{\mathsf{P}}$. The fact that $\tilde{\mathsf{P}}$ allows an ordering allows also an ordering of $\tilde{\mathcal{A}}$:

$$\tilde{A} < \tilde{B} \Leftrightarrow \exists \epsilon > 0, \ \tilde{A}(x)\tilde{u} \leq \tilde{B}(x)\tilde{u} - \epsilon, \ a.e.$$

for all $\tilde{u} \in \tilde{\mathsf{P}}$.
The next Lemma shows that the rotation number

$$\rho : \tilde{\mathcal{A}} \to \mathsf{R}$$

is continuous and monotone.

Lemma 7.4 *Assume $\tilde{A} < \tilde{B} < \tilde{C}$.*
a) There exists a neighborhood $\tilde{\mathcal{N}}$ of $\tilde{B}$ with

$$\tilde{A} < \tilde{\mathcal{N}} < \tilde{C}$$

b)

$$\rho(\tilde{A}) \leq \rho(\tilde{B}) \leq \rho(\tilde{C}) \ .$$

c) If $\rho(\tilde{A}) = \rho(\tilde{C})$, there exists a neighborhood $\tilde{\mathcal{N}}$ of $\tilde{B}$ such that for $\tilde{B}_1 \in \tilde{\mathcal{N}}$ $\rho(\tilde{A}) = \rho(\tilde{\mathcal{N}}) = \rho(\tilde{C})$.

Proof.

a) If $\||\tilde{B}_1 - \tilde{B}\||| \le \epsilon$ then for every $\tilde{u} \in \tilde{P}$ and almost all $x \in X$ $|\tilde{B}_1(x)\tilde{u} - \tilde{B}(x)\tilde{u}| \le \epsilon$.

b) From $\tilde{A} < \tilde{B} < \tilde{C}$ follows $\tilde{A}^n < \tilde{B}^n < \tilde{C}^n$ and so $\phi(\tilde{A}^n) - \pi < \phi(\tilde{B}^n) < \tilde{B}^n) + \pi$.

c) Using $a)$, there exists a neighborhood $\tilde{\mathcal{N}}$ of $\tilde{B}$ such that for $\tilde{B}_1 \in \tilde{\mathcal{N}}$

$$\tilde{A} < \tilde{B}_1 < \tilde{C} \ .$$

From $b)$ follows

$$\rho(\tilde{A}) \le \rho(\tilde{B}_1) \le \rho(\tilde{C}) \ .$$

The claim follows from the assumption

$$\rho(\tilde{A}) = \rho(\tilde{C}) \ .$$

$\square$

Proof of the proposition. We have already seen the $\Rightarrow$ direction. Assume now that $\rho_A(\beta) = 0$ for $\beta \in I$, where $I = [-\epsilon, \epsilon]$. The aim is to construct a strictly invariant cone bundle.

Define

$$\begin{aligned} \tilde{A}_1 &= \tilde{R}(-\epsilon/2)\tilde{A} \ , \\ \tilde{A}_2 &= \tilde{R}(\epsilon/2)\tilde{A} \ . \end{aligned}$$

For $\delta > 0$, we denote with $\mathcal{N}_i$ the δ balls around $\tilde{A}_i$. Because $\mathcal{P}$ is dense in $\mathcal{A}$ (see [Kni 1]), also $\tilde{\mathcal{P}}$ is dense in $\tilde{\mathcal{A}}$ and there exists $\tilde{B}_i \in \tilde{\mathcal{N}}_i$ with $\tilde{B}_i \in \tilde{\mathcal{N}}_i \cap \tilde{\mathcal{P}}$.

Because of the above lemma c) we have for $\delta > 0$ small enough $\rho(\tilde{B}_i) = \rho(\tilde{A}_i) = 0$. Denote by $\overline{w}_1$ one of the two co-invariant direction fields of $\tilde{B}_1$ and look at the cone field

$$E = [\overline{w}_1, \overline{w}_1 + \epsilon - \delta] \ .$$

We claim that every $T \times B_2$ invariant probability measure μ on $X \times P$ satisfies $\mu(E) = 0$. To prove this we use that

$$\tilde{B}_1(x)\tilde{u}_1(x) = \tilde{u}_1(x)$$

while

$$\tilde{B}_1(x)\tilde{u}_1(x) \ge \tilde{u}_1(x) + \epsilon - \delta$$

so that

$$\rho(\tilde{B}_2) \ge \tilde{B}_1 + \mu(E)2\pi \ .$$

Choose a generic point $x \in X$ for which the rotation number $\rho(\tilde{A}, x)$ exists and take a point $\tilde{u}_1(x) \in \tilde{P}$ on the B_1-co-invariant direction field. We compare the orbits of $(x, \tilde{u})$ under both skew-products

$$T \times \tilde{B}_1, \ T \times \tilde{B}_2 \ .$$

If after n steps, the orbit of $(x, \tilde{u})$ of $(T \times \tilde{B}_1)^n$ has hit $k(n)$ times the set E then we must have $\phi(\tilde{B}_2^n) \geq \phi(\tilde{B}_1^n) + (k-1)2\pi$. According to Birkhoff's ergodic theorem the sequence $k(n)/n$ converges for $n \to \infty$ to $\mu(E)$ and we get

$$\rho(\tilde{B}_2) \geq \tilde{B}_1 + \mu(E)2\pi .$$

Especially for the $T \times \tilde{B}_2$ invariant measure μ_2 with support on the direction field $\overline{w}_2$ of B_2, we have $\mu_2(E) = 0$. The sector bundle $C = [w_1, w_2]$ is strictly invariant for $B := R(\epsilon/2)B_1$ and it is mapped into the sector bundle

$$[\overline{w}_1 + \epsilon/4, \overline{w}_2 - \epsilon/4]$$

if δ is small enough. Because $|||A - B||| \leq \delta$, also the sector bundle $[\overline{w}_1, \overline{w}_2]$ is mapped into $[\overline{w}_1 + \epsilon/8, \overline{w}_2 - \epsilon/8]$ by A for δ small enough. This shows $A \in \mathcal{S}$. $\quad\square$

8 Overview: spectra of cocycles

Denote by $\mathcal{X}$ the crossed product of $L^\infty(X, M(2, \mathbb{C}))$ with the dynamical system. Elements in $\mathcal{X}$ have different kind of spectra. First of all, they have the spectrum as elements in the C^* algebra $\mathcal{X}$. There is a representation of $\mathcal{X}$ in $\mathcal{B}(L^2(X, \mathbb{C}^2))$ defined by

$$Ku = \sum_n K_n u(T^n)$$

and a representation of $\mathcal{X}$ in $\mathcal{B}(l^2(\mathbb{C}^2))$ defined for almost all $x \in X$ given by

$$(K(x)u)_n = \sum_n K_{n-m}(T^m x)u_n .$$

Both representations give spectra and if T is ergodic, there exists a set Σ such that the spectrum of $K(x)$ is Σ for almost all $x \in X$ (see the proof in [Cyc 87]). In general, when the dynamical system is no more ergodic define

$$\Sigma = \{z \in \mathbb{C} \mid \exists Y_z, \ m(Y_z) > 0, \ \text{such that} \ \forall x \in Y_z \ z \in \sigma(K(x))\} .$$

We call Σ the *individual spectrum* of K. It is in general different from the spectrum of K as an element of the C^* algebra $\mathcal{X}$ or from the spectrum of K as an operator on $L^2(X, \mathbb{C}^2)$.

Interesting special operators in $\mathcal{X}$ are random Jacobi operators on the strip $L = A\tau + (A\tau)^* + B$ or cocycles $A\tau^*$ which are also called *weighted composition operators, weighted translation operators* or *transfer operators*. An important special case is $A = 1$, when $K = \tau^*$ is Koopman's operator for the dynamical system (X, T, m).

Cocycles $A\tau^*$ are discrete versions of linear differential operators $d/dt - a(t)$, where a spectral theory has been developed by Sacker, Sell [Sac 78], (see also [Joh 87]). We will consider here operators defined by a function $A \in L^\infty(X, M(2, \mathbb{R}))$ or a function $A \in L^\infty(X, SL(2, \mathbb{R}))$.

In the case of $A \in L^\infty(X, SL(2, \mathbb{R}))$, one can define the following other type of spectra for cocycles.

- **The Sacker-Sell spectrum** is defined by

$$\{z \in \mathbb{C} \mid |z| \cdot A \text{ has exponential dychotomy}\} \ .$$

Exponential dichotomy will be defined later and means roughly speaking that the operator acting on $L^2(X, \mathbb{C}^2)$ has a stable and/or unstable fiber bundle invariant.

- We define a **Herman spectrum** by

$$\left\{z \in \mathbb{C} \mid A(z) = \begin{pmatrix} \frac{z+z^{-1}}{2} & \frac{z-z^{-1}}{2i} \\ -\frac{z-z^{-1}}{2i} & \frac{z+z^{-1}}{2} \end{pmatrix} A \text{ is hyperbolic}\right\} \ .$$

Parameterized cocycles with $|z| = 1$ have been considered by Herman [Her 83] who mentioned there that the Lyapunov exponent can be written with an *abstract Thouless formula* as

$$\lambda(z) = \int_{\mathbb{C}} \log(|z - z'| \ dk(z') + g(z)$$

with a harmonic function g and a measure dk having support on the set, where $A(z)$ is not uniformly hyperbolic. Having in mind the Schrödinger case, where the support of the measure (density of states) is the spectrum of the operator, it is natural to think of the support of dk as a spectrum also.

- We define a **Schrödinger spectrum** by

$$\left\{z \in \mathbb{C} \mid \forall C, \ [C(T)AC^{-1}]_{22} = 0 \Rightarrow C(T)AC^{-1} + \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \text{ is hyperbolic}\right\} \ .$$

This definition makes sense, when the dynamical system (X, T, m) is aperiodic, because there exists then always $C \in L^\infty(X, SL(2, \mathbb{R}))$ with $[C(T)AC^{-1}]_{22} = 0$. In other words, the Schrödinger spectrum of a cocycle is just the set of points z in the complex plane such that a conjugation of the cocycle is a transfer cocycle for a discrete Schrödinger operator having z in the spectrum.

The Schrödinger and the Sacker-Sell spectrum are invariants of the cocycle because conjugation doesn't change it. On the other hand, the Herman spectrum, which is lying on the unit circle for real cocycles is not an invariant of the conjugacy class.

