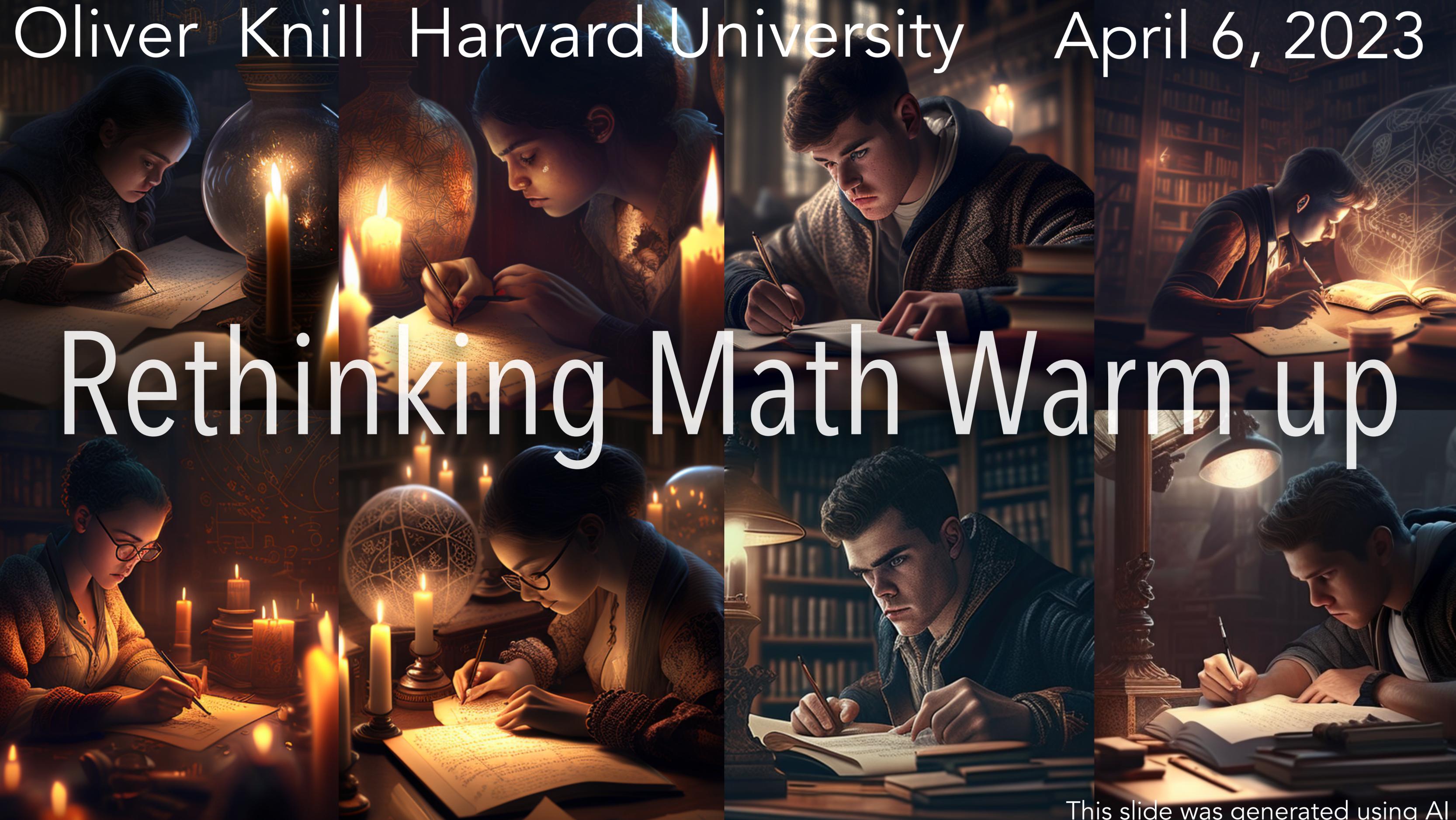


Oliver Knill Harvard University

April 6, 2023



# Rethinking Math Warm up

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Oliver Knill Harvard University

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History



# History

# Preceptors

Robin Gottlieb 1983  
Otto Bretscher 1991  
Robert Winters 1994  
Tamara Lefcourt 1999  
John Boller 1999  
Andy Engelward 1999  
John Mackey 2000  
Dale Winter 2000  
Oliver Knill 2000  
Tom Judson 2002  
Matt Leingang 2003  
Dereck Bruff 2003  
Angela Classen 2004  
Bret Benesh 2006  
Janet Chen 2006  
John Hall 2007

Peter Garfield 2008  
Juliana Belding 2008  
Jameel Al-Aidros 2008  
Rehana Patel 2009  
Meghan Anderson 2011  
Meredith Hegg 2012  
Sarah Chisholm 2013  
YuWen Hsu 2014  
Rosalie Belanger 2014  
Brendan Kelly 2015  
Neha Gupta 2016  
Dusty Grundmeyer 2015  
Matt Demers 2015  
Kate Penner 2016  
Hakim Walker 2017  
Stepan Paul 2017

Emily Braley 2017  
Caroline Junkins 2017  
Drew Zemke 2018  
Jill Guerra 2018  
David Freund 2019  
Voula Collins 2019  
Reshma Menon 2019  
Rose Bongers 2020  
Roderic Guigo 2021  
Amadeus Martin 2021  
Dylan Quintana 2021  
Allechar Serrano 2021  
Matthew Cavallo 2021  
Erica Dinkins 2022  
Hannah Constantin 2022



