

RESEARCH ARTICLE

Shortest closed curve to contain a sphere in its convex hull

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Abstract

We show that in Euclidean 3-space any closed curve which contains the unit sphere within its convex hull has length $L \geq 4\pi$, and characterize the case of equality. This result generalizes the authors' recent solution to a conjecture of Zalgaller. Furthermore, for the analogous problem in n dimensions, we include the estimate $L \geq Cn\sqrt{n}$ by Nazarov, which is sharp up to the constant C .

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1 | INTRODUCTION

The *convex hull* of a set X in Euclidean space \mathbf{R}^3 is the intersection of all convex sets that contain X . The *inradius* of X is the supremum of the radii of spheres which are contained in X . Here, we show the following:

Theorem 1.1. *Let $\gamma : [a, b] \rightarrow \mathbf{R}^3$ be a closed rectifiable curve of length L , and r be the inradius of the convex hull of γ . Then,*

$$L \geq 4\pi r. \quad (1)$$

Equality holds only if, up to a reparameterization, γ is simple, $C^{1,1}$, lies on a sphere of radius $\sqrt{2}r$, and traces consecutively 4 semicircles of length πr .

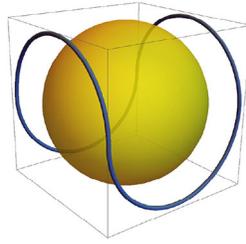


FIGURE 1 The baseball curve.

In 1996, V. A. Zalgaller [18, 22] conjectured that the above theorem holds subject to the additional assumption that γ lie outside a sphere S of radius r within its convex hull. The length minimizer, called the *baseball curve*, together with S , is shown in Figure 1.

Zalgaller’s conjecture was proved recently in [15] following earlier work in [13]. Here, we refine the methods introduced in those papers to establish the more general result above. Our approach will be similar to that in [15]. We start by setting $r = 1$ and assuming that γ has the smallest length among closed curves, which contain the unit sphere S^2 within their convex hull [15, section 2]. The *horizon* of γ is the measure in S^2 counted with multiplicity of the set of points $p \in S^2$, where the affine tangent plane $T_p S^2$ intersects γ :

$$H(\gamma) := \int_{p \in S^2} \#\gamma^{-1}(T_p S^2) dp.$$

Since γ is closed, one quickly sees that $\#\gamma^{-1}(T_p S^2) \geq 2$ for almost every $p \in S^2$ [13, Lemma 7.1]. Hence, $H(\gamma) \geq 8\pi$. The *efficiency* of γ is given by

$$E(\gamma) := \frac{H(\gamma)}{L(\gamma)}.$$

So to establish (1), it suffices to show $E(\gamma) \leq 2$. To this end, we note that for any partition of γ into subcurves γ_i ,

$$E(\gamma) = \sum_i \frac{H(\gamma_i)}{L(\gamma)} = \sum_i \frac{L(\gamma_i)}{L(\gamma)} E(\gamma_i).$$

So it suffices to construct a partition with $E(\gamma_i) \leq 2$. Similar to [15], this is achieved by *unfolding* γ into the plane (Section 3), and identifying a collection of subcurves of γ we call *spirals* (Section 4); however, these operations need to be generalized here as they were defined only for curves with $|\gamma| \geq 1$ in [15]. Furthermore, we will show that if $E(\gamma) = 2$, then $|\gamma| \geq 1$. So the rigidity of (1) follows from Zalgaller’s conjecture established in [15], and completes the proof of Theorem 1.1 (Section 5).

For curves in \mathbf{R}^2 , the isoperimetric inequality quickly yields $L \geq 2\pi r$ as the analog of (1). We will include in the Appendix a version of (1) by F. Nazarov for curves in \mathbf{R}^n , which is obtained by covering the unit sphere S^{n-1} with certain slabs, and applying the correlation inequality [16, 19] to their Gaussian volume. This approach has implications for covering problems for the sphere by congruent disks [5], and yields a new proof of a result of Tikhomirov [21] (Note A.4). There are

many natural optimization problems for a convex hull of space curves, which remain open, including other questions of Zalgaller [22], which are closely related to well-known problems of Bellman [2–4] in operations research and search theory [1, 12]; see also [13, 15, 17] and references therein.