We will consider here only the Sacker Sell and not the Herman spectrum and not the Schrödinger spectrum. The aim is to show that the Sacker-Sell spectrum is the same as the spectrum of K as an operator on $L^2(X, \mathbb{C}^2)$.

For more information on cocycles treated as operators, we refer to [Gro 90] and the extensive work in [Lat 91].

9 The individual spectrum and Lyapunov exponents

Define
$$\mathcal{W} = \{K \in \mathcal{X} \mid K = A\tau^*\} .$$

There is a simple relation between the individual spectrum of $K = A\tau$ and the Lyapunov exponent of A:

Proposition 9.1 *Assume (X, T, m) to be ergodic. Given $A \in \mathcal{X}, K = A\tau$, where $A \in L^\infty(X, M(2, R))$. Then the Lyapunov exponent of A, $\lambda(A)$ is related to the spectral radius of $K(x)$ through*

$$e^{\lambda(A)} = \mathrm{rspec}(K(x))$$

for almost all $x \in X$.

Proof. We observe $K^n(x) = A^n(x)\tau^n$. Furthermore, we have the definitions

$$\log(\mathrm{rspec}(K(x))) = \lim_{n \to \infty} n^{-1} \log \|K^n(x)\| = \lim_{n \to \infty} n^{-1} \log \|A^n(x)\| = \lambda(A) .$$

$\square$

This proposition tells us, that the problem of Lyapunov exponents is a spectral problem. In the theory of random discrete Schrödinger operators, the Thouless formula gives another, deeper relation between Lyapunov exponents and the density of states.

In the finite case $|X| < \infty$, the spectrum of K as an operator $K = A\tau^*$ on the finite dimensional space $L^2(X)$ is a discrete point spectrum which can be calculated explicitly:

Proposition 9.2 *Assume $|X| = N$ is finite and T is a cyclic permutation of N elements in X. Every $K = A\tau^*$ has then pure point spectrum $\mathrm{spec}(K) = \{\lambda_i^{1/N}\}$, where λ_i are the eigenvalues of A^N.*

Proof. Denote by $u_i \in \mathbf{C}^2$ the eigenvector to the eigenvalue λ_i of A^N, If μ_i is one of the N roots of $\lambda_1^{1/N}$ then $n \mapsto v_i(n) = \mu_i^{n-1} A^{n-1} u_i$ is an eigenvector of K to the eigenvalue μ_i. $\qquad\qquad\square$

Two elements K, L in $\mathcal{X}$ are *conjugated*, if there exists a $M \in \mathcal{X}$, such that $K = MLM^{-1}$. In the case of cocycles $K = A\tau^*, L = B\tau^* \in \mathcal{W}$, a special conjugation is defined, when M is only a multiplication operator $M = C\tau^0$. We say then, that $K = M^{-1}LM, L$ are *cohomologous*. The relation is

$$A = C^{-1}BC(T^{-1}) .$$

It follows, that cohomologous cocycles have the same spectrum and the same individual spectrum.

The investigation of the spectra of cocycles is a generalization of the spectrum of the unitary operator τ, which plays an important tool in ergodic theory.

What is the individual spectrum of a cocycle? Let us look first at the simplest case when the cocycle is trivial $L = \tau^*$. It can be seen (with help of a Fourier transformation) that the individual spectrum is then the unit circle: the operator τ is acting on $l^2(\mathbf{C}^2)$ and is a product of operators τ acting on $l^2(X, \mathbf{C})$. The later operator is the shift

$$(\tau(x)u)_n = u_{n-1} .$$

The Fourier transform $F : u \mapsto \hat{u} \in L^2(\mathsf{T})$ diagonalises the operator $\tau(x)$

$$F\tau(x)F^{-1}\hat{u}(s) = e^{is}\hat{u}(s) .$$

We see that in this case $L = \tau$, the individual spectrum does not contain any information about the dynamical system. In contrary to the operator $\tau(x)$, the operator τ acting on $L^2(X)$ can have interesting type of spectra.

The *approximate point spectrum* called $\sigma_{ap}(L(x))$ of $L(x)$ is defined as

$$\{\lambda \in \mathbf{C} \mid \exists u_n \in l^2(Z), \ |u_n| = 1, |L(x)u_n - \lambda u_n| \to 0\} .$$

Because the boundary of the spectrum is always contained in the approximate point spectrum, it is never empty.

Proposition 9.3 *The individual spectrum of a cocycle is rotational symmetric.*

Proof. The individual spectrum of the unitary operator τ is the unit circle and is the same than the approximate point spectrum. It is enough to prove, that for every $\lambda \in \sigma_{ap}(\tau(x))$ and $\mu \in \sigma_{ap}(L(x))$, the complex number $\lambda \cdot \mu$ is in $\sigma_{ap}(L(x))$. Let u_n, v_n

be two sequences in $l^2(Z^d)$ such that $|L(x)u_n - \lambda u_n| \to 0$ and $|\tau(x)v_n - \mu v_n| \to 0$. We have

$$
\begin{aligned}
|L(x)u_n v_n - \lambda \mu u_n v_n| &= |A(x)u_{n+1}v_{n+1} - \lambda \mu u_n v_n| \\
&\leq |A(x)u_{n+1}v_{n+1} - \lambda u_n \mu v_{n+1}| + |\lambda u_n \mu v_{n+1} - \lambda u_n \mu v_n| \\
&= |Au_{n+1} - \lambda u_n| + |\lambda| \cdot |v_{n+1} - \mu v_n|
\end{aligned}
$$

and this goes to 0 for $n \to \infty$. Therefore $\lambda \cdot \mu$ is also in the approximate point spectrum of $L(x)$. $\qquad\square$

We conclude that the individual spectrum is determined by its radial component.

10　The Sacker-Sell spectrum

There is a discrete version of the Sacker-Sell spectrum for cocycles. We will show here that the Sacker-Sell spectrum is the same as the spectrum of the operator acting on $L^2(X, \mathbf{C}^2)$. There is something to prove because the definition of the Sacker-Sell spectrum was adapted to the original definition of Sacker-Sell in the case of cocycles over flows instead of mappings. What we will have to prove essentially is that hyperbolicity=exponential dichotomy.

We say, a cocycle $A\tau^\sim \in \mathcal{W}$ has *exponential dichotomy* , if $L^2(X, \mathbf{C}^2)$ is the direct product of two measurable sub-bundles $\mathcal{H}^+ \oplus \mathcal{H}^-$ satisfying: for $w^\pm \in H^\pm$, there $\exists \Gamma > 0, \alpha < 1$ such that for almost all $x \in X$ and all $n \in \mathbb{N}$

$$
\begin{aligned}
|A^{-n}(x)w^+(x)| &\leq \Gamma \alpha^n |w^+(x)| , \\
|A^n(x)w^-(x)| &\leq \Gamma \alpha^n |w^-(x)| .
\end{aligned}
$$

Call $\mathcal{D} \subset \mathcal{W}$ the set of cocycles having exponential dichotomy.

For the real Banach manifold $\mathcal{A}$ of operators $K\tau$ satisfying $K \in L^\infty(X, SL(2, \mathbf{R}))$ is defined the open set

$$
\mathcal{S} = \{A \in \mathcal{A} \mid \exists C \in \mathcal{A}, \exists \epsilon > 0 \; [C(T)AC^{-1}]_{ij} \geq \epsilon\} .
$$

We have shown already that every $A \in \mathcal{S}$ is conjugated to a diagonal cocycle: there exists $C \in \mathcal{A}, \alpha < 1$ with

$$
C(T)AC^{-1} = \text{Diag}(\gamma, \gamma^{-1})
$$

and $\gamma(x) \leq \alpha < 1$. We proved also that $\mathcal{S} = \mathcal{D} \cap \mathcal{A}$.

For complex cocycles $A \in \mathcal{A} = L^\infty(X, SL(2, \mathbb{C}))$, we define

$$\mathcal{S} = \{A \in \mathcal{A} \mid \exists C \in \mathcal{A}, \alpha < 1 \text{ with } C(T)AC^{-1} = \mathrm{Diag}(\gamma, \gamma^{-1}) \text{ and } |\gamma(x)| \leq \alpha < 1 .$$

For complex cocycles $A \in \mathcal{M} = L^\infty(X, M(2, \mathbb{C}))$, we define $\mathcal{S}$ to be the union of

$$
\begin{aligned}
\mathcal{S}^0 &= \{A \in \mathcal{A} \mid \exists C \in \mathcal{A}, \exists \alpha < 1, C(T)AC^{-1} = \mathrm{Diag}(\gamma, \gamma^{-1}), |\gamma(x)| \leq \alpha < 1\} \\
\mathcal{S}^+ &= \{A \in \mathcal{M} \mid \exists C \in \mathcal{A}, \exists \alpha < 1, \|C(T)AC^{-1}(x)\| \leq \gamma(x), |\gamma(x)| \leq \alpha < 1\} \\
\mathcal{S}^- &= \{A \in \mathcal{M} \mid \exists C \in \mathcal{A}, \exists \alpha < 1, \|C(T)AC^{-1}(x)\| \geq \gamma^{-1}(x) |\gamma(x)| \leq \alpha < 1\}.
\end{aligned}
$$

Lemma 10.1 $\mathcal{S} = \mathcal{D}$.

Proof. Every $A \in \mathcal{S}$ has exponential dichotomy.
Less trivial is the converse direction, namely that exponential dichotomy implies that $A \in \mathcal{S}$. The idea to the proof of this is essentially due to J.Mather [Mat 68] who characterized Anosov diffeomorphisms.

If A has exponential dichotomy on $L^2(X, \mathbb{C}^2)$, then there exists a splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ which is invariant under the operator A and the spectral radius of A restricted to $\mathcal{H}^+$ is smaller than 1. Also the spectral radius of A^{-1} restricted to $\mathcal{H}^-$ is smaller than 1. (Both $\mathcal{H}^+$ or $\mathcal{H}^-$ can also be trivial.)