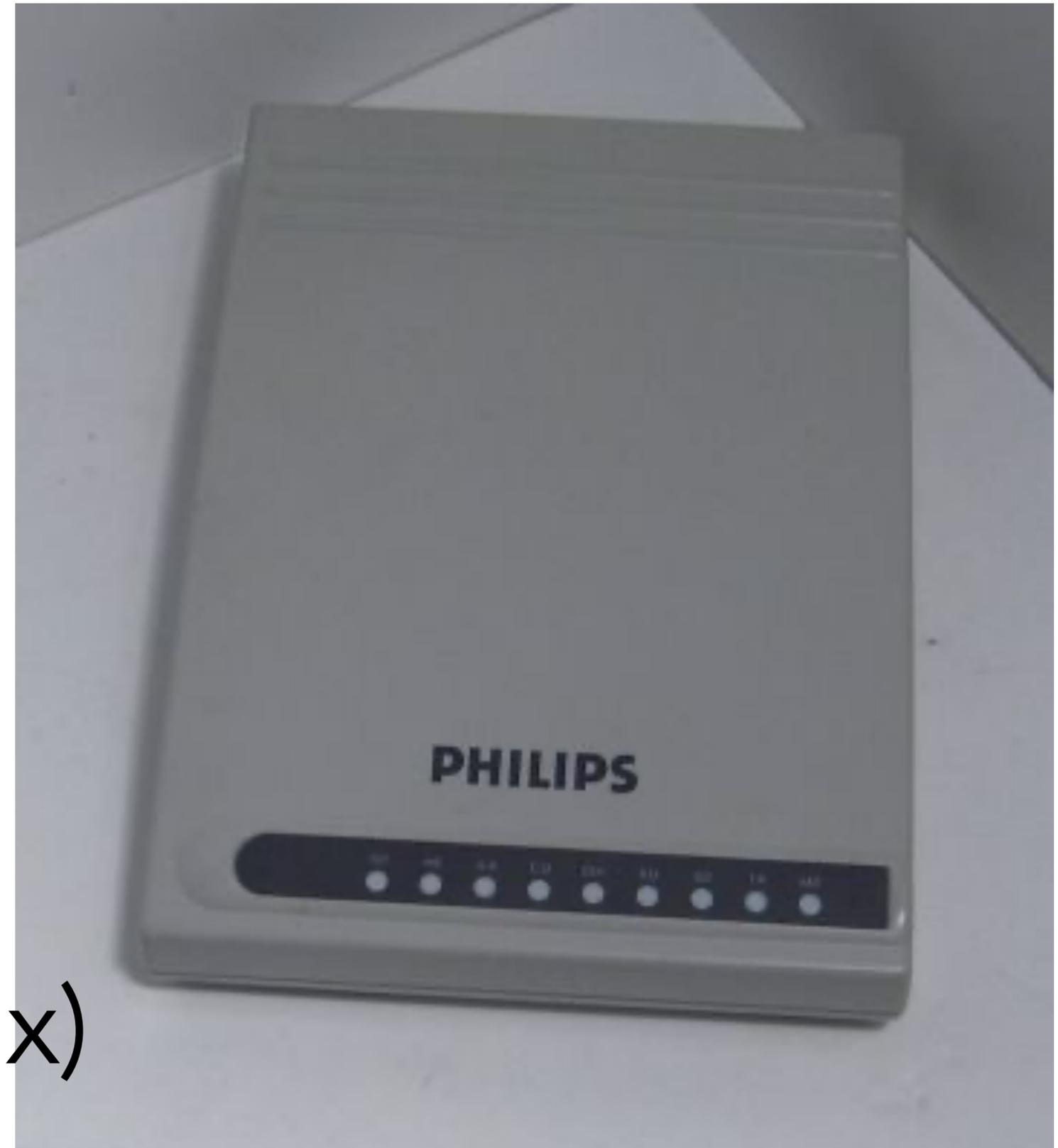
2017



Changes ....

# In 2001:

we still used  
telephone modems to  
connect to Harvard  
(scripts needed to be  
self programmed in linux)



# Developments

word wide web HTML5  
flash social CMS  
java Twitter AI bots

Educational TV VHS Computer Games YouTube  
DVD flash Shorts  
TikTok

since 30ies: slide projectors iphone 3Dprint Zoom  
since 40ies: overhead projectors Oculus Metaverse  
since 60ies: PRS power point keynote FPV drones

Computer algebra systems Maxima Matlab Wolfram  
Reduce Macyma Cayley Mathematica, Maple, alpha desmos  
Magma

Pocket Programmable Graphing Calculators Ipad  
calculators calculators calculators with CAS Alpha

ELIZA AI winter Google Chatbot Sofia Siri Alexa ChatGPT  
Midjourney

1970

1980

1990

2000

2010

2020

It is personal ... .

Olivetti P6040



Olivetti P6040, 1977  
16 character diode display!  
was our school computer at high school