2 | MINIMAL INSPECTION CURVES

\mathbf{R}^n denotes the n -dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$, norm $|\cdot| := \langle \cdot, \cdot \rangle^{1/2}$, and origin o . A *curve* is a continuous rectifiable mapping $\gamma : [a, b] \rightarrow \mathbf{R}^n$ with length $L = L(\gamma)$. We also use γ to refer to its image $\gamma([a, b])$. If $\gamma(a) = \gamma(b)$, then we say that γ is *closed* and identify $[a, b]$ with the topological circle $\mathbf{R}/(b - a)$. Rectifiable curves may be parameterized with constant speed [6], which we assume is the case throughout this work. In particular, all curves below are Lipschitz continuous, and thus differentiable almost everywhere, with $|\gamma'| = L/(b - a)$; see [15, section 2] and references therein for basic facts on rectifiable curves. We say γ is a (*generalized*) *inspection curve* provided that γ is closed and its convex hull, $\text{conv}(\gamma)$, contains the unit sphere \mathbf{S}^2 . It follows from Arzela–Ascoli theorem that there exists an inspection curve γ whose length achieves the minimum value among all inspection curves [15, section 2]. Then, γ will be called a *minimal* inspection curve. We let *int*, *cl*, and ∂ , stand, respectively, for interior, closure, and boundary.

Lemma 2.1. *Let $\gamma : \mathbf{R}/L \rightarrow \mathbf{R}^3$ be a minimal inspection curve. Suppose $\gamma(t) \in \text{int}(\text{conv}(\gamma))$, for some $t \in \mathbf{R}/L$. Then, there exists a connected open set $U \subset \mathbf{R}/L$, with $t \in U$, such that γ maps $\text{cl}(U)$ injectively to a line segment with end points on $\partial \text{conv}(\gamma)$. In particular, $\gamma(t) = o$ for at most finitely many $t \in \mathbf{R}/L$.*

Proof. Let U be the component of $\gamma^{-1}(\text{int}(\text{conv}(\gamma)))$, which contains t . If $\gamma|_{\text{cl}(U)}$ does not trace a line segment, we may shorten γ by replacing $\gamma(\text{cl}(U))$ with the line segment connecting the end points of $\gamma(\text{cl}(U))$. But this operation preserves $\text{conv}(\gamma)$, as it preserves the points of γ on $\partial \text{conv}(\gamma)$. Hence, we obtain an inspection curve shorter than γ , which is impossible. If $\gamma(t) = o$, then $L(\gamma|_U) \geq 2$, since $\gamma(U)$ contains a diameter of \mathbf{S}^2 . So there can be only finitely many such points, since γ is rectifiable. \square

We say that t is a *regular* point of a curve γ provided that γ is differentiable at t and $\gamma'(t) \neq 0$. Then, the *tangent line* of γ at t is well defined. Since we assume that curves are parameterized with constant speed, they are regular almost everywhere. Furthermore, by Lemma 2.1, all points $t \in \mathbf{R}/L$ with $\gamma(t) \in \text{int}(\text{conv}(\gamma))$ of a minimal inspection curve γ are regular.

Lemma 2.2. *Let $\gamma : \mathbf{R}/L \rightarrow \mathbf{R}^3$ be a minimal inspection curve, $t \in \mathbf{R}/L$ be a regular point of γ , and ℓ be the tangent line of γ at t . Suppose that ℓ intersects $\text{int}(\text{conv}(\gamma))$. Then, there exists an open interval $U \subset \mathbf{R}/L$, with $t \in U$, which is mapped injectively by γ into $\ell \cap \text{int}(\text{conv}(\gamma))$.*

Proof. If $\gamma(t) \in \partial \text{conv}(\gamma)$, then either $\gamma'(t)$ or $-\gamma'(t)$ points outside $\text{conv}(\gamma)$. Hence, for some s close to t , $\gamma(s)$ lies outside $\text{conv}(\gamma)$, which is impossible. So $\gamma(t) \in \text{int}(\text{conv}(\gamma))$, in which case, Lemma 2.1 completes the proof. \square

Combining the last two observations we obtain the following:

Proposition 2.3. *Let $\gamma : \mathbf{R}/L \rightarrow \mathbf{R}^3$ be a minimal inspection curve. Then, there exists an open set $U \subset \mathbf{R}/L$ such that tangent lines of γ on U do not pass through o . Furthermore if $U \neq \mathbf{R}/L$, then $\mathbf{R}/L \setminus U$ is the disjoint union of a finite number of closed intervals each mapped by γ into a line segment, which passes through o and ends on $\partial \text{conv}(\gamma)$.*