Assume first that the splitting is not trivial. It follows, that the two Lyapunov exponents are different. According to the multiplicative ergodic theorem, there exists another invariant splitting $\mathcal{H} = \mathcal{V}^+ \oplus \mathcal{V}^-$ into two measurable sub-bundles. We show that $\mathcal{V}^\pm = \mathcal{H}^\pm$. Given $v \in \mathcal{H}^+ \subset L^\infty(X, \mathbb{C}^2)$. For each $x \in X$ $(A^n v)(x)$ is approaching $v^+(x)$ where $v^+ \in \mathcal{V}^+$. This implies that $v(x) = v^+(x)$ for all x and so $v \in \mathcal{V}^+$. We have shown that $\mathcal{H}^+ \subset \mathcal{V}^+$. In the same way, also $\mathcal{H}^- \subset \mathcal{V}^-$. From $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \subset \mathcal{V}^+ \oplus \mathcal{V}^- = \mathcal{H}$ we get $\mathcal{H}^\pm = \mathcal{V}^\pm$. For $w^\pm \in W^\pm$, there $\exists \Gamma > 0, \alpha < 1$ such that for almost all $x \in X$ and all $n \in \mathbb{N}$

$$
\begin{aligned}
|A^{-n}(x)w^+(x)| &\leq \Gamma \alpha^n |w^+(x)| , \\
|A^n(x)w^-(x)| &\leq \Gamma \alpha^n |w^-(x)| .
\end{aligned}
$$

Let $w^\pm(x)$ be a unit vector in $W^\pm(x)$. Define for $x \in X$ the matrix $B(x)$ through

$$
\begin{aligned}
B(x)e^+ &= (\sum_{n=0}^\infty |A^{-n}(x)w^+(x)|\alpha^{-n/2})^{-1}e^+(x) , \\
B(x)e^- &= (\sum_{n=0}^\infty |A^n(x)w^-(x)|\alpha^{-n/2})^{-1}e^-(x) .
\end{aligned}
$$

Because $|A^{-n}(x)w^+(x)| \leq \alpha^n \Gamma$ and $|A^n(x)w^-(x)| \leq \alpha^n \Gamma$, both sums

$$
\begin{aligned}
(\sum_{n=0}^\infty & |A^{-n}(x)w^+(x)|\alpha^{-n/2})^{-1} \\
(\sum_{n=0}^\infty & |A^n(x)w^-(x)|\alpha^{-n/2})^{-1}
\end{aligned}
$$

converge and the limit is positive. With $C(x) = B(x)/\det(B(x))$ we can do the diagonalisation $D(x) = C(T(x))^{-1}A(x)C(x) = D(\gamma(x), \gamma^{-1}(x)) = \text{Diag}(\gamma, \gamma^{-1})$ and get $|\lambda(x)| = |D(x)e^{(1)}| = |B(T(x))^{-1}A(x)B(x)e^{(1)}| \leq \alpha^{1/2}$.

In the case when either $\mathcal{H}^+$ or $\mathcal{H}^-$ is the whole space fiber bundle, we proceed similarly. Assume that $\mathcal{H}^+ = L^2(x, \mathbb{C}^2)$ (the other case is parallel). Define for $x \in X$ the matrix $B(x)$ through

$$
\begin{aligned}
B(x)e^+ &= (\sum_{n=0}^{\infty} |A^{-n}(x)e^+(x)|\alpha^{-n/2})^{-1}e^+(x) \\
B(x)e^- &= (\sum_{n=0}^{\infty} |A^{-n}(x)e^-(x)|\alpha^{-n/2})^{-1}e^-(x) \ .
\end{aligned}
$$

Because $|A^{-n}(x)e^{\pm}(x)| \leq \alpha^n\Gamma$, both sums converge and the limit is positive. With $C(x) = B(x)/\det(B(x))$ we have $C(T(x))^{-1}A(x)C(x) = D(x)\gamma(x)$ with $||D(x)|| \leq 1$ and $|\lambda(x)| \leq \alpha^{1/2}$. $\qquad\qquad\Box$

$A \in \mathcal{W}$ is called *hyperbolic*, if the spectrum of A as an operator on $L^2(X, \mathbb{C}^2)$ does not intersect the circle

$$\{z \in \mathbb{C} \mid |z| = 1\}.$$

Corollary 10.2 $A \in \mathcal{W}$ *is hyperbolic as an operator on* $L^2(X, \mathbb{C}^2)$ *if and only if A has exponential dichotomy.*

The *Sacker-Sell spectrum* of a cocycle $A \in \mathcal{A}$ is

$$\sigma_{Sacker-Sell} = \{z \in \mathbb{C} \mid zA \notin \mathcal{D}\}.$$

This spectrum gives nothing new:

Proposition 10.3 *The Sacker-Sell spectrum of A is modulo $|z| = 1$ equal to the spectrum of $A\tau$ as an operator on* $L^2(X, \mathbb{C}^2)$.

Proof. We know that $A \in \mathcal{S}$ if and only if $z \notin \sigma(A)$ for all $|z| = 1$. This implies that $zA \in \mathcal{S} = \mathcal{D}$ if and only if $|z| \notin |\sigma(A)|$. $\qquad\qquad\Box$

11 Questions

Some questions.

• In the theory of discrete one-dimensional Jacobi-operators one deals with *transfer matrices*

$$\mathcal{B} = \{A(x) \in \mathcal{A} \,, | \, A = \begin{pmatrix} b & -1 \\ 1 & 0 \end{pmatrix}, b \in L^{\infty}(X)\} \ ,$$

and we write $A_E = \begin{pmatrix} b+E & -1 \\ 1 & 0 \end{pmatrix}$. In [Dei 83] is proved the following result in the theory of Jacobi matrices: given an interval $I = (E_1, E_2)$ in $\mathbb{R}$ and $A \in \mathcal{B}$ such that for $E \in I$ we have $\lambda(A_E) = 0$ then

$$| \cos(\rho(E_2)) - \cos(\rho(E_1))| \geq (E_2 - E_1)/2 .$$

In this case there exists $E \in I$ such that $\lambda(A_E) > 0$.

This result suggests that also for general $A \in \mathcal{A}$, we can expect that a slow change of the rotation number along the Herman circle $AR(\beta)$ implies positive Lyapunov exponents. More precicly we conjecture that if

$$|\rho(A(\beta_2)) - \rho(A(\beta_1))| < |\beta_2 - \beta_1| ,$$

there exists $\beta \in [\beta_1, \beta_2]$ such that

$$\lambda(A(\beta)) > 0 .$$

This is true if X is a finite set.

• What are the invariant sets for the mapping $\phi_A : \mathcal{U} \to \mathcal{U}$ if A is not in $\mathcal{P}$? There can be exactly two fixed points in the case $A \in \mathcal{P}$, (which are hyperbolic in the case $A \in \mathcal{S}$.) There can be exactly one fixed point (example: $A(x)$ has diagonal elements 1 and $A_{12} = 2$), or uncountably many fixed points like for example $A(x) = 1$) or no fixed point like for example

$$A \in \mathcal{O} = L^\infty(X, SO(2, \mathbb{R}))$$

is not cohomologous to 1.) Every $A \in \mathcal{A}$ such that $\phi_A : \mathcal{U} \to \mathcal{U}$ has 3 separated fixed points is conjugated to 1. Can there exist other types of invariant sets?

• What are the cohomology classes of cocycles in $\mathcal{S}$? Because cocycles in $\mathcal{S}$ can be conjugated to diagonal cocycles, the question is equivalent to find the cohomology group $\mathcal{H}^1(T, \mathbb{R})$.

• We would like to know the various spectra of a cocycle. What spectral types can occur? Cocycles are not normal in general. Does there exist an analogue of a density of states, a measure dk with support on the individual spectrum of the cocycle?

• Is the Herman spectrum of a cocycle lying on the unit circle? An analogy with the spectrum of a Jacobi operator would suggest that the rotation number $\beta \mapsto \rho(\beta)$ on the Herman circle is continuous.

• Is there a relation between the spectrum (or spectral type) of the cocycle $A\tau^*$ and the random Jacobi operator $L = A\tau^* + (A\tau^*)^*$?

• Can one use the spectra of cocycles to distinguish different abstract dynamical systems (X, T_1, m), (X, T_2, m)? The union of all spectra of cocycles in $L^\infty(X, \{a, b\})$ with $a, b \in \mathsf{R}$ would be a possible invariant of a dynamical system.

References

[Cyc 87] H.L.Cycon,R.G.Froese,W.Kirsch,B.Simon. *Schroedinger Operators.*
Texts and Monographs in Physics, Springer, 1987

[Dei 83] P.Deift, B.Simon. *Almost periodic Schroedinger operators.*
Commun. Math. Phys., 90:389-411, 1983

[Den 76] M.Denker,C.Grillenberg,K.Sigmund. *Ergodic theory on compact spaces.*
Lecture Notes in Mathematics,No. 527,Springer 1976

[Gro 90] K.L.Gromov. *Spectral Properties of Unitary Weighted Translation Operators.*
Functional Analysis and Applications24:233-235, 1990

[Her 83] M.Herman. *Une méthode pour minorer les exposants de Lyapunov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2.*
Comm.Math.Helv., 58:453-502, 1983

[Joh 87] R.Johnson, K.Palmer, G.Sell. *Ergodic properties of linear dynamical systems.*
SIAM J. Math. Anal., 18:1-33, 1987

[Kni 1] O.Knill. *The upper Lyapunov exponents of $SL(2, R)$ cocycles: Discontinuity and the problem of positivity.*
Lect. Notes Math., 1486, Springer 1991

[Kni 2] O.Knill. *Positive Lyapunov exponents for a dense set of bounded measurable $SL(2, R)-$ cocycles.*
Ergod. Th. Dynam. Sys., 12, 319-331, 1992

[Lat 91] Yu.D.Latushkin, A.A. Stepin. *Weighted translation operators and linear extensions of dynamical systems.*
Russian Math.Surveys, 46:95-165, 1991

[Man 83] R.Mané. Oseledec's theorem from a generic viewpoint.
Proceedings of the International Congress of Mathematicians, Warschau, 1983

[Man 83a] R.Mané. *The Lyapunov exponents of generic area preserving diffeomorphism.*
unpublished.

[Mat 68] J.Mather. *Characterization of Anosov Diffeomorphisms.*
Indag. Math., 30:479-483, 1968

[Rue 79] D.Ruelle. *Ergodic theory of differentiable dynamical systems.*
Publ.math. de l'IHES, 50:27-58, 1979

[Rue 85] D.Ruelle. *Rotation numbers for diffeomorphisms and flows*
Ann. de l'Inst. Henri Poincare, Physique theoretique, 42:109-115, 1985

[Rue 79a] D.Ruelle. *Analycity Properties of the Characteristic Exponents of Random Matrix Products.*
Advances in Mathematics, 32:68-80, 1979

[Sac 78] R.Sacker, G.Sell. *A spectral theory for linear differential systems.*
J.Diff.Equ., 27:320-358, 1978

[Woj 85] M.Wojtkowski. *Invariant families of cones and Lyapunov exponents.*
Ergod. Th. Dynam. Sys.,5:145-161,1985.