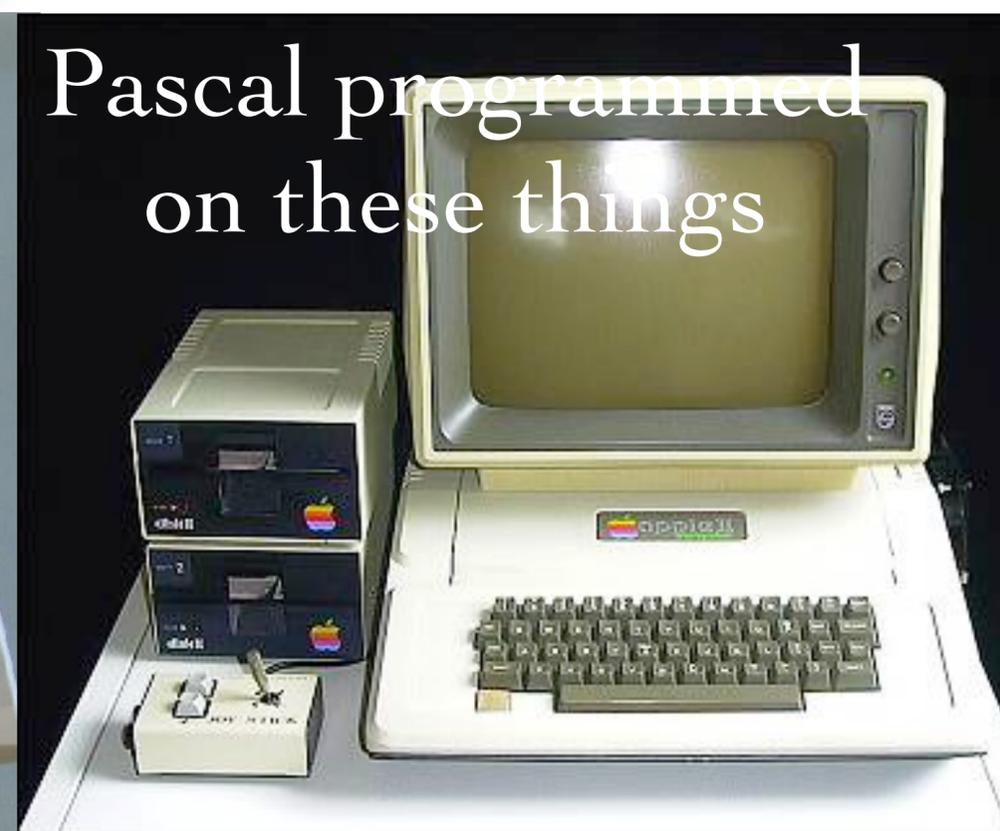
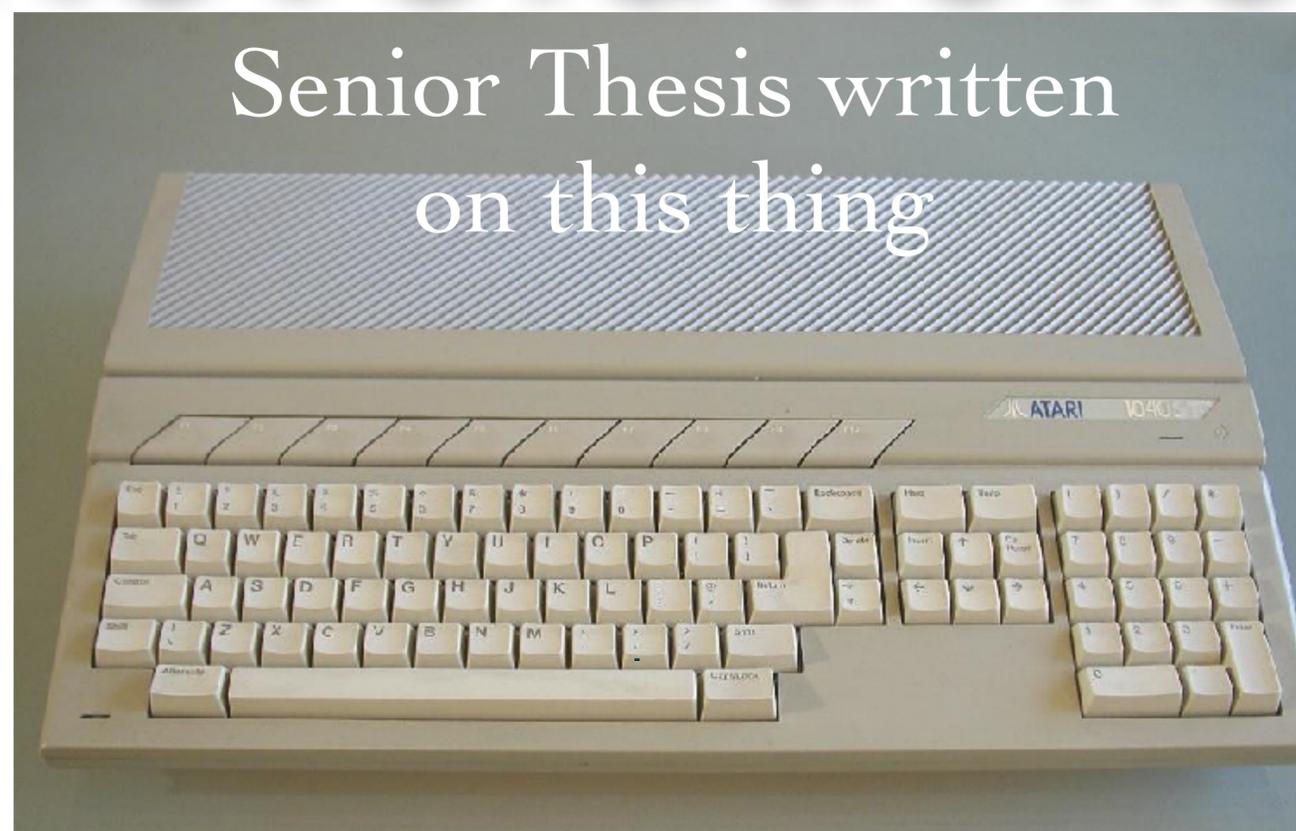
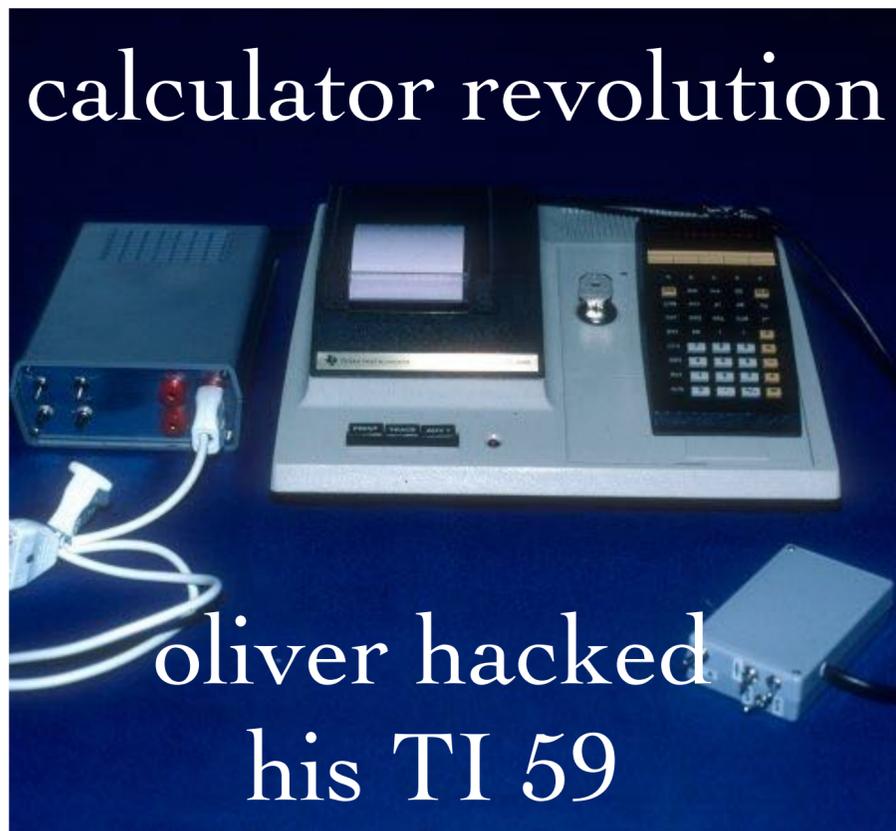
oliver

High school Schaffhausen





# Revolutions



WWW revolution, DVD games, apps



Wikis, blogs, e-books, video, social media, content management systems



CAS, email, digital photography, 3D printing, AI bots

# My own personal firsts

- 1972 first time calculator used
- 1977 first programs on calculator
- 1981 first real computer owned
  - 1983 first websurf (telnet)
- 1987 first time Mathematica used
- 1991 first gopher websurf
  - 2001 run webwork at Harvard
- 2003 built chatbot for classroom CAS
  - 2006 first youtube created
- 2010 first tablet: ipad
- 2014 first hybrid taught at extension
- 2017 first course taught with zoom
- 2023 first youtube short
- 2023 slides generated AI assisted
- 1978 first remote course taken (TV)
- 1984 first time email used
- 1988 first modem connection to school
- 1988 Latex for paper creation
- 1994 Computer algebra in classroom
- 1994 first course website made (calculus)
- 2012 first twitter for courses
- 2013 action cams in in classroom
- 2013 3D printing for classroom
- 2013 first Pecha Kucha talk
- 2018 first Computer Games in Classroom
- 2022 first Chat GPT assistance

Changes are fast ...