Proof. Let X be the union of all closed intervals $I \subset \mathbf{R}/L$ such that $\gamma(I)$ is a line segment, which passes through o and ends on $\partial \text{conv}(\gamma)$. By Lemma 2.1, there are at most finitely many such intervals. Thus, X is closed. Let $U := \mathbf{R}/L \setminus X$. By Lemma 2.2, no tangent line of γ at a regular point of U may pass through o , which completes the proof. □

3 | UNFOLDING

Let $\gamma : \mathbf{R}/L \rightarrow \mathbf{R}^3$ be a minimal inspection curve. We will always assume that 0 is a local minimum point of $|\gamma|$. By Lemma 2.1, γ passes through o at most finitely many times, which, if they exist, will be denoted by $0 =: t_0, \dots, t_m := L$. Then, the projection $\bar{\gamma} : \mathbf{R}/L \rightarrow \mathbf{S}^2$, given by $\bar{\gamma} := \gamma/|\gamma|$ is well defined on $\mathbf{R}/L \setminus \{t_k\}$. Furthermore, since by Proposition 2.3, γ traces line segments near t_k , $\bar{\gamma}$ is Lipschitz on each interval (t_{k-1}, t_k) . Thus, $\bar{\gamma}$ is differentiable almost everywhere on \mathbf{R}/L . Consequently, the arclength function

$$\theta(t) := \int_0^t |\bar{\gamma}'(s)| ds$$

is well defined on $[0, L]$ (θ measures the “cone angle” [7] or “vision angle” [8] of γ from the point of view of o). The *unfolding* of γ is the planar curve $\tilde{\gamma} : [0, L] \rightarrow \mathbf{R}^2$ defined as

$$\tilde{\gamma}(t) := |\gamma(t)|e^{i(\theta(t)+(k-1)\pi)}, \quad \text{for } t \in [t_{k-1}, t_k].$$

Note that $|\gamma| = |\tilde{\gamma}|$, and whenever γ passes through o , then $\tilde{\gamma}$ will pass through o as well on a line segment. As in [15], we may also compute that

$$|\tilde{\gamma}'| = ||\gamma'| + i|\gamma|\theta'|, \quad \text{and} \quad \theta' = |\bar{\gamma}'| = \frac{1}{|\gamma|^2} \sqrt{|\gamma|^2|\gamma'|^2 - \langle \gamma, \gamma' \rangle^2}, \tag{2}$$

almost everywhere. It follows that, for almost all $t \in [0, L]$, $|\tilde{\gamma}'| = |\gamma'| = 1$. So $\tilde{\gamma}$ is parameterized by arclength, and $L(\gamma) = L(\tilde{\gamma})$. Hence, by [15, Cor. 3.2], $E(\gamma) = E(\tilde{\gamma})$ since points of γ with $|\gamma| \leq 1$ make no contribution to $E(\gamma)$. Furthermore, the angles $\alpha := \angle(\gamma, \gamma')$ and $\tilde{\alpha} := \angle(\tilde{\gamma}, \tilde{\gamma}')$ are defined almost everywhere, and

$$\alpha = \cos^{-1}(|\gamma'|) = \cos^{-1}(|\tilde{\gamma}'|) = \tilde{\alpha}. \tag{3}$$

Lemma 3.1. *Let $\gamma : \mathbf{R}/L \rightarrow \mathbf{R}^3$ be a minimal inspection curve. Then, $\tilde{\gamma}$ is locally one-to-one.*

Proof. Let U be as in Proposition 2.3. Then, γ and γ' are linearly independent at all regular points of U . So (2) shows $\theta' > 0$ almost everywhere on U , via Cauchy–Schwarz inequality. Hence, θ is strictly increasing on U , which yields that $\tilde{\gamma}$ is star-shaped with respect to o in a neighborhood of each point of U . Since, by Proposition 2.3, γ traces a line segment on each component of $\mathbf{R}/L \setminus U$,

$|\gamma|$ is strictly monotone on each of these components. Hence, $\tilde{\gamma}$ is one-to-one on each component of $\mathbf{R}/L \setminus U$, since $|\tilde{\gamma}| = |\gamma|$. Finally, $\tilde{\gamma}$ is one-to-one in a neighborhood of each point of ∂U , since $\tilde{\gamma}$ is locally star-shaped on U and it maps each component of $\mathbf{R}/L \setminus U$ to a line passing through o . \square

A planar curve $\gamma : [a, b] \rightarrow \mathbf{R}^2$ is *locally convex* provided that it is locally one-to-one and each point $t \in [a, b]$ has a neighborhood $U \subset [a, b]$ such that $\gamma(U)$ lies on the boundary of a convex set. A *side* of a line $\ell \subset \mathbf{R}^2$ is one of the two closed half spaces determined by ℓ . A *local supporting line* ℓ for γ at t is a line passing through $\gamma(t)$ with respect to which $\gamma(U)$ lies on one side. If $\gamma(U)$ lies on a side of ℓ , which contains o , then we say that ℓ lies *above* γ . Finally, if γ is locally convex and through each point of it there passes a local support line, which lies above γ , then we say that γ is *locally convex with respect to* o . Note that if γ is locally convex with respect to o and passes through o , then γ must trace a line segment near o .