[Woj 86] M.Wojtkowski. *Principles for the Design of Billiards with Nonvanishing Lyapunov Exponents.*
Comm. Math. Phys.,105:391-414,1986.

[You 86] L.-S.Young. *Random Perturbations of Matrix Cocycles.*
Ergod. Th. Dynam. Sys.,6:627-637,1986.

Epilog: Dynamical systems in mathematics

Contents

The language of dynamical systems has entered in different parts of mathematics and physics. The reason is, that the widest definition of a dynamical system, an action of a group or semi-group on a space is so wide, that it can appear everywhere. Also it seems intuitively to be better to work with a group as *time* instead of dealing with static objects (a little essay on this can be found in [Rue 82]). A change of language allows also interesting generalizations of classical problems or give a new view on old questions.

We list some examples, where dynamical systems appear naturally. Our choice of examples should show that generalizations of mathematical structures were obtained by just looking at a theory in a more "dynamic way". Some examples should also indicate how dynamical systems enter naturally in other a priori unrelated domains of mathematics. We restrict ourself to dynamical systems with time Z or Z^d.

1 Stochastic Processes

A *stochastic process* is obtained by taking any abstract dynamical system (X, T, m), a measurable function $f : x \mapsto R$ and forming the sequence of random variables $f_n = f(T^n)$. Independent identically distributed random variables f_n on a probability space (I, ν) are given in the language of dynamical systems by a *Bernoulli shift* $(X = I^Z, T, m = \nu^Z)$ and a measurable function $f : X \to R$ and $f_n = f(T^n)$.

Many results in probability have generalizations. For example, *Birkhoff's ergodic theorem* generalizes the *law of large numbers*. This nowadays little step in reviewing stochastic processes clarified in past the "ergodic hypotheses" standing at the beginning of statistical mechanics.

2 Random walks

A *random walk* on a d dimensional lattice is obtained by taking an abstract dynamical system (X, T, m), $2d$ translations $A_i = A_{i+d}^{-1}$, $i = 1 \ldots d$ on R^d and a partition $X = \bigcup_{i=1}^{2d} Y_i$. The map

$$A(x) = A_i \Leftrightarrow x \in Y_i$$

defines the random walk

$$n \mapsto A^n(x) = A(T^{n-1}x)A(T^{n-2}x) \ldots A(x) \subset R^d$$

on the orbit of x. The classical random walk is obtained, when the partition $(Y_i)_{i=1}^d$ is a generator of the Bernoulli shift (X, T, m) with $2d$ symbols.

3 Percolation

Given any $\mathbb{Z}^d$ dynamical system $(X, T_1, \ldots T_d, m)$ and a measurable set $Y \subset X$. The points on the orbit

$$O(x) = \{T^n x = T_1^{n_1} T_2^{n_2} \ldots T_d^{n_d} x \mid n \in \mathbb{Z}^d\}$$

which belong to Y are called *activated sites* on the lattice $O(x)$. Connected components of activated points are called *clusters*. Choosing a one-parameter family of sets Y_p with $m(Y_p) = p$ and $Y_p \subset Y_q$ ($p < q$), one finds a critical point p_c, above which there is an infinite cluster and below which, there are only finite clusters. See [Mee 90] for examples of this *dependent percolation*. Usually, *percolation problems* deal with a lattice (for example $\mathbb{Z}^d$) and the rule that each site is activated independently with probability p.

4 Thermodynamic formalism

Parts of the thermodynamic formalism for *Statistical mechanics* on a lattice $\mathbb{Z}^d$ can be extended to statistical mechanics of a $\mathbb{Z}^d$ action on a compact metric space. This theory has been developed by Bowen [Bow 75], Ruelle [Rue 78] and others. One can see it as a step to push the mathematics of equilibrium statistical mechanics into more deterministic domains.

5 Renormalization

A dynamical system in infinite dimensional spaces are often called a renormalisation group. The infinite dimensional space is then a large space like a space of interval maps [Lan 82], circle maps, a space of two dimensional twist maps [Mac 83], two dimensional area preserving maps [Eck 84] or a space of quantum field theories [Fer 92]. *Universality* phenomena can be explained by the existence (most of the time only conjectured) of hyperbolic fixed points periodic orbits or hyperbolic invariant sets.

6 Combinatorial number theory

The *multiple Birkhoff recurrence theorem* for a $\mathbb{Z}^d$ action $(X, T_1, \ldots, T_d)$ on a compact metric space X states that there exists a point $x \in X$ and a sequence $n_i \to \infty$ such that $T^{n_i} x \to x$ for all $i = 1, \ldots d$. Fürstenberg [Fur 81] showed, how this result can be used to prove *Van der Waerdens theorem* which tells that any partition of $\mathbb{N}$ into two disjoint sets $A \cup B$ has the property, that one of the two sets contains arithmetic progressions of arbitrary length. He obtained also various generalizations of this combinatorial result by ergodic theoretical tools or methods from topological dynamics.

7 Encryption

Iteration of a dynamical system with sensitive dependence on initial conditions can be used to render *clear text* into *encrypted code*. The iteration of a dynamical system works as a *trap door*. It is easy to mix up things by letting the time run forward. The reconstruction, however is difficult to perform, if one does not know the dynamical system (the key). In practice, one works with dynamical systems over finite fields. Encryption schemes like DES (Data Encryption Standard) (see [Den 82]) base on the fact, that iterating a simple dynamical system produce good codes. DES for example consists (roughly speaking) of an iteration of a twist map over a finite field.

8 Discrete logarithm problem

The *discrete logarithm problem* for a dynamical system (X, T) is the problem to find for a given pair of points $x, y \in X$ a number $n \in \mathbb{Z}$ such that

$$T^n x = y$$

or to tell that no such number exists. The classical discrete logarithm problem (used for example for secrete key exchanges like the Diffie-Helleman scheme) is obtained with the dynamical system $(\mathbb{Z}_N^*, T)$

$$T : x \mapsto a \cdot x \bmod N .$$

It is the question to find the number n in the equation

$$a^n = b \bmod N .$$

9 Pseudo random number generators

One problem in finding good *pseudo random number generators* is the problem of find a finite dynamical system with *large periodic orbits*. Many attacks to encryption systems base on the hope, that a relatively small number of iterations of the encryption give the decryption. Pseudo random generators are also used in algorithms to factor large numbers. Example: Pollard's rho method uses the *discrete quadratic map* $z \mapsto z^2 + a$ over a finite field.

References

[Bow 75] R.Bowen. *Equilibrium States and the ergodic theory of Anosov diffeomorphisms.*
Lect. Notes. Math., 470, Springer 1975

[Bra 88] G.Brassard. *Modern Cryptology.*
Lecture Notes in Computer Science,325, Springer 1988

[Den 82] D.Denning. *Cryptography and data security.*
Addison-Wesley Publishing Company, Reading, 1982

[Eck 84] J.P. Eckmann, H.Koch, P.Wittwer. *A computer-assisted proof of universality for are-preserving maps.*
Momoirs of the American Mathematical Society47/289:111-121 1984

[Fer 92] R.Fernández, J.Fröhlich, A.Sokal. Random Walks, critical phenomena and triviality in Quantum field theory.
Texts and Monographs in Pysics, Springer, Berlin 1992

[Fur 81] H.Fürstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory.*
Princeton University Press, Princeton, New Jersey, 1981

[Lan 82] O.Lanford III. *A computer-assisted proof of the Feigenbaum conjectures.*
Bulletin of the American Mathematical Society6:427-434, 1982

[Mac 83] R.S Mackay. *A renormalisation approach to invariant circles in area-preserving maps.*
Physica D7:283-300, 1983

[Mee 90] R.W.Meester. *Two models of dependent percolation.*
Dissertation Technische Universiteit Delft, 1990

[Rue 78] D.Ruelle. *Thermodynamic Formalism.*
Addison-Wesley Publishing Company, London, 1978

[Rue 82] D.Ruelle. *The Obsessions of Time*
Commun. Math. Phys., 85:3-5, 1982

Bibliography

[Adl 81] M.Adler. *On the Bäcklund Transformation for the Gelfand-Dickey Equations.*
Commun. Math. Phys., 80:517-527, 1981

[Aha 90] D.Aharonov, U.Elias. *Invariant curves around a parabolic fixed point at infinity.*
Ergod. Th. Dynam. Sys., 10:209-229, 1990

[Alv 91] L.Alvarez-Gaume. *Integrability in Random Matrix Models.*
CERN preprint June, 1991

[Alv 91a] L.Alvarez-Gaume, C.Gomez, J.Lacki. *Integrability in random matrix models.*
Physics Letters B, 253:56-62, 1991

[Akc 65] M.A.Akcoglu,R.V.Chacon. *Generalized eigenvalues of automorphisms.*
Proc.Amer.Math.Soc., 16:676-680, 1965

[Ang 90] S.Angenent. Monotone recurrence relations their Birkhoff orbits and topological entropy.
Erg. Th. Dyn., Sys., 10:15-41, 1990

[Ang 92] S.Angenent. *A remark on the topological entropy and invariant circles of an area preserving twist map.*
In "Twist mappings and their Applications", IMA Volumes in Mathematics, Vol. 44., Eds. R.Mc Gehee, K.Meyer, 1992

[Anz 51] H.Anzai. *Ergodic Skew Product Transformations on the Torus.*
Osaka Math. J., 3:83-99, 1951.