2021



2022



Old Series

# 2004

Remember: The web as we know was at that time when the web was still young. We just transitioned from Modem connections to broad band. No iphones, social networks existed yet. Facebook went live just in this year. MQC was in Memorial Hall, and we were all younger ...

## Math Warm Up Series

Math Warm Up Series of 2004, calculus faculty ran optional course specific Preview/Review Sessions about various topics:



**MOV, Ogg Webm.**

**Robin Gottlieb: Logarithms I**



**MOV, Ogg Webm**

**Robin Gottlieb: Logarithms II**



**MOV, Ogg Webm.**

**Amanda: Derivatives**



**MOV, Ogg Webm.**

**Angels, Arrows, and Limits in Calculus**



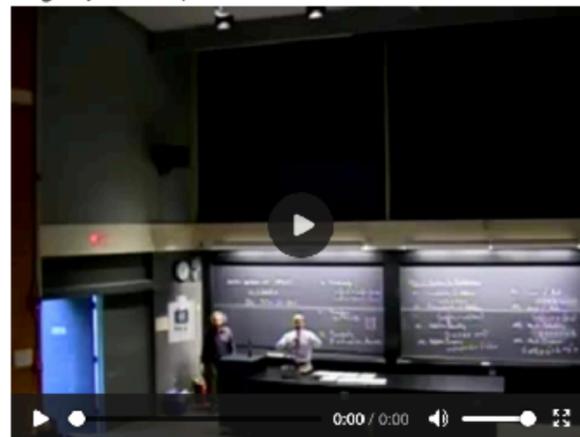
**MOV, Ogg Webm**

**Using Math Help Resources Effectively**



**MOV, Ogg Webm.**

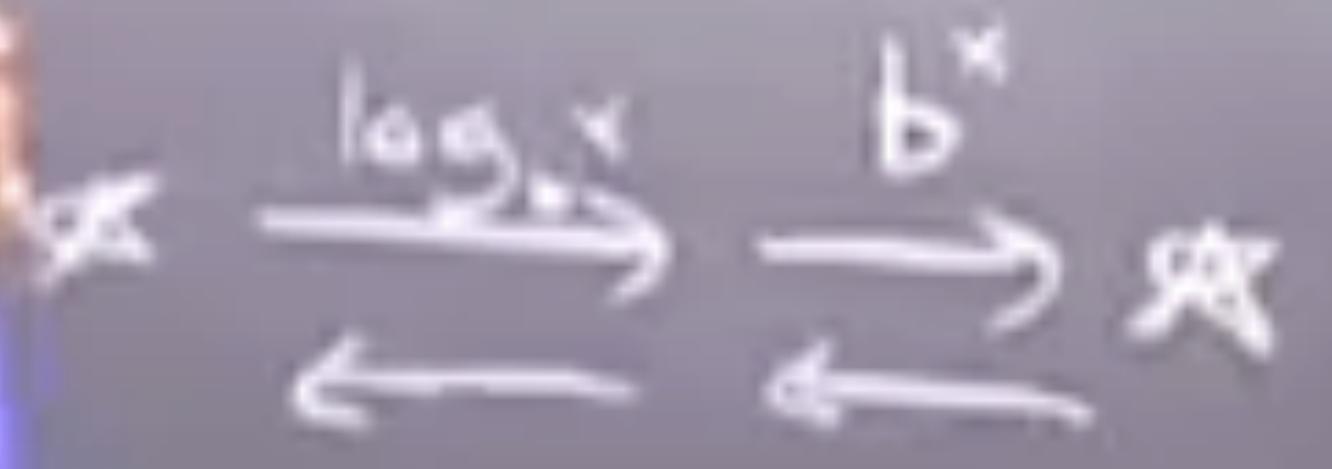
**Complex Numbers**



<https://legacy-www.math.harvard.edu/mwus/index.html>







$$1/x$$

$$10^{\log x} = x$$

$$\log_{10} 10^x = x$$

$$e^{\ln x} = x$$

$$\ln e^x = x$$

$u + 0 = u$   
 $u + 0 = u + (0+0)$  A3  
 $u + 0 = (u+0) + (0+0)$  A3  
 $u + 0 = (u+0) + 0$  A4 (A3)

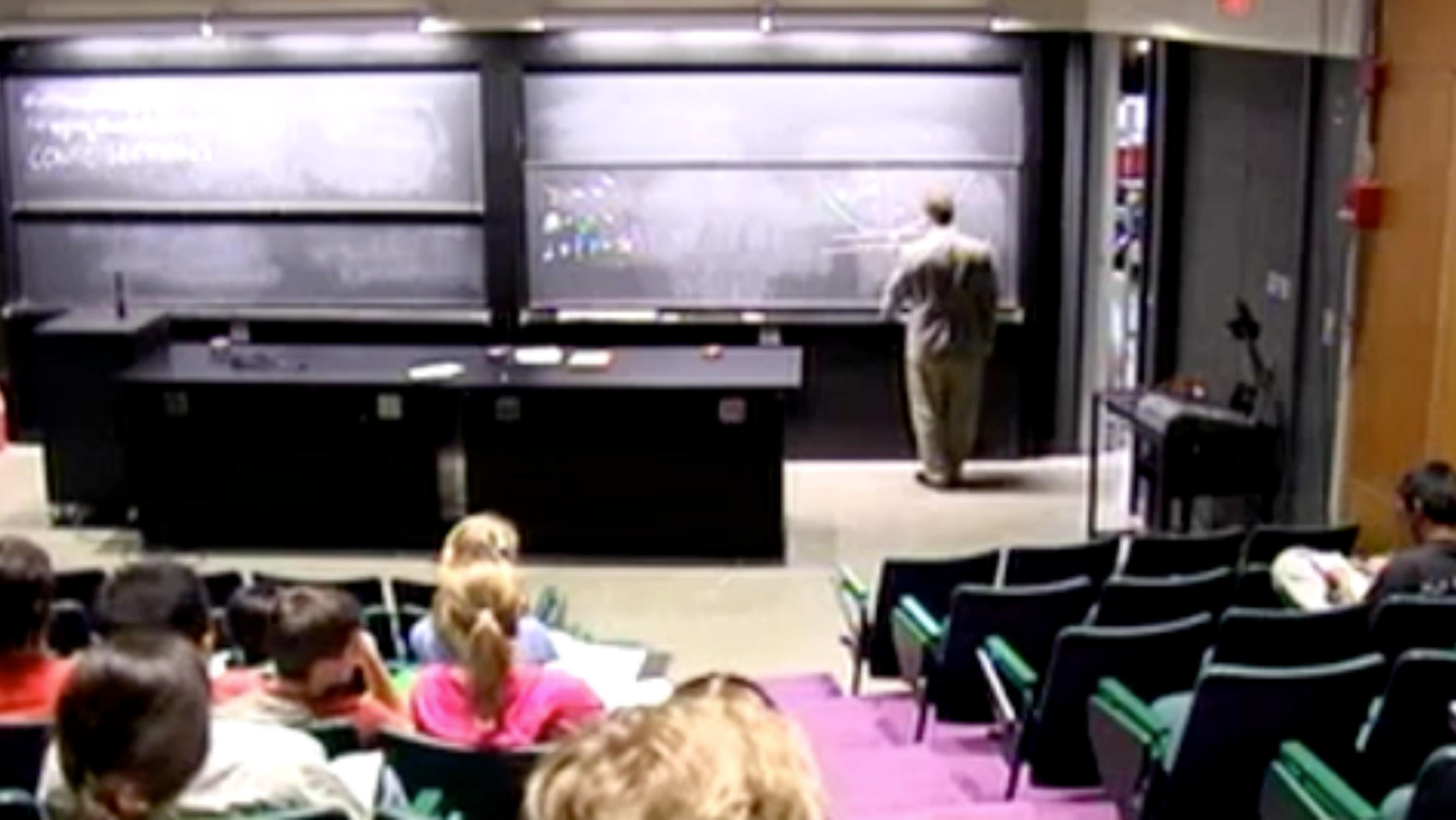
$0 + u = u$  A3  
 $0 + u = 0 + (0+u)$  A3  
 $0 + u = (0+0) + u$  A4  
 $0 + u = 0 + u$  A3  
 کیا فرق