Lemma 3.2. *Let $\gamma : \mathbf{R}/L \rightarrow \mathbf{R}^3$ be a minimal inspection curve. Then, $\tilde{\gamma}$ is locally convex with respect to o .*

Proof. Let U be as in Proposition 2.3, and $t \in U$. By Lemma 3.1, there exists a neighborhood V of t in U on which $\tilde{\gamma}$ is one-to-one. Furthermore, $\tilde{\gamma}(V)$ is star-shaped with respect to o . So connecting the end points of $\tilde{\gamma}(V)$ to o by line segments yields a simple closed curve. It is shown in the proof of [15, Proposition 4.3] that this curve bounds a convex set, due to minimality of γ . Thus, $\tilde{\gamma}$ is locally convex with respect to o on U . Next, suppose $t \in \partial U$, and let V be a small neighborhood of t in $\text{cl}(U)$. By Proposition 2.3, $\tilde{\gamma}$ connects one end point of $\tilde{\gamma}(V)$ to o by tracing a line segment. Connect the other end point of $\tilde{\gamma}(V)$ to o by another line segment. Then, the resulting simple closed curve again bounds a convex set by the argument in the proof of [15, Proposition 4.3]. So $\tilde{\gamma}$ is locally convex with respect to o on $\text{cl}(U)$. Finally, $\tilde{\gamma}$ is locally convex with respect to o on the complement of $\text{cl}(U)$, since these regions are mapped to line segments, by Proposition 2.3. \square

4 | SPIRAL DECOMPOSITION

If $\gamma : [a, b] \rightarrow \mathbf{R}^2$ is a locally convex curve, parameterized with constant speed, then its one sided derivatives, γ'_{\pm} , are well-defined everywhere and are nonvanishing [14, Lemma 5.1]. Set $\gamma'(a) := \gamma'_+(a)$. We say that $\gamma : [a, b] \rightarrow \mathbf{R}^2$ is a (*generalized*) *spiral* provided that (i) γ is locally convex with respect to o , (ii) $|\gamma|$ is nondecreasing, and (iii) $\langle \gamma(a), \gamma'(a) \rangle = 0$. A spiral is called *strict* if $|\gamma|$ is increasing. A *spiral decomposition* of a curve $\gamma : [a, b] \rightarrow \mathbf{R}^2$ is a collection U_i of mutually disjoint open subsets of $[a, b]$ such that (i) $\gamma|_{\text{cl}(U_i)}$ is a strict spiral, after switching the direction of $\gamma|_{\text{cl}(U_i)}$ if necessary, and (ii) $|\gamma|' = 0$ almost everywhere on $[a, b] \setminus \cup_i \text{cl}(U_i)$.

Lemma 4.1. *Let $\gamma : \mathbf{R}/L \rightarrow \mathbf{R}^3$ be a minimal inspection curve. Then, $\tilde{\gamma}$ admits a spiral decomposition.*

Proof. The argument follows the same outline as in [15, Proposition 5.2], with minor modifications. Recall that we assume 0 is a local minimum point of $|\gamma|$. If $|\gamma(0)| > 0$, then it follows that $\tilde{\alpha}(0) = \tilde{\alpha}(L) = \pi/2$. Otherwise, $|\tilde{\gamma}(0)| = |\tilde{\gamma}(L)| = 0$, since $|\gamma| = |\tilde{\gamma}|$. Let X be the set of points $t \in [0, L]$ such that $\tilde{\gamma}$ has a local support line at $\tilde{\gamma}(t)$, which is orthogonal to $\tilde{\gamma}(t)$, or $|\tilde{\gamma}(t)| = 0$. Then, $0, L \in X$ and $|\tilde{\gamma}|' = 0$ almost everywhere on X . Also note that X is closed, since the limit of any