[Are 83] H.Aref. *Integrable, chaotic and turbulent vortex motion in two dimensional flows.*
Ann. Rev. Fluid. Mech., 15:345-389, 1983

[Arn 88] V.Arnold. *Geometrical methods in the theory of ordinary differential equations.*
Springer Grundlehren No. 250, New York, 1988

[Arn 89] V.I. Arnold. *Mathematical methods of classical mechanics.*
Springer, New York (etc.), 1989

[Aub 92] S.Aubry. *The concept of anti-integrability: definition, theorems and appli-*
cations to the standard map.
In "Twist mappings and their Applications", IMA Volumes in Mathematics,
Vol. 44., Eds. R.Mc Gehee, K.Meyer, 1992

[Aub 90] S.Aubry, G.Abramovici. *Chaotic Trajectories in the Standard map. The*
concept of anti-integrability.
Physica D, 43:199-219, 1990

[Aub 92] S.Aubry, R.S.MacKay, C.Baesens. *Equivalence of uniform hyperbolicity*
for symplectic twist maps and phonon gap for Frenkel-Kontorova models.
Physica D, 56:123-134, 1992

[Avr 90] J.Avron, P. van Mouche, B.Simon. *On the measure of the spectrum for*
the almost Mathieu model.
Commun. Math. Phys., 132:103-118, 1990

[Bag 88] L.Baggett. *On circle valued cocycles of an ergodic measure preserving*
transformation.
Israel J. Math., 61:29-38, 1988.

[Bak 84] G.A.Baker, D.Bessis, P.Moussa. *A family of almost periodic Schrödinger*
operators.
Physica A, 124:61-78, 1984

[Ban 88] V.Bangert. *Mather Sets for Twist Maps and Geodesics on Tori.*
Dynamics Reported, 1:1-55, 1988

[Ban 92] J.Banks, J.Brooks,G.Cairns,G.Davis,P.Stacey. *On Devaney's definition of*
chaos.
Americ. Math. Month.,99:332-334, 1992

[Bar 83] M.F.Barnsley, J.S.Geronimo, A.N. Harrington. *Geometry, electrostatic*
measure and orthogonal polynomials on Julia sets for polynomials.
Ergodic Th. Dynam. Sys., 3:509-520, 1983

[Bar 83a] M.F.Barnsley, J.S.Geronimo, A.N. Harrington. *Infinite-Dimensional Ja-*
cobi Matrices associated with Julia sets.
Proc. Amer.Math.Soc, 88:625-630, 1983

[Bar 83b] M.F.Barnsley, J.S.Geronimo, A.N. Harrington. *On the Invariant Sets of*
a Family of Quadratic Maps.
Commun. Math. Phys., 88:479-501, 1983

[Bar 85] M.F.Barnsley, J.S.Geronimo, A.N. Harrington. *Almost periodic Jacobi Matrices associated with Julia sets for Polynomials.*
Commun. Math. Phys., 99: 303-317, 1985

[Bar 88] M.Barnsley. *Fractals everywhere.*
Academic Press, London, 1988

[Bbgp 92] A.Bobenko,M.Bordemann,C.Gunn,U.Pinkall. *On two integrable cellular automata.*
SFB 288 Preprint No.25, Technische Universität Berlin

[Bel 85] J.Bellisard. *K theory of C* algebras in solid state physics.*
Lecture Notes Phys., 257: 99-156, 1986

[Bel 82] J.Bellisard, D.Bessis, P.Moussa. *Chaotic States of Almost Periodic Schrödinger operators.*
Phys. Rev. Lett., 49:700-704, 1982

[Bel 92] J.Bellisard. *Le papillon de Hofstadter.*
Astérisque, 206 Séminaire Bourbaki 44ème année.

[Ben 78] G.Benettin,J.Strelcyn. *Numerical experiments on the free motion of a point mass moving in a plane convex region: Stochastic transition and entropy.*
Phys. Review A, 17:, 1978

[Ber 86] Yu.M.Berezansky. *The integration of semi-infinite Toda Chain by means of inverse spectral problem.*
Reports on mathematical physics, 24:21-47, 1986

[Ber 86b] Yu.M.Berezansky, M.I.Gekhtman,M.E.Shmoish. *Integration of some chains of nonlinear difference equations by the method of the inverse spectral problem.*
Ukrainian. Math. J., 38:74-78, 1986

[Bes 82] D.Bessis, M.L.Mehta, P.Moussa. *Orthogonal polynomials on a family of Cantor sets and the problem of iterations of quadratic mappings.*
Lett math. Phys., 6: 123-140, 1982

[Bla 84] P.Blanchard. *Complex analytic dynamics on the Riemann sphere.*
Bull. (New Ser.) of the Amer. Math. Soc., 11:85-141, 1984

[Bob 93] A.Bobenko,N.Kutz,U.Pinkall. *The Discrete Quantum Pendulum.*
SFB 288 Preprint No.42, 1993

[Bog 88] O.I.Bogoyavlensky. *Five Constructions of Integrable Dynamical Systems.*
Acta Appl. Math., 13:227-266, 1988

[Bog 90] O.I.Bogoyavlensky. *On Integrable Generalizations of Volterra Systems.*
in Analysis, et cetera, ed. by P.Rabinowitz, E.Zehnder, Academic Press,
Boston 1990

[Bog 76] O.I.Bogoyavlensky. *On Perturbations of the Periodic Toda lattice.*
Commun. Math. Phys., 51:201-209, 1976

[Bow 75] R.Bowen. *Equilibrium States and the ergodic theory of Anosov diffeomor-phisms.*
Lect. Notes. Math., 470, Springer 1975

[Bow 75] R.Bowen, D.Ruelle. *The Ergodic Theory of Axiom A Flows.*
Invent. Math., 29:181-202, 1975

[Bra 88] G.Brassard. *Modern Cryptology.*
Lecture Notes in Computer Science, 325, Springer 1988

[Bra 81] M.Braun. *Mathematical remarks on the van Allen radiation belt: A survey of old and new results.*
SIAM Review, 23:61-93, 1981

[Bra 81a] M.Braun. *Invariant curves, homoclinic points and ergodicity in area-preserving mappings.*
SIAM J.Math.Anal, 12:630-638, 1981

[Bro 65] H.Brolin. *Invariant sets under itaration of rational functions.*
Arkiv Math., 6: 103-144, 1965

[Bro 75] M.Brown, W.D.Neuman. *Proof of the Poincaré-Birkhoff fixed point theo-rem.*
Michigan Math. J. 24: 21-31, 1975

[Cap 91] H.Capel, F.Nijhoff, V.Papageorgiou. *Complete integrability of Lagrangian mappings and lattices of KdV type.*
Physics Letters A, 155:377-387, 1991

[Car 87] N.Caranicolas, Ch.Vozikis. Chaos in a quartic dynamical model.
Celestial Mechanics, 40:35-49, 1987

[Car 87] R.Carmona, S.Kotani. *Inverse Spectral Theory for Random Jacobi Matri-ces.*
Journal Statistical Physics, 46:1091-1114, 1987

[Car 90] R.Carmona, J.Lacroix. *Spectral Theory of Random Schrödinger Operators.*
Birkhäuser, Boston, 1990

[Cel 88] A. Celletti A., L. Chierchia. *Construction of analytic KAM surfaces and effective stability bounds.*
Commun. Math. Phys., 118:119-161, 1988

[Con 89] G.Contopoulos, B.Barbanis. Lyapunov characteristic numbers and the structure of phase space.
Astron. Astrophys., 222:329-343, 1989

[Con 89] A.Connes. *Compact metric spaces, Fredholm modules, and hyperfiniteness.*
Ergod. Th. Dynam. Sys 9:207-220, (1989)

[Con 90] A.Connes. *Geometrie non commutative.*
InterEditions, Paris 1990

[Cor 82] I.P.Cornfeld, S.V.Fomin, Ya.G.Sinai. *Ergodic Theory.*
Springer, New York, 1982

[Cra 89] W.Craig. The Trace Formula for Schroedinger Operators on the Line.
Commun. Math. Phys., 126:379-407, 1989

[Chu 84] M.T.Chu. *On the global convergence of the Toda lattice for real normal matrices and ists applications to the eigenvalue problem.*
SIAM J. Math. Anal. 15 (1), 1984

[Chu 84a] M.T.Chu. *The generalized Toda flow, the QR algorithm and the center manifold theory.*
SIAM J. Discrete Math., 5:187-201, 1984

[Chu 86] M.T.Chu. *A differential equation approach to the singular value decomposition of bidiagonal matrices.*
Linear algebra and its applications, 80:71-79, 1986

[Chu 89] V.Chulaevsky. *Lyapunov Exponents and Anderson Localisation for Discrete Schroedinger Operators.*
Preprint FIM ETH Zuerich, 1989

[Chu 91] V.Chulaevsky, Ya.G.Sinai. *The exponential localization and structure of the spectrum for 1D quasi-periodic discrete Schrödinger operators.*
Reviews in Mathem. Phys., 3:241-284, 1991

[Cyc 87] H.L.Cycon,R.G.Froese,W.Kirsch,B.Simon. *Schroedinger Operators.*
Texts and Monographs in Physics, Springer, 1987

[Dam 91] P.Damianou. *About the Toda and Volterra system.*
Physics Lett. A, 1991

[Dei 78] P.Deift. *Applications of a commutation formula.*
Duke Math. J., 45: 267-310, 1978

[Dei 91] P.Deift, J.Demmel, L.Li,C.Tomei. *The bidiagonal singular value decomposition and Hamiltonian mechanics.*
SIAM J. Numer. Anal., 28:1463, 1991

[Deif 80] P.Deift,F.Lund,E.Trubowitz. *Nonlinear Wave Equations and Constrained Harmonic Motion.*
Commun. Math. Phys., 74:141-188, 1980

[Deif2 80] P.Deift,L.C.Li,C.Tomei. *Toda flows with infinitly many variables.*
Journal of Functional analysis, 64:358-402, 1985

[Deif3 89] P.Deift,L.C.Li,C.Tomei. *Matrix Factorizations and Integrable Systems.*
Commun. Pure Appl. Math., XLII: 443-521, 1989

[Deif4 89] P.Deift,L.C.Li,C.Tomei. Ordinary differential equations and the symmetric eigenvalue problem.
SIAM J. Num. Anal., 20:1-21, 1983

[Deif5 89] P.Deift,L.C.Li. *Generalized Affine Lie Algebras and the Solution of a Class of Flows Associated with the QR Eigenvalue Algorithm.*
Commun. Pure Appl. Math., XLII: 963-991, 1989

[Dei 83] P.Deift,T.Nanda,C.Tomei. *Ordinary differential equations and the symmetric eigenvalue problem.*
SIAM J. Numer. Anal., 20:1-21, 1983

[Deif5 83] P.Deift, B.Simon. *Almost periodic Schroedinger operators.*
Commun. Math. Phys., 90:389-411, 1983