تعمیراتی اصول  
 • تعمیراتی اصول  
 • تعمیراتی اصول  
 • تعمیراتی اصول

FOTU  
 $(a+b) + c = a + (b+c) = (a+b) + c = a + (b+c)$

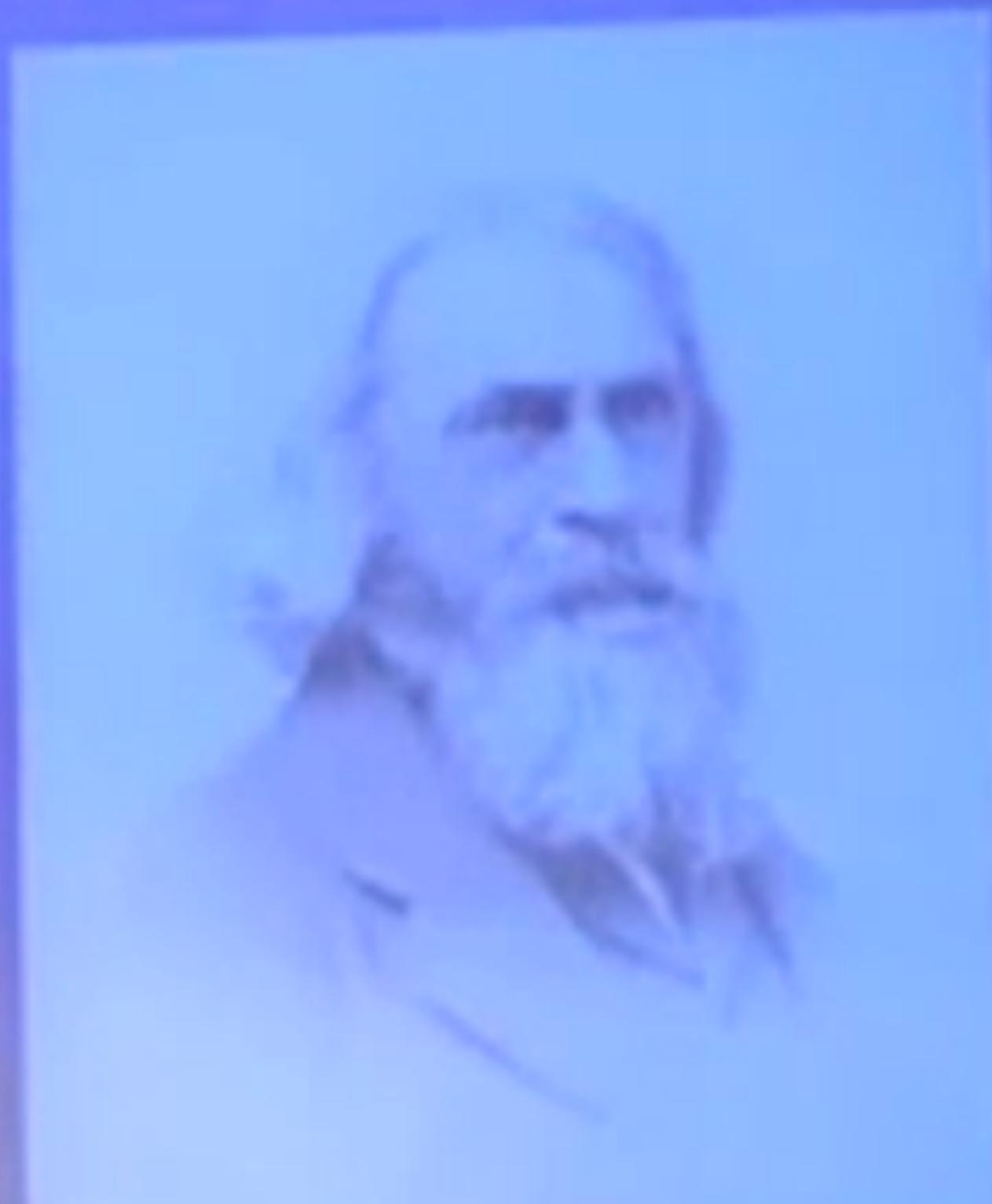






# Benjamin Peirce: At Harvard

Gentlemen, that is surely true, it is absolutely paradoxical, we cannot understand it, and we don't know what it means. but we have proved it, and