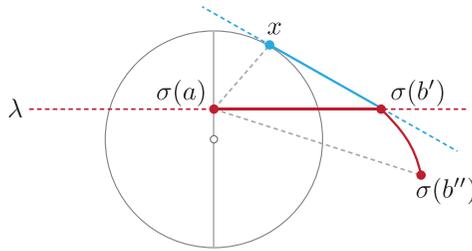


FIGURE 2 Construction of the competing curve.

sequence of support lines of a convex body is a support line, and the set of points with $|\tilde{\gamma}(t)| = 0$ is compact. Consequently each component U of $[0, L] \setminus X$ is an open subinterval of $[0, L]$. It remains to show that $\tilde{\gamma}|_{\text{cl}(U)}$ is a spiral. By Lemma 3.2, $\tilde{\gamma}|_{\text{cl}(U)}$ is locally convex with respect to o . Furthermore, as argued in the proof of [15, Proposition 5.2], $|\tilde{\gamma}'|$ is always positive or always negative at differentiable points of $|\tilde{\gamma}|$ on U . So we may suppose that $|\tilde{\gamma}|$ is increasing on U , after switching the direction of $\tilde{\gamma}|_{\text{cl}(U)}$ if necessary. Finally, let $x \in \partial U$ be the initial point of $\tilde{\gamma}|_{\text{cl}(U)}$. If $|\tilde{\gamma}(x)| = 0$, then $\tilde{\gamma}|_{\text{cl}(U)}$ is a spiral. If $|\tilde{\gamma}(x)| > 0$, it follows that $\tilde{\gamma}(x)$ is orthogonal to $\tilde{\gamma}'_+(x)$, which again shows that $\tilde{\gamma}|_{\text{cl}(U)}$ is a spiral and completes the proof. \square

Let \mathbf{S}^1 denote the unit circle in \mathbf{R}^2 . The last observation quickly yields the following:

Lemma 4.2. *Let $\gamma, \tilde{\gamma}$ be as in Lemma 4.1 and $\sigma : [a, b] \rightarrow \mathbf{R}^2$ be a spiral in the decomposition of $\tilde{\gamma}$. Let $t \in [a, b]$ be a regular point of both σ and γ , and ℓ be the tangent line of σ at t . Suppose that ℓ crosses \mathbf{S}^1 . Then, $\sigma([a, t])$ lies on ℓ .*

Proof. Let $\bar{\ell}$ be the tangent line of γ at t . If ℓ crosses \mathbf{S}^1 , then $\bar{\ell}$ crosses \mathbf{S}^2 , by (3). In particular, $\bar{\ell}$ intersects the interior of $\text{conv}(\gamma)$. Then, Lemma 2.2 completes the proof. \square

The key point in the proof of Theorem 1.1 is the following:

Proposition 4.3. *Let $\sigma : [a, b] \rightarrow \mathbf{R}^2$ be a spiral in the unfolding of a minimal inspection curve. Then, $E(\sigma) \leq 2$. Furthermore, if $|\sigma(a)| < 1$, then $E(\sigma) < 2$.*

Proof. If $|\sigma(a)| \geq 1$, then $E(\sigma) \leq 2$ by [15, Proposition 2.7]. So we assume $|\sigma(a)| < 1$. We may also assume $|\sigma(b)| > 1$ for otherwise $H(\sigma) = 0$, which yields $E(\sigma) = 0$. Let b' be the supremum of $t \in [a, b]$ such that $\sigma([a, t])$ is a line segment. By Lemma 2.1, $|\sigma(b')| \geq 1$. We may assume that $\sigma(a)$ lies on the nonnegative portion of the y -axis, and $\sigma([a, b'])$ lies to the right of the y -axis, see Figure 2. If $b' < b$, then we may choose $b' < b'' < b$ such that $\sigma([b', b''])$ is convex, and lies to the right of the y -axis. Since σ is locally convex with respect to o , $\sigma([b', b''])$ lies below the line λ spanned by $\sigma([a, b'])$, if $|\sigma(a)| > 0$. If $|\sigma(a)| = 0$, we may still assume that $\sigma([b', b''])$ lies below λ after a reflection. Consider the line that passes through $\sigma(b')$ and is tangent to the upper half of \mathbf{S}^1 , say at a point x . Let τ be the curve obtained by joining the line segment $x\sigma(b')$ to the beginning of $\sigma|_{[b', b]}$. We will show that (i) τ is a spiral and (ii) $E(\sigma) < E(\tau)$. Then, we are done, because $E(\tau) \leq 2$ since its initial height is ≥ 1 .