[Deim 77] K.Deimling. *Ordinary Differential Equations in Banach Spaces.*
Lecture Notes in Math., No. 596, Springer 1977

[Del 83] F.Delyon, B.Souillard. *The Rotation Number for Finite Difference Operators and its Properties.*
Commun. Math. Phys., 89:415-426, 1983

[Den 76] M.Denker,C.Grillenberg,K.Sigmund. *Ergodic Theory on Compact Spaces.*
Lecture Notes in Math., No. 527, Springer 1976

[Den 82] D.Denning. *Cryptography and data security.*
Addison-Wesley Publishing Company, Reading, 1982

[Der 93] J.-M. Derrien. *Critères d'ergodicité de cocycles en escalier. Exemples.*
C.R.Acad.Sci.Paris Série I316:73-76, 1993

[Dev 80] R.Devaney. *Triple Collision in the Planar Isosceles Three Body Problem.*
Invent. Math.60:249-267, 1980

[Dev 84] R.Devaney. *A piecewise linear model for the zones of instability of an area preserving map.*
Physica D10:387-393, 1984

[Dev 89] R.Devaney. *An introduction to chaotic dynamical systems.*
Redwood City, California (etc.) : Addison-Wesley, 1989

[Die 68] J.Dieudonné. *Foundations of Modern Analysis, Tome I.*
Academic Press,New York and London 1968

[Dix 81] J.Dixmier. *Von Neumann Algebras.*
North-Holland mathematical Library, Vol 27, Amsterdam, 1981

[Don 88] V.Donnay. *Geodesic flow on the two-sphere, Part I: Positive measure entropy.*
Ergod. Th. Dynam. Sys., 8:531-553, 1988

[Don 88] V.Donnay. *Geodesic flow on the two-sphere, Part II: Ergodicity.*
Lect. Notes. Math., 1342: 112-153, 1988

[Don 91] V.Donnay. *Using Integrability to produce Chaos: Billiards with positive Entropy.*
Commun. Math. Phys., 141:225-257, 1991

[Dua 82] R.Douady. *Application du théorme des tores invariants.*
These Université Paris VII, 1982

[Eck 84] J.P. Eckmann, H.Koch, P.Wittwer. *A computer-assisted proof of universality for are-preserving maps.*
Momoirs of the American Mathematical Society: 47/289:111-121, 1984

[Ehl 82] F.Ehlers, H.Knörrer. *An algebro-geometric interpretation of the Bäcklund transformation for the Korteweg-de Vries equation.*
Comment. Math. Helv., 57:1-10, 1982

[Eil 47] S.Eilenberg,S. McLane. *Cohomology theory in abstract groups.*
Annals of Mathematics, 48:51-78, 1947.

[Eli 91] L.H.Eliasson. *Ergodic Skew-Systems on* $T^d \times SO(3, R)$.
Preprint Forschungsinstitut für Mathematik ETH Zürich.

[Ere 90] A.Eremenko, M.Lyubich. *The dynamics of analytic transformations.*
Leningrad Math. J.,3:563-634, 1990

[Fad 86] L.D.Faddeev,L.A.Takhtajan. *Hamiltonian methods in the theory of Solitons.*
Springer, 1986

385

[Fel 77]   J.Feldmann, C.Moore. *Ergodic equivalence relations, cohomology and von Neumann algebras I,II.*
Transactions of the Americal Mathematican Society, 234: 289-359, 1977

[Fer 92]   R.Fernández, J.Fröhlich, A.Sokal. Random Walks, critical phenomena and triviality in Quantum field theory.
Texts and Monographs in Pysics, Springer, Berlin 1992

[Fla 74]   H.Flaschka. *On the Toda Lattice II.*
Progress of Theoretical Physics, 51:703-716, 1974

[Fon 90]   E.Fontich. *Transversal Homoclinic Points of a Class of Conservative Diffeomorphisms.*
J.Diff.Equ., 87:1-27, 1990

[For 91]   J.Fornaess, N.Sibony. *Random iterations of rational functions.*
Ergod. Th. Dynam. Sys., 11:687-708, 1991

[Fri 70]   N.A.Friedman. *Introduction to Ergodic Theory.*
Van Nostrand-Reinhold, Princeton, New York, 1970

[Fro 90]   J.Fröhlich, T.Spencer, P.Wittwer. *Localization for a Class of One Dimensional Quasi-Periodic Schrödinger Operators.*
Commun. Math. Phys., 132:5-25, 1990

[Fur 63]   H.Furstenberg. *Non-commuting random products.*
Trans.Amer.Math.Soc.,108:377-428, 1963

[Fur 81]   H.Fürstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory.*
Princeton University Press, Princeton, New Jersey, 1981

[Gal 81]   L.Galgani, A.Giorgilli,J.M.Strelcyn. *Chaotic Motions and Transition to stochasticity in the classical problem of the heavy rigid body with a fixed point.*
Il nuovo cimento, 61 B, No 1 , 1981

[Gel 92]   V.Gelfreich, V.Lazutkin,C.Simo, M.Tabanov. *Fern-like structures in the wild set of the standard and semistandard maps in C^2.*
International J. of Bif. and Chaos,2: 353-370, 1992

[Gen 90]   C. Genecand. *Transversal homoclinic points near elliptic fixed points of area preserving diffeomorphism of the plane.*
Diss. ETHZ, 1990

[Ger 91]   A.Gerasimov, A.Marshakov, A.Mironov, A.Morozov, A.Orlov. *Matrix models of two dimensional gravity and Toda theory.*
Nuclear Physics B, 357:565-618, 1991

[Ges 93] F.Gesztesy, Z.Zhao. *Critical and subcritical Jacobi operators defined as Friedrichs extensions.*
J.Diff. Equ., 103:68-93, 1993

[Ghe 92] J-M. Ghez. *Schrödinger Operators generated by substitutions.*
Report No. 3, Institut für Angewandte Analysis und Stochastik, Berlin, 1992

[Gol 92] C. Golé. *A new proof of Aubry-Mather's theorem.*
Math. Z., 210:441-448, 1992

[Gol 92a] C.Golé. *Monotone maps of $T^n \times R^n$ and their periodic orbits.*
In "Twist mappings and their Applications", IMA Volumes in Mathematics, Vol. 44., Eds. R.Mc Gehee, K.Meyer 1992

[Gol 92] I.Ya.Goldsheid, E.Sorets. *Lyapunov Exponents of the Schrödinger Equation with Quasi-Periodic Potential on a Strip.*
Commun. Math. Phys., 145:507-513, 1992

[Gol 92a] I.Ya.Goldsheid, E.Sorets. *Lyapunov Exponents of the Schrödinger Equation with Certain Classes of Ergodic Potentials.*
Preprint Nr. 147 Institut für Mathematik Ruhr-Universität Bochum, März 1992

[Goo 88] J.Goodman, P.D.Lax. *On dispersive Difference Schemes. I.*
Commun. Pure Appl. Math., XLI, p. 591, 1988

[Gor 85] D.Goroff. *Hyperbolic sets for twist maps.*
Ergod. Th. & Dynam. Sys., 5:337-339, 1985

[Gro 90] K.L.Gromov. *Spectral Properties of Unitary Weighted Translation Operators.*
Functional Analysis and Applications24:233-235, 1990

[Gro 90] D.Gross, A.Migdal. *Nonperturbative two dimensional quantum gravity.*
Physical review letters, 64: 127-130, 1990

[Gui 87] V.Guillemin. *Some inverse spectral results on Riemannian manifolds.*
AMS Lecture, Mimeographed copy of a preprint.

[Gut 92] E.Gutkin, N.Simanyi. *Dual polygonal billiards and necklace dynamics.*
Commun. Math. Phys., 143:431-449, 1992

[Hah 32] H.Hahn. *Reelle Funktionen, 1.Teil, Punktfunktionen.*
Akademische Verlagsgesellschaft, Leipzig 1932

[Hal 88] G. Hall. *A remark on the multiplicity of monotone periodic orbits.*
Ergod.Th. Dyn. Sys., 8:109-118, 1988

[Hal 56]   P.Halmos. *Lectures on Ergodic Theory.*
           The mathematical society of Japan, 1956

[Hay 87]   A.Hayli, T.Dumont, J.M.Ollangier, J.M. Strelcyn. *Quelques résultats nou-
           veaux sur les billard de Robinik.*
           J.Phys. A : Math. Gen.: 2337-3249, 1987

[Hen 64]   M.Henon, C.Heiles. *The applicability of the third integral of motion some
           numerical experiments.*
           Astronomical Journal, 69:73-79, 1964

[Hen 83]   M.Henon, J.Wisdom. *The Benettin-Strelcyn oval billiard revisited.*
           Physica D8:157-169, 1983

[Her 81]   M.R.Herman.   *Construction d'un difféomorphisme minimal d'entropie
           topologique non nulle.*
           Ergodic Theory & Dynamical Systems, 1:65-76, 1981

[Her 83]   M.Herman.   *Une méthode pour minorer les exposants de Lyapunov et
           quelques exemples montrant le caractère local d'un théorème d'Arnold et
           de Moser sur le tore de dimension 2.*
           Commun.Math.Helv., 58:453-502, 1983

[Her 83a] M.Herman.   *Sur les courbes invariantes par les difféomorphismes de
           l'anneau.*
           Astérisque, 103-104, Société mathématique de france 1983

[Hil 02]   D.Hilbert. *Mathematische Problem.*
           Gesammelte Abhandlungen III, Berlin 1935

[Hof 92]   A.Hof. *Some remarks on discrete aperiodic Schrödinger operators.*
           Preprint ETH Zürich, December 1992.