# Trig Handout

## Trigonometry Review Problems

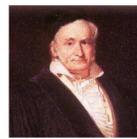
1. The gravitational pull of the sun and moon on large bodies of water produces tides. Tides generally rise and fall twice every 25 hours. The range between high and low tide varies greatly with location. On the Pacific coast of America this range can be as much as 15 feet. The Bay of Fundy in New Brunswick has an extremely dramatic range of 45 feet. Can we model the tide level as a function of time?
2.
  - (a) For what values of  $x$  is  $\sin x = 0$ ?
  - (b) For what values of  $x$  is  $\cos x = 0$ ?
  - (c) How can we simplify  $\sin(x + 2\pi)$ ?
  - (d) How can we simplify  $\cos(x - 6\pi)$ ?
3.
  - (a) Suppose that an angle measuring  $\theta$  radians subtends an arc of a circle of radius  $r$  units. What is the length of that arc?
  - (b) Suppose that an angle measuring  $\alpha$  degrees subtends an arc of a circle of radius  $r$  units. What is the length of that arc?
4. Use the symmetry of the unit circle to find the values of each of the following

# Example Handout

## An hour on complex numbers Harvard University, 9/23/04, O.Knill

"The shortest path between two truths in the real domain passes through the complex domain."  
Jacques Hadamard (1865-1963)

THE SYMBOL I. Introducing the symbol  $i = \sqrt{-1}$  and extending all usual calculation rules using  $i^2 = -1$  leads to the algebra of complex numbers  $z = a + ib$ . For example,  $z = 17 - 12i$  is a complex number. Real numbers like  $z = 3.2$  are considered complex numbers too. The mathematician Johann Carl Friedrich Gauss (1777-1855) was one of the first to use complex numbers seriously in his research even so in as late as 1825 still claimed that "the true metaphysics of the square root of -1 is elusive".



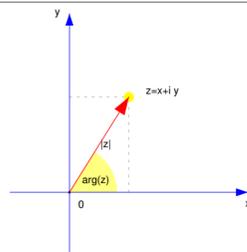
## EULER FORMULA. The identity

$$\cos(\theta) + i \sin(\theta) = e^{i\theta}$$

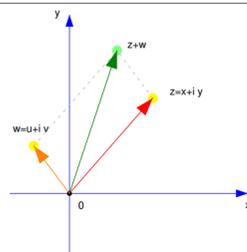
can be seen by staring at the power series  $\cos(x) = 1 - x^2/2! + x^4/4! - \dots$ ,  $\sin(x) = x - x^3/3! + x^5/5! - \dots$  and  $e^x = 1 + x + x^2/2! + x^3/3! + \dots$ . If you should have defined  $\sin(x) = (e^x - e^{-x})/(2i)$ ,  $\cos(x) = (e^x + e^{-x})/2$ , the proof of the Euler formula is obvious.



POLAR REPRESENTATION. Because complex numbers  $z = x + iy$  can be realized as vectors  $(x, y)$  in the plane, we can represent them in polar coordinates  $z = x + iy = r \cos(\theta) + ir \sin(\theta)$ . Euler's formula gives  $z = r e^{i\theta}$ . The plane is also called the **complex plane** or the **Gauss plane**.

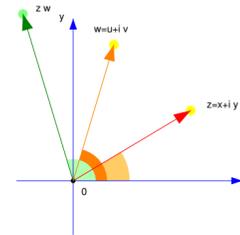


ADDITION.  $z = x + iy, w = u + iv$   $z + w = (x + u) + i(u + v)$ . Adding  $-w = u + iv$  to  $z$  is called subtraction and denoted by  $z - w$ . In the Gauss plane, addition can be done by drawing the parallelogram spanned by the vectors  $(x, y)$  and  $(u, v)$  to get the vector  $(x + u, y + v)$ .



$$\text{Examples: } (5 + 7i) + (3 - 4i) = 8 + 3i. \\ (3 + i) - (2 + i) = 1.$$

MULTIPLICATION. With  $z = x + iy$  and  $w = u + iv$  define  $zw = (xu - yv) + i(xv + uy)$ . Because multiplying  $z = r e^{i\theta}$  and with  $w = s e^{i\phi}$  gives  $zw = r s e^{i(\theta + \phi)}$ , we see that the length of the product  $|zw|$  is the product of the lengths  $|z||w|$  of the  $z$  and  $w$  and that the polar angle  $\theta + \phi$  of  $zw$  is the sum of the polar angles  $\theta$  and  $\phi$  of  $z$  and  $w$ .



$$\text{Examples: } (3 + 2i)(1 - i) = 5 - i. \\ (1 + i)^2 = 2i.$$

DE MOIVRE MAGIC. With  $z = e^{i\theta}$ , we have  $z^n = e^{in\theta}$  and so

$$(\cos(\theta) + i \sin(\theta))^n = e^{in\theta}$$

Writing out the real and imaginary part leads to interesting identities. For example, for  $n = 3$ , we get  $\cos(3\theta) + i \sin(3\theta) = (\cos(\theta) + i \sin(\theta))^3 = \cos^3(\theta) - 3 \cos(\theta) \sin^2(\theta) + i (3 \cos^2(\theta) \sin(\theta) - \sin^3(\theta))$ . Comparing real and imaginary parts gives identities which would be harder to derive without this magical stunt.



AN AMAZING FORMULA.

$$1 + e^{i\pi} = 0$$

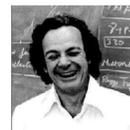
combines the constants  $0, 1, e, \pi$  in a wonderful way. It is considered one of the 5 most beautiful formulas in mathematics. Richard Feynmann called it as a 15 year old

"the most remarkable formula in math".

In the book of E. Kasner and J. Newman, "Mathematics and the Imagination", Benjamin Peirce is quoted after proving this formula here in front of a Harvard class:

Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means. But we have proved it, and therefore, we know it is the truth.

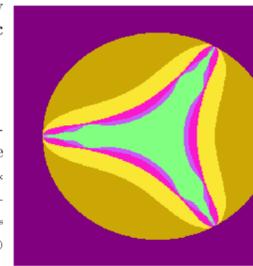
We can only repeat that statement modifying the start of the sentence to "Ladies and Gentlemen" of course.



MANDELBAR SET. The same construction can be done by replacing the quadratic map with the **conjugate quadratic map**

$$f_c : z \mapsto \bar{z}^2 + c.$$

The corresponding set is called the **Mandelbar set**. People are interested in this set because its topological properties are different from the one believed to be true for the Mandelbrot set. This is out of the scope but the last big open problem in complex dynamics: one believes that the Mandelbrot set is locally connected meaning that every neighborhood of a point contains a connected open neighborhood. The mandelbar set does not have this property. There are points, where the set resembles the graph of the function  $f(x) = \sin(1/x)$  which is not locally connected at  $(0, 0)$ .