First, we check that τ is a spiral. This is obvious if $b' = b$. So assume $b' < b$, and let $b' < b'' < b$ be as defined above. It suffices to check that τ is locally convex at $\sigma(b')$. Connect the end points of

the portion $x\sigma(b'')$ of τ to $\sigma(a)$ to obtain a closed curve Γ . Note that Γ is simple since $x\sigma(b')$ lies above λ while $\sigma([b', b''])$ lies below it. Let θ be the interior angle of Γ at $\sigma(b')$. We need to show $\theta \leq \pi$. To this end, let $t_i \in (b', b'')$ be a sequence of regular points of σ converging to b' , and ℓ_i be tangent lines of σ at t_i . Then, ℓ_i converge to a support line of $\sigma([b', b''])$ at $\sigma(b')$, which we call ℓ . By Lemma 4.2, ℓ_i do not cross S^1 . Consequently, ℓ does not cross S^1 either. So ℓ also supports $x\sigma(b')$. Hence, ℓ is a support line of Γ at $\sigma(b')$, which yields that $\theta \leq \pi$ as desired.

It remains to check that $E(\sigma) < E(\tau)$. To see this, consider the triangle $\sigma(a)x\sigma(b')$. The interior angle of this triangle at x is $\geq \pi/2$, since $\sigma(a)$ lies on the nonnegative portion of the y -axis. Hence, $|x\sigma(b')| < |\sigma(a)\sigma(b')|$, which yields $L(\tau) < L(\sigma)$. On the other hand, tangent planes of S^2 intersect $\mathbf{R}^2 \simeq \mathbf{R}^3 \times \{0\} \subset \mathbf{R}^3$ in lines that do not cross S^1 , and any such line has exactly the same number of transverse intersections with σ as it does with τ . Hence, $H(\tau) = H(\sigma)$ by definition of horizon. So $E(\sigma) < E(\tau)$ as desired. □

5 | PROOF OF THEOREM 1.1

Set $r = 1$ and let $\gamma : \mathbf{R}/L \rightarrow \mathbf{R}^3$ be a minimal inspection curve, as discussed in Section 2. To establish (1), it suffices to show then that $E(\gamma) \leq 2$, as outlined in Section 1. In Section 3, we established that $E(\gamma) = E(\tilde{\gamma})$ where $\tilde{\gamma} : [0, L] \rightarrow \mathbf{R}^2$ is the unfolding of γ . By Lemma 4.1, $\tilde{\gamma}$ admits a spiral decomposition, generated by a collection of mutually disjoint open sets $U_i \subset [0, L]$, $i \in I$. Set $U_0 := [0, L] \setminus \cup_i \text{cl}(U_i)$, and let $\tilde{\gamma}_i := \tilde{\gamma}|_{\text{cl}(U_i)}$, $\tilde{\gamma}_0 := \tilde{\gamma}|_{U_0}$. As in the proof of Zalgaller’s conjecture in [15, section 10], we have

$$E(\tilde{\gamma}) = \frac{H(\tilde{\gamma})}{L(\tilde{\gamma})} = \frac{1}{L(\tilde{\gamma})} \sum_i H(\tilde{\gamma}_i) = \frac{1}{L(\tilde{\gamma})} \left(L(\tilde{\gamma}_0)E(\tilde{\gamma}_0) + \sum_i L(\tilde{\gamma}_i)E(\tilde{\gamma}_i) \right). \tag{4}$$

By Lemma 2.2, every point $t \in [0, L]$ with $|\gamma(t)| < 1$ lies on a line segment in γ with end points on S^2 , and thus $\tilde{\gamma}(t)$ belongs to a strict spiral (with origin of the spiral corresponding to the midpoint of that line segment). So $|\tilde{\gamma}_0| \geq 1$. Then, as described in [15, section 10], $E(\tilde{\gamma}_0) \leq 2$. Furthermore $E(\tilde{\gamma}_i) \leq 2$ for all i by Proposition 4.3. So $E(\tilde{\gamma}) \leq 2$ by (4), as desired. To characterize the case of equality in (1), note that by (4), if $E(\tilde{\gamma}) = 2$, then $E(\tilde{\gamma}_i) = 2$. Consequently, by Proposition 4.3, $|\tilde{\gamma}_i| \geq 1$. So $|\tilde{\gamma}| \geq 1$, which yields $|\gamma| \geq 1$. Hence, by the proof of Zalgaller’s conjecture [15, Theorem 1.1], γ is the baseball curve.