[Hua 82]   L.K. Hua. *Introduction to Number theory.*
           Springer, Berlin, 1982

[Joh 82]   R.Johnson, J.Moser. *The Rotation Number for Almost Periodic Potentials.*
           Commun. Math. Phys., 84:403-438, 1982

[Joh 84]   R.A.Johnson.   *Lyapunov numbers for the almost periodic Schroedinger
           equation.*
           Illinois J. Math., 28:397-419, 1984

[Joh 86]   R.A.Johnson. *Exponential Dichotomy, Rotation Number and Linear Dif-
           ferential Operators with Bounded Coefficients.*
           J. Differential Equations, 61:54-78, 1986

[Joh 87] R.Johnson, K.Palmer, G.Sell. *Ergodic properties of linear dynamical systems.*
SIAM J. Math. Anal., 18:1-33, 1987

[Kac 75] M.Kac,P. van Moerbeke. *A complete solution of the periodic Toda problem.*
Proc. Nat. Acad. Sci. USA, 72, No.8, p.2879-2880, 1975

[Kam 93] S.Kamvissis. *On the long time behavior of the doubly infinite Toda lattice under initial data decaying at infinity.*
Commun. Math. Phys., 153:479-519, 1993

[Kat 80] A. Katok. *Lyapunov Exponents, entropy, and periodic orbits for diffeomorphisms.*
Publ.Math. IHES, 51:137-172, 1980

[Kat 82] A. Katok. *Some remarks on Birkhoff and Mather twist map theorems.*
Ergod. Th. & Dynam. Sys., 2:185-194, 1982

[Kat 86] A.Katok, J.-M.Strelcyn. *Invariant manifolds, Entropy and Billards : Smooth maps with Singularities.*
Lecture Notes in Math., 1222, Springer 1986

[Kif 82] Y.Kifer. *Perturbations of Random Matrix Products.*
Z.Wahrsch. theorie verw. Gebiete, 61:83-95, 1982

[Kir 67] A.A. Kirillov. *Dynamical systems, factors and representations of groups.*
Russian Math. Surveys, 22:63-75, 1967

[Kni 2] O.Knill. *Positive Lyapunov exponents for a dense set of bounded measurable $SL(2,R)-$ cocycles.*
Ergod. Th. Dynam. Sys., 12, 319-331, 1992

[Kni 1] O.Knill. *The upper Lyapunov exponents of $SL(2,R)$ cocycles: Discontinuity and the problem of positivity.*
Lect. Notes Math., 1486, Springer 1991

[Kni 3] O.Knill. *Isospectral Deformations of Random Jacobi Operators.*
Commun. Math. Phys., 151:403-426, 1993

[Kni 4] O.Knill. *Factorisation of Random Jacobi Operators and Bäcklund transformations.*
Commun. Math. Phys., 151:589-605, 1993

[Kni 5] O.Knill. *An analytic map containing the standard map family.*
Submitted to Nonlinearity

[Kni 6] O.Knill. *Embedding abstract dynamical systems in monotone twist maps.*
To be submitted

[Koo 86] H.Kook, J.D.Meiss. *Periodic orbits for reversible, symplectic mappings.*
Physica D, 35:65-86, 1986

[Kos 79] B.Kostant. *The solution to a generalized Toda lattice and representation theory.*
Adv. in Math., 34:195-338, 1979

[Kot 89] S.Kotani. *Jacobi matrices with random potentials taking finitely many values.*
Rev. in Mathematical Phys., 1:129-133, 1989

[Kot 88] S.Kotani, B.Simon. *Stochastic Schroedinger Operators and Jacobi Matrices on the Strip.*
Commun. Math. Phys, 119:403-429, 1988

[Lak 81] V.Lakshmikantham, S.Leela. *Nonlinear Differential equations in abstract spaces.*
Pergamon Press, Oxford, 1981

[Lan 68] O.Lanford III. *The classical mechanics of one-dimensional systems of infinitly many particles.*
Commun. Math. Phys.9:176-191, 1968

[Lan 75] O.Lanford III. *Time evolution of large classical systems.*
Dynamical Systems: Theory and Applications, Springer Lecture notes of Physics,38:1-111, 1975

[Lan 82] O.Lanford III. *A computer-assisted proof of the Feigenbaum conjectures.*
Bulletin of the American Mathematical Society, 6:427-434, 1982

[Lan 83] O.Lanford III. *Introduction to the mathematical theory of dynamical systems.*
Les Houches Session XXXVI, Editors G.Iooss, R.Helleman, R.Stora, North-Holland Publishing Company, Amsterdam 1983

[Lan 85] O.Lanford III. *Introduction to hyperbolic sets.*
in "Regular and Chaotic Motion in Dynamical systems"
Eds. G.Velo, A.S.Wightman, Nato ASI Ser. B, 118:73-102, 1985

[Lan 92] O.Lanford III. *Lectures on dynamical systems.*
Notes from lectures held at ETH, 1990-1992.

[Las 90] J.Laskar. *The chaotic motion of the solar system: a numerical estimate of the size of the chaotic zones.*
Icarus, 88:266-291, 1990

[Las 93] Y.Last. *A relation between a.c. spectrum of ergodic Jacobi matrices and the spectra of periodic approximants.*
Commun. Math. Phys., 151:183-192, 1993

[Las 93] Y.Last. Zero measure spectrum for the almost Mathieu operator.
Technion preprint March, 1993

[Laz 84] V.F.Lazutkin, D.Ya. Terman. *Percival Variational Principle for Invariant Measures and Commensurate-Incommensurate Phase Transitions in One Dimensional Chains.*
Commun. Math. Phys, 94:511-522, 1984

[Led 82] F.Ledrappier. *Quelques Proprietés des Exposants Caracteristiques.*
Lecture Notes in Math., No. 1097, Springer 1982

[Led 84] F.Ledrappier. *Positivity of the exponents for stationary sequences of matrices.*
Lecture Notes in Math., No. 1186,Springer 1984

[Led 91] F.Ledrappier, L.-S. Young. *Stability of Lyapunov exponents.*
Ergod. Th. Dynam. Syst, 11:469-484, 1991

[Lem 89] M.Lemańczyk. *On the Weak Isomorphism of Stricly Ergodic Homeomorphisms.*
Monatsh. Math., 108:39-46, 1989

[Li 87] L.C.Li. *Long time behaviour of an infinite particle system.*
Commun. Math.Phys., 110:617-623, 1987

[Liu 61] L.A. Liusternik, V.J.Sobolev. *Elements of Functional analysis.*
Frederick Ungar Publishing Company, New York, 1961

[Liv 72] A.Livsic. *Cohomology of dynamical systems.*
Izv. Akad. Nauk SSSR, 36:1278-1301, 1972

[Lju 83] M.Ju. Ljubich. *Entropy properties of rational endomorphisms of the Riemann sphere.*
Ergod. Th. Dynam.Sys., 3:351-385, 1983

[Mac 63] G. W. Mackey. *Ergodic theory, group theory and differential geometry*
Proc. Nat. Acad. Sci. US, 50:1184-1191, 1963

[Mac 83] R.S Mackay. *A renormalisation approach to invariant circles in area-preserving maps.*
Physica D, 7:283-300, 1983

[Mac 92] R.S. Mackay, J.D. Meiss. *Cantori for symplectic maps near the anti-integrable limit.*
Nonlinearity, 5:149-160, 1992

[Man 81] R.Mané. *A proof of Pesin's formula.*
Erg. Th. Dyn. Syst., 1: 95-102 (1981)

[Man 83] R.Mané. *Oseledec's theorem from a generic viewpoint.*
Proceedings of the International Congress of Mathematicians, Warschau, 1983

[Man 83a] R.Mané. *The Lyapunov exponents of generic area preserving diffeomorphism.*
Unpublished

[Man 85] R.Mané. *On the Bernoulli property for rational maps.*
Ergod. Th. Dynam.Sys., 5:71-88, 1985

[Man 77] Yu. I. Manin. *A course in mathematical Logic.*
Graduated texts in Mathematics 53, New York, Springer, 1977

[Mar 91] E.J.Martinec. *On the Origin of Integrability in Matrix Models.*
Commun. Math. Phys., 138:437, 1991

[Mat 68] J.Mather. *Characterization of Anosov Diffeomorphisms.*
Indag. Math., 30:479-483, 1968

[Mat 82] J.Mather. *Existence of quasi-periodic orbits for twist homeomorphism of the annulus.*
Topology, 21:457-467, 1982

[Mat 84] J.Mather. *Amount of rotation about a point and the Morse Index.*
Commun. Math. Phys., 94:141-153, 1984

[McK 78] H.P. McKean. *Integrable systems and algebraic curves.*
Lecture Notes in Math., No. 755, Springer 1978

[McK 76] H.P.McKean, E.Trubowitz. *Hill's Operator and Hyperelliptic Function Theory in the Presence of Infinitely many Branch Points.*
Commun. Pure Appl. Math., XXIX, 143-226, 1976

[McK 89] H.P.McKean. *Is there an infinite dimensional algebraic geometry? Hints from KdV.*
Proc. of Symposia in Pure Mathematics, 49:27-38, 1989

[Mee 90] R.W.Meester. *Two models of dependent percolation.*
Dissertation Technische Universiteit Delft, 1990

[Mer 85] K.D.Merrill. *Cohomology of step functions under irrational rotations.*
Israel J. Math., 52:320-340, 1985.

[Mil 91] J.Milnor. *Dynamics in one complex variable.*
Introductory Lectures, SUNY, 1991

[Moo 78] C.Moore,K.Schmidt. *Coboundaries and homomorphisms for non-singular actions and a problem of H.Helson.*
Proc. London Math.Soc., 40:443-475, 1980

[Mor 76] P.v.Moerbeke. *The spectrum of Jacobi Matrices.*
Invent. Math., 37:45-81, 1976

[Mor 78] P.v.Moerbeke. *About isospectral deformations of discrete Laplacians.*
Lecture Notes in Mathematics, No. 755, Springer 1978

[Mos 73] J. Moser. *Stable and random Motion in dynamical systems.*
Princeton University Press, Princeton, 1973

[Mos 75] J.Moser. *Three Integrable Hamiltonian Systems Connected with Isospectral Deformations.*
Adv. in Math.,l 16:197-220, 1975

[Mos 75] J.Moser. *Finitly many mass points on the Line under the influence of an exponential potential. An integrable system.*
Lecture Notes in Physics, 38, 1975

[Mos 84] J.Moser. *Drei Kapitel über integrable Hamiltonsche Systeme.*
ETH lecture, Summer 1984

[Mos 86] J. Moser. *Recent Developments in the theory of Hamiltonian systems.*
SIAM Review, 28:459-485, 1986

[Mos 87] J. Moser. *On the construction of invariant curves and Mather sets via a regularized variational principle.*
in P.H.Rabinowitz et al (eds.) "Periodic Solutions of Hamiltonian Systems and Related Topics":221-234, 1987

[Nak 90] Y.Nakamura. *Moduli Space of SU(2) Monopoles and Complex Cyclic-Toda Hierarchy.*
Commun. Math.Phys, 128:509-520, 1990

[Ner 88] M.G.Nerurkar. *On the construction of smooth ergodic skew-products.*
Ergod. Th. Dynam. Syst., 8:311-326, 1988