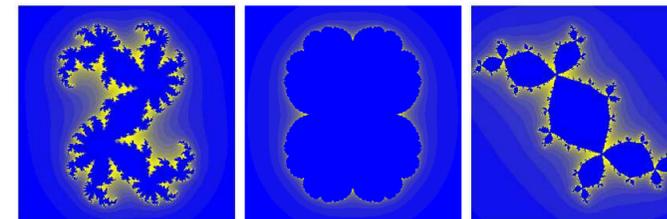


JULIA AND FATOU SETS. The map  $f_c$  leaves a set  $J_c \subset C$  called **Julia set** and its complement  $F_c$  the **Fatou set** invariant.

The Julia set is in general a fractal and at in a complicated way mixed with the Fatou set. It is ironic that Gaston Julia (1893-1978) and Pierre Fatou (1879-1929) were not very well spoken on each other. They competed both for the 1918 'grand priz' of the academie of sciences and produced similar results leading to a priority dispute. Julia was wounded in world war I. He lost his nose and had to wear a leather strap across his face for the rest of his life. He carried on his mathematical researches in hospital.



EXAMPLES OF JULIA SETS.



"The dragon"  $c=0.36+0.1i$  "The cauliflower"  $c=0.25$  "The Douady rabbit"  $c=-0.121 + 0.739i$

Julia sets are in general **fractals** meaning that their dimension is between 1 and 2. By the way, one knows that the boundary of the Mandelbrot set has dimension 2. It is a very complicated set.

## PROBLEMS.

- 1) What is  $(1 + 3i)/(4 + i)$ ?
- 2) What is  $i^{100002}$ ?
- 3) Find all the roots of  $x^2 + 3x + 3$ .
- 4) Find all the solutions of  $x^2 + 9 = 0$ .
- 5) Verify that  $\sin(ix) = i \sinh(x)$ , where  $\sinh(x) = (e^x - e^{-x})/2$ .
- 6) Find all the third roots of  $3 + 4i$ .
- 7) Find  $\log(1 + \sqrt{3}i)$ .
- 8) Derive  $2 \sin(x) \cos(x) = \sin(2x)$ ,  $\cos^2(x) - \sin^2(x) = \cos(2x)$  from de Moivre.
- 9) You iterate the map  $f(z) = z^2 + i$ . What is  $f(f(f(f(z))))$  for  $z = 1$ ?
- 10) Find the prime factorization of the Gaussian integer  $z = 5 + 7i$ .

## APPENDIX: COMPLEX NUMBERS AND GRAPHICS CALCULATORS.

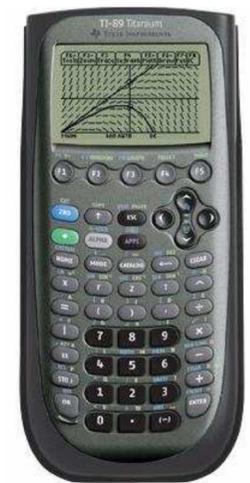
**Pocket Calculators:** While smaller pocket calculators like the TI-30 give an error message when trying to compute  $\sqrt{-1}$ , more advanced calculators like the TI-89 know the arithmetic of complex numbers. The symbol  $i$  is accessible as

< 2nd >< catalog >

(you see the  $i$  symbol above the < catalog > key). Note that graphics calculators should be considered rather a toy. They are not very appropriate on the college level. The graphic abilities of calculators like TI-89 Titanium are very limited. There are many reasons to learn a real computer algebra system (CAS).

**Advantages of graphics calculators:** always on and ready, cheap (50-150 dollars). Often familiar to students from high school.

**Disadvantages of graphics calculators:** small memory, poor graphics, complicated interface, difficult to connect with other programs or other computers. Soon obsolete.



# 2007

**Update:** Videos are now available!

**Tuesday, September 18th:** [Math 1a refresher for Math 1b](#) from 3-5 pm in Hall A, featuring Amanda McCoy

Has it been a while since you took Calculus AB? Former Math 1b course assistant Amanda McCoy will run through the basics from Math 1a that you need to succeed in Math 1b. From tangent lines to derivatives, and from integrals to the Fundamental Theorem of Calculus, Amanda will discuss the big ideas that you should brush up on before entering Math 1b.

**Wednesday, September 19th:** [Precalculus](#) from 3-5 pm in Hall A with Matthew Leingang

Good precalculus skills are a must for Math 1a and beyond. Preceptor Matthew Leingang will remind you of the trigonometry, exponentials, and logarithms that you will need to know before starting the semester.

**Thursday, September 20th:** [Problem Solving](#) from 3-5 pm in Hall A with Robin Gottlieb

What happens when you don't know how to solve a math problem? It could be that you forgot how to do the problem, or it could be that you have encountered a math problem that you have never seen before. Professor Robin Gottlieb will give you a mental toolbox that you help you think like a mathematician to figure out problems that you do not know how to immediately solve. This course is valuable for anyone who is planning on taking any of the calculus classes.

[Math 1b refresher for Math 19a](#) from 7-9 pm in Hall E with Nicole Ali

Are you unsure of what you will need to know to enter Math 19a? Current and former Math 19a course assistant Nicole Ali will highlight all of the important ideas from Math 1b that will pop up again in Math 19a.

**Friday, September 21st:** [Algebra](#) from 3-5pm in Hall A with course assistant Larissa de Lima

Whether you are preparing for the algebra test or just brushing up on algebra skills that have been lying dormant over the summer, former Math X course assistant Larissa de Lima will run you through the basics of how to correctly do algebra.

# Earlier Adventures



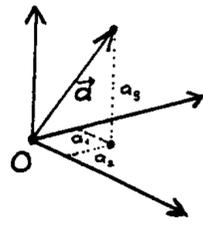
This slide was generated using AI

# WARMUP 1992

I was participating at a warm-up series at ETHZ in 1992 for incoming students. Here is my handout (review of distances). Students then came in with much higher preparation levels.

### Vektoren

Rechtwinkliges Koordinatensystem mit Ursprung  $O$



### Addition, Skalare Multiplikation

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

$$\vec{a} - \vec{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

#### Kommutativität

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

#### Assoziativität

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

#### Nullvektor, Inverses

$$\vec{0} = (0, 0, 0)$$

$$\vec{a} + (-\vec{a}) = \vec{0}$$

#### Distributivität

$$\lambda \cdot (\vec{a} + \vec{b}) = \lambda \cdot \vec{a} + \lambda \cdot \vec{b}$$

$$(\lambda + \mu) \cdot \vec{a} = \lambda \cdot \vec{a} + \mu \cdot \vec{a}$$

### Skalarprodukt

$$\vec{a} \cdot \vec{b} = \langle \vec{a}, \vec{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$$