APPENDIX: HIGHER DIMENSIONS

Here, we establish a higher dimensional version of (1) due to Fedor Nazarov:

Theorem A.1 (Nazarov). *Let $\gamma : [a, b] \rightarrow \mathbf{R}^n$ be a curve of length L , and r be the inradius of the convex hull of γ . Then,*

$$L \geq Cn\sqrt{n}r, \tag{A.1}$$

where $C > 0$ is an absolute constant.

By *absolute constant*, here we mean that C does not depend on n or γ . A Hamiltonian path in the edge graph of the *cross polytope*, that is, the unit ball with respect to the L^1 -norm in \mathbf{R}^n , gives

an example of a curve with $L \leq 2n\sqrt{2nr}$ [2]. Thus, (A.1) is sharp up to the constant C . To establish (A.1), we may set $r = 1$. Furthermore, we may assume that n is even. Indeed suppose that (A.1) holds for even n . If n is odd and bigger than 1, then we may project γ into \mathbf{R}^{n-1} to obtain $L \geq C(n-1)^{3/2} \geq (C/2)n^{3/2}$. Finally, it is enough to show that if $L \leq Cn\sqrt{n}$, for some absolute constant C , then the inradius of $\text{conv}(\gamma) \leq 1$, which means that there exists $u \in \mathbf{S}^{n-1}$ such that $\langle \gamma(t), u \rangle \leq 1$ for all $t \in [a, b]$. Equivalently, if $L \leq 2n\sqrt{n}$, then $\langle \gamma(t), u \rangle \leq C/2$. In summary, it suffices to show the following:

Proposition A.2. *Let $\gamma : [a, b] \rightarrow \mathbf{R}^{2n}$ be a curve of length $\leq 2n\sqrt{n}$. Then, there exists $u \in \mathbf{S}^{2n-1}$ such that $\langle \gamma(t), u \rangle \leq C$ for all $t \in [a, b]$.*

To prove the above proposition, we again assume that γ has constant speed. Let $t_i \in [a, b]$, $i = 0, \dots, n$, be equidistant points with $t_0 := a$, $t_n := b$, and set $s_i := (t_{i-1} + t_i)/2$ for $i = 1, \dots, n$. Let H be an n -dimensional subspace of \mathbf{R}^{2n} , which is orthogonal to each $\gamma(s_i)$, and $\bar{\gamma}$ be the projection of γ into H . Then, $\bar{\gamma}|_{[t_{i-1}, s_i]}$, $\bar{\gamma}|_{[s_i, t_i]}$ are curves of length $\leq \sqrt{n}$ with one end at o , since γ has constant speed. So, identifying H with \mathbf{R}^n , we have reduced Proposition A.2 to the following:

Proposition A.3. *Let $\gamma_i : [a, b] \rightarrow \mathbf{R}^n$, $i = 1, \dots, 2n$, be curves of length $\leq \sqrt{n}$ with $\gamma_i(a) = o$. Then, there exists $u \in \mathbf{S}^{n-1}$ such that $\langle \gamma_i(t), u \rangle \leq C$ for all $t \in [a, b]$.*

To prove the last proposition, we employ the standard Gaussian measure, which is defined for Borel sets $A \subset \mathbf{R}^n$ as

$$\mu(A) := \frac{1}{(\sqrt{2\pi})^n} \int_A e^{-|x|^2/2} d\lambda(x),$$

where λ is the n -dimensional Lebesgue measure. We also record that if K_i are a family of convex sets that are symmetric with respect to o , then

$$\mu\left(\bigcap_i K_i\right) \geq \prod_i \mu(K_i) \tag{A.2}$$

by the Gaussian correlation inequality [16, 19]. Here, we need this fact only for slabs, which had been established in [20].

Proof of Proposition A.3. We set $[a, b] = [0, 1]$ and assume that γ_i have constant speed. For every $t \in [0, 1]$ and i there exist vectors $v_{ik}(t) \in \mathbf{R}^n$, such that

$$\gamma_i(t) := \sum_{k=1}^{\infty} v_{ik}(t), \quad \text{and} \quad |v_{ik}(t)| \leq \frac{\sqrt{n}}{2^k}.$$