[Neu 32] J. von Neumann. *Zur Operatorenmethode in der Mechanik.*
Ann. Math. 33: 587-642, 1932

[Orn 70]   D.S.Ornstein. *Factors of Bernoulli shifts are Bernoulli shifts.*
           Adv. Math., 5:349-364, 1970

[Pag 89]   E. Le Page. *Régularité du plus grand exposant caractéristique des produits
           de matrices aléatoires indépendantes et applications.*
           Ann. Inst. Henri Poincaré, 25:109-142, 1989

[Par 86]   M.Paramio, J.Sesma. *A continued fraction Procedure to determine invari-
           ant directions in twist mappings.*
           Physics Letters A, 117:333-336, 1986

[Par 80]   W. Parry. *Topics in ergodic theory.*
           Cambridge University Press, 1980

[Part 77] K.R.Parthasarathy, K.Schmidt. *On the cohomology of a hyperfinite action.*
           mboxMonatsh. Math., 84:37-48, 1977

[Per 90]   A.M.Perelomov. *Integrable Systems of classical Mechanics and Lie Alge-
           bras I.*
           Birkhäuser, Boston 1990

[Per 80]   I.C. Percival. *Variational principle for invariant tori and cantori.*
           Amer. Instit. of Physics Conf. Proc., 57:310-320, 1980

[Pes 77]   Ya.B.Pesin. *Characteristic Lyapunov Exponents and Smooth Ergodic The-
           ory.*
           Russian Math. Surveys, 32:55-114, 1977

[Pet 83]   Peterson K.: *Ergodic theory.*
           Cambridge Studies in advanced mathematics 2, Cambridge University press,
           1983

[Prz 82]   F.Przytycki.   *Examples of conservative diffeomorphisms of the two-
           dimensional torus with coexistence of elliptic and stochastic behaviour.*
           Ergod. Th. Dynam.Sys, 2:439-463, 1982

[Qui 91]   G.Quispel,H.Capel,V.Papageorgiou,F.Nijhoff. *Integrable mappings derived
           from soliton equations.*
           Physica A, 173:243-266, 1991

[Rag 78]   M.S.Raghunathan. *A proof of Oseledec's Multiplicative Ergodic Theorem.*
           Israel Journal of Mathematics, 32:356-362,1978

[Ree 80]   M.Reed, B.Simon. *Methods of modern mathematical physics I, Functional
           analysis.*
           Academic Press, Orlando, 1980

[Rob 83] M.Robnik. *Classical dynamics of a family of billiards with analytic bound-aries.*
J.Phys.A : Math. Gen., 16:3971-3986, 1983

[Rud 62] W.Rudin. *Fourier Analysis on groups.*
Interscience tracts in pure and applied mathematics 12, Interscience Publishers, 1962

[Rue 78] D.Ruelle. *Thermodynamic Formalism.*
Addison-Wesley Publishing Company, London, 1978

[Rue 79] D.Ruelle. *Ergodic theory of differentiable dynamical systems.*
Publ. math. de l'IHES, 50:27-58, 1979

[Rue 82] D.Ruelle. *Repellers for real analytic maps.*
Ergod. Th. Dynam. Sys., 2:99-107, 1982

[Rue 85] D.Ruelle. *Rotation numbers for diffeomorphisms and flows.*
Ann. de l'Inst. Henri Poincare, Physique theoretique, 42:109-115, 1985

[Rue 79a] D.Ruelle. *Analycity Properties of the Characteristic Exponents of Random Matrix Products.*
Advances in Mathematics, 32:68-80, 1979

[Rue 82] D.Ruelle. *The Obsessions of Time.*
Commun. Math. Phys., 85:3-5, 1982

[Rue 87] D.Ruelle. *Chaotic evolution and strange attractors.*
Cambridge university press, Cambridge 1997

[Rue 90] D.Ruelle. *Deterministic chaos: the science and the fiction.*
Proc. R.Soc. Lond. A, 427:241-248, 1990

[Sac 78] R.Sacker, G.Sell. *A spectral theory for linear differential systems.*
J.Diff.Equ., 27:320-358, 1978

[Sal 89] D. Salamon, E. Zehnder. *KAM theory in configuration space.*
Commun. Math. Helv., 64: 84-132, 1989

[San 86] V.S.Sanders. *An invitation to von Neumann algebras.*
Springer, New York, 1986

[Sch 87] R.Schmid. *Infinite dimensional Hamiltonian systems.*
Bibliopolis, edizioni di filosofia e scienze, Napoli, 1987

[Sch 89] K. Schmidt. *Algebraic ideas in ergodic theory.*
Regional conference Series in mathematics No. 76, AMS Providence 1989

[Sie 69]   C.L.Siegel. *Topics in complex function theory Vol 2.*
           Wiley-Interscience, New York, 1969

[Sha 90]   A.B.Shabat, R.I.Yamilov. *Symmetries of nonlinear chains.*
           Leningrad Math. J., 2:377-400, 1990

[Sim 84]   B.Simon. *Fifteen Problems in Mathematical Physics.*
           Perspectives in mathematics,   Anniversary   of   Oberwolfach   1984.
           Birkhäuser, Basel, 1984

[Sim 83]   B.Simon. *Kotani Theory for One Dimensional Stochastic Jacobi Matrices.*
           Commun. Math. Phys., 89:227-234, 1983

[Sim 90]   B.Simon. *Absence of Ballistic Motion.*
           Commun. Math. Phys., 134:209-212, 1990

[Sor 91]   E.Sorets, T.Spencer. *Positive Lyapunov Exponents for Schrödinger Operators with Quasi-Periodic Potentials.*
           Commun. Math. Phys., 142: 543-566, 1991

[Spe 86]   T.Spencer. *Some rigorous results for random and quasiperiodic potentials.*
           Phys. A, 140:70-77, 1986

[Ste 71]   A.M.Stepin. *Cohomologies of automorphism groups of a Lebesgue space.*
           Functional Anal. Appl., 5:167-168, 1971

[Str 89]   J.-M.Strelcyn. *The Coexistence Problem for conservative dynamical systems:A review.*
           Colloquium Mathematicum,LXII:331-345, 1991

[Sur 89]   Y.B.Suris. *Integrable mappings of the standard type.*
           Funct. Analy. Appl., 23:74-76, (1989)

[Sur 90]   Y.Suris. *Discrete time generalized Toda lattices: complete integrability and relation with relativistic Toda lattices.*
           Physics Letters A, 145:113-119, 1990

[Sur 91]   Y.Suris. *Generalized Toda chains in discrete time.*
           Leningrad Math. J., 2:339-352, 1991

[Sym 80]   W.Symes. *Systems of Toda Type, Inverse Spectral Problems, and Representation Theory.*
           Invent. Math., 59, 13-51, 1980

[Sym 82]   W.Symes. *The QR algorithm and scattering for the finite nonperiodic Toda lattice.*
           Physica D, 4:275-280, 1982

[Tab 93] S.Tabachnikov. *Commuting dual billiard maps.*
Preprint FIM Febr. 1993

[Tod 80] H.Toda. *Theory of nonlinear lattices .*
Springer. 1981

[Tsu 58] M. Tsuji. *Potential theory in modern function theory.*
Chelsea publishing company, New York, 1958

[Vee 69] W.Veech. *Strict ergodicity in zero dimensional dynamical systems and the Kronecker-Weyl theorem mod 2.*
Trans.Amer.Math.Soc., 140:1-33, 1969

[Ven 89] S. Venakides. *The Continuum Limit of Theta Functions.*
Commun. Pure Appl. Math., 42:711-728, 1989

[Viv 87] F.Vivaldi, A.Shaidenko. *Global Stability of a class of discontinuous dual billiards.*
Commun. Math. Phys., 110:625-640, 1987

[Wed 75] M.Wedati, M.Toda. *Bäcklund transformation for the exponential lattice.*
J. Phys. Soc. Japan, 39:1196, 1975

[Wal 82] P.Walters. *An introduction to ergodic theory.*
Graduate texts in mathematics 79, New York, Springer, 1982

[Wis 87] J.Wishdom. *Chaotic behaviour in the solar system.*
in "Dynamical chaos", Proceedings of a Royal Society discussion Meeting, Eds. M.Berry, I.Percival, N.Weiss, Princeton, 1987

[Woj 85] M.Wojtkowski. *Invariant families of cones and Lyapunov exponents.*
Ergodic Theory & Dynamical Systems, 5:145-161, 1985.

[Woj 86] M.Wojtkowski. *Principles for the Design of Billiards with Nonvanishing Lyapunov Exponents.*
Commun. Math. Phys., 105:391-414,1986.

[Wol 91] Wolfram Research. *Mathematica Version 2.0.*
Illinois 1991

[You 86] L.-S.Young. *Random perturbations of matrix cocycles.*
Ergodic Theory & Dynamical Systems, 6:627-637, 1986.

[You 92] L.-S.Young. *Some Examples of nonuniformly hyperbolic cocycles.*
Preprint, Arizona, 1992

[Zeh 73] E. Zehnder. *Homoclinic points near elliptic fixed points.*
Commun. Pure Appol. Math., 26:131-182, 1973

[Zha 93] X-H.Zhao, K-H.Kwek,J-B.Li, K-L.Huang *Chaotic and resonant stream-lines in the ABC flow* SIAM J. Appl. Math, 53:71-77, 1993

Index

Acknowledgments

Curriculum Vitae

I was born in Zürich, on October 22, 1962. I attended the primary and secundary schools in Uhwiesen near the Rheinfalls and the High School in Schaffhausen, graduating in 1982 with a diploma type B (latin).

In 1982 I participated at the competition "Schweizer Jugend forscht" with a work about partition functions in combinatorial number theory.

I studied mathematics during $1982 - 1987$ at the ETH with emphasis on theoretical physics. In this time I did also my military service.

My diploma work in 1987 was guided by Prof. J. Moser. The topic were twist mappings arising in the Störmerproblem, which is the problem to understand the motion a charged particle in a magnetic dipole field.

Since 1987, I was assistant at the ETH with a interruption of one year (1988-1989), when I spent an academic year at the Technion (Haifa) with a "Max und Silvia Uscher Wolf" fellowship.

I began my doctorate under the supervision of Prof. O. Lanford III in the fall of 1989.

In my free time I like running, biking, paragliding or playing piano.