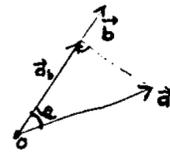
#### Kommutativität

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

#### Linearität

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

$$\vec{a} \cdot (\lambda \vec{b}) = \lambda \vec{a} \cdot \vec{b}$$



#### Norm, Einheitsvektor

$$\vec{a}_e = \frac{\vec{a}}{|\vec{a}|}, |\vec{a}| = \sqrt{a \cdot a}$$

#### Projektion auf Vektor $\vec{b}$

$$\vec{a}_b = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \cdot \vec{b}$$

#### Winkel

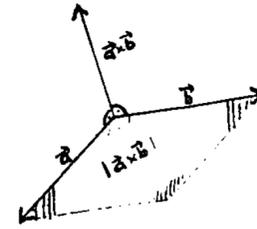
$$|\vec{a} \cdot \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cos \phi$$

#### Orthogonale Vektoren

$$\vec{a} \cdot \vec{b} = 0$$

### Vektorprodukt

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$



#### Antikommutativität

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

#### Linearität

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\vec{a} \times (\lambda \vec{b}) = \lambda \vec{a} \times \vec{b}$$

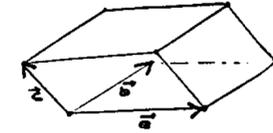
#### Parallelogrammfläche

$$|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \cdot \sin(\phi)$$

#### Parallele Vektoren

$$\vec{a} \times \vec{b} = 0$$

### Spatprodukt



$$[\vec{a}, \vec{b}, \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}) = c_1(a_2 b_3 - a_3 b_2) + c_2(a_3 b_1 - a_1 b_3) + c_3(a_1 b_2 - a_2 b_1)$$

#### Zyklische Symmetrie

$$[\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}]$$

$$[\vec{a}, \vec{b}, \vec{c}] = -[\vec{b}, \vec{a}, \vec{c}]$$

#### Linearität

$$\lambda [\vec{a}, \vec{b}, \vec{c}] = [\vec{a}, \vec{b}, \lambda \vec{c}]$$

$$[\vec{a}, \vec{b}, \vec{c} + \vec{d}] = [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{d}]$$

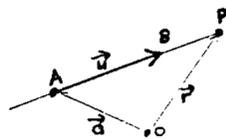
#### Volumen des Parallelepipeds (Spats)

$$|[\vec{a}, \vec{b}, \vec{c}]|$$

#### Koplanare Vektoren

$$[\vec{a}, \vec{b}, \vec{c}] = 0$$

### Geraden



#### Parameterdarstellung

$$\vec{r} = \vec{a} + \lambda \vec{u}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

#### Zweipunkteform

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

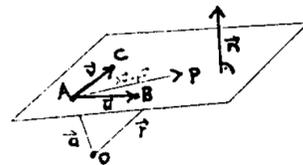
#### Punktnormalenform, Hesseform

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

#### Vektorproduktform

$$(\vec{r} - \vec{a}) \times \vec{u} = 0$$

### Ebenen



#### Parameterdarstellung

$$\vec{r} = \vec{a} + \lambda \vec{u} + \mu \vec{v}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \mu \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

#### Gleichung

$$ax + by + cz = d$$

#### Normale

$$\vec{n} = (a, b, c)$$

#### Dreipunkteform

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) + \mu(\vec{c} - \vec{a})$$

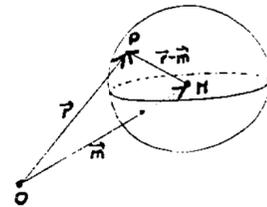
#### Punktnormalenform

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

#### Normale zu $ax + by + cz = d$

$$\vec{n} = (a, b, c)$$

### Kugeln



#### Vektorgleichung

$$(\vec{r} - \vec{m}) \cdot (\vec{r} - \vec{m}) = r^2$$

$$\text{Kugelmittelpunkt } \vec{m} = (m_1, m_2, m_3)$$

#### Radius $r$

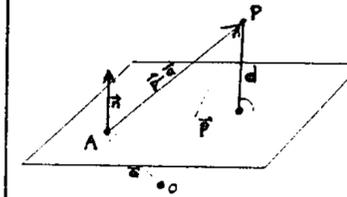
#### Koordinatengleichung

$$(x - m_1)^2 + (y - m_2)^2 + (z - m_3)^2 = r^2$$

#### Einheitskugel

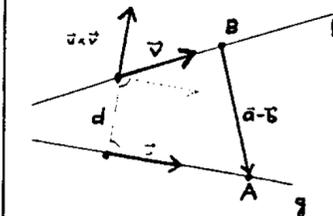
$$x^2 + y^2 + z^2 = 1$$

### Abstandsformeln



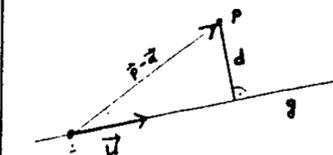
#### Punkt (P) - Ebene (E)

$$d = |(\vec{p} - \vec{a}) \cdot \vec{n}_e|$$



#### Gerade (g) - Gerade (h)

$$d = |(\vec{a} - \vec{b}) \cdot (\vec{u} \times \vec{v})_e|$$



#### Punkt (P) - Gerade (g)

$$d = |\vec{u}_e \times (\vec{p} - \vec{a})|$$



This slide was generated using AI

**ADVENTURE ALGEBRA**



2017



Modern Numbers  
Prime Numbers  
A Computational Perspective

Real and Complex Dynamical Systems  
Mathematical  
Book of Prime Records

Harmonic Analysis and the Theory of Probability  
Using Active Learning

GENIUS  
JAMES GLEICK  
COPING WITH CHAOS

THE MATH MYTH  
Visualizing Mathematics with 3D Printing

THE HISTORY OF STATISTICS  
THE EMERGENCE OF PROBABILITY

Convex Polytopes

Final mathematics

$\frac{1}{2}$   
 $\frac{1}{4}$   
 $\frac{1}{8}$

$\frac{3}{4}$

$\frac{1}{10}$   
 $\frac{1}{10}$   
 $\frac{1}{10}$   
 $\frac{1}{10}$

$\frac{1}{12}$   
 $\frac{1}{12}$   
 $\frac{1}{12}$

$\frac{1}{8}$   
 $\frac{1}{8}$   
 $\frac{1}{8}$

$\frac{1}{6}$   
 $\frac{1}{6}$   
 $\frac{1}{6}$

$\frac{1}{4}$   
 $\frac{1}{4}$   
 $\frac{1}{4}$   
 $\frac{1}{4}$

$\frac{1}{2}$   
 $\frac{1}{2}$   
 $\frac{1}{2}$

0.5  
1.0





THE HISTORY OF STATISTICS  
THE EMERGENCE OF PROBABILITY

THE MATH MYTH

Visualizing Mathematics with 3D Printing

Convex Polytopes