To generate these vectors, set $t_0 := 0$, and let $t_k := t_{k-1} - 1/2^k$, if $t < t_{k-1}$, and $t_k := t_{k-1} + 1/2^k$ otherwise. Then, we set $v_{ik}(t) := \gamma_i(t_k) - \gamma_i(t_{k-1})$. Note that each $v_{ik}(t)$ is chosen from a set V_{ik} ,

of cardinality 2^{k-1} , which is independent of t . Now consider the slabs

$$S(v) := \left\{ x \in \mathbf{R}^n \mid |\langle x, v \rangle| \leq \frac{\sqrt{n}}{k^2} \right\}, \quad v \in V_{ik},$$

which have width $2(\sqrt{n}/k^2)/|v| \geq 2(2^k/k^2)$, and set

$$A := \bigcap_{i=1}^{2n} \bigcap_{k=1}^{\infty} \bigcap_{v \in V_{ik}} S(v).$$

By Fubini's theorem, and a standard estimate for the Gaussian integral,

$$\mu(S(v)) \geq \frac{1}{\sqrt{2\pi}} \int_{-a_k}^{a_k} e^{-t^2/2} dt \geq 1 - e^{-a_k^2/2},$$

where $a_k := 2^k/k^2$. So by (A.2),

$$\mu(A) \geq \prod_{i=1}^{2n} \prod_{k=1}^{\infty} \prod_{v \in V_{ik}} \mu(S(v)) \geq \left(\prod_{k=1}^{\infty} (1 - e^{-a_k^2/2})^{2^{k-1}} \right)^{2n}.$$

Since $\ln(1 - e^{-x}) \geq -2e^{-x}$ for $x \geq 32/81$, which is the smallest value of $a_k^2/2$ (achieved for $k = 3$), we have

$$\begin{aligned} \prod_{k=1}^{\infty} (1 - e^{-a_k^2/2})^{2^{k-1}} &= \exp \left(\sum_{k=1}^{\infty} 2^{k-1} \ln(1 - e^{-a_k^2/2}) \right) \\ &\geq \exp \left(- \sum_{k=1}^{\infty} 2^k e^{-a_k^2/2} \right) =: \sqrt{\delta} > 0. \end{aligned}$$

So we conclude that $\mu(A) \geq \delta^n$ where $\delta > 0$ is an absolute constant. Next note that, if B_r^n is the ball of radius r centered at o in \mathbf{R}^n , with volume $|B_r^n|$, then

$$\mu(B_r^n) \leq \frac{|B_r^n|}{(\sqrt{2\pi})^n} = \left(\frac{\sqrt{e} r}{\sqrt{n}} \right)^n \frac{|B_{\sqrt{n}}^n|}{(\sqrt{2\pi})^n (\sqrt{e})^n} \leq \left(\frac{\sqrt{e} r}{\sqrt{n}} \right)^n \mu(B_{\sqrt{n}}^n) \leq \left(\frac{\sqrt{e} r}{\sqrt{n}} \right)^n.$$

So if $r := \delta \sqrt{n}/\sqrt{e}$, then $\mu(B_r^n) \leq \delta^n \leq \mu(A)$. Consequently, $A \not\subset \text{int}(B_r^n)$, which means that there exists $u_0 \in A$ with $|u_0| \geq r$. Now setting $u := u_0/|u_0|$, we have

$$\langle \gamma_i(t), u \rangle = \sum_{k=1}^{\infty} \langle v_{ik}, u \rangle \leq \frac{1}{r} \sum_{k=1}^{\infty} \langle v_{ik}, u_0 \rangle \leq \frac{\sqrt{e}}{\delta \sqrt{n}} \sum_{k=1}^{\infty} \frac{\sqrt{n}}{k^2} \leq \frac{2\sqrt{e}}{\delta} =: C,$$

as desired. □

Note A.4. When γ_i in Proposition A.3 trace line segments, we obtain the following result in discrete geometry: If $N \leq 2n$ points in \mathbf{R}^n contain \mathbf{S}^{n-1} within their convex hull, then at least one of them has distance $\geq \sqrt{n}/C$ from o . Equivalently, if $N \leq 2n$ disks of geodesic radius ρ cover \mathbf{S}^{n-1} , then $\cos(\rho) \leq C/\sqrt{n}$, which had been observed earlier by Tikhomirov [21]. Furthermore, proof of Proposition A.3 allows an estimate for C as follows. If γ_i trace line segments, we may set $k = 1$. Then, $\mu(S(v)) \geq \int_{-2}^2 e^{-t^2/2} dt / \sqrt{2\pi} \geq 0.95$. So $\delta = (0.95)^2$, which yields $C = \delta/(2\sqrt{e}) \simeq 3.65$. It has been conjectured that the optimal value of C is 1, which would correspond to the case where the points form the vertices of a cross polytope [5, Conjecture 1.3]. This has been shown only for $n = 3$ [10], see [11, p. 34], and $n = 4$ [9].

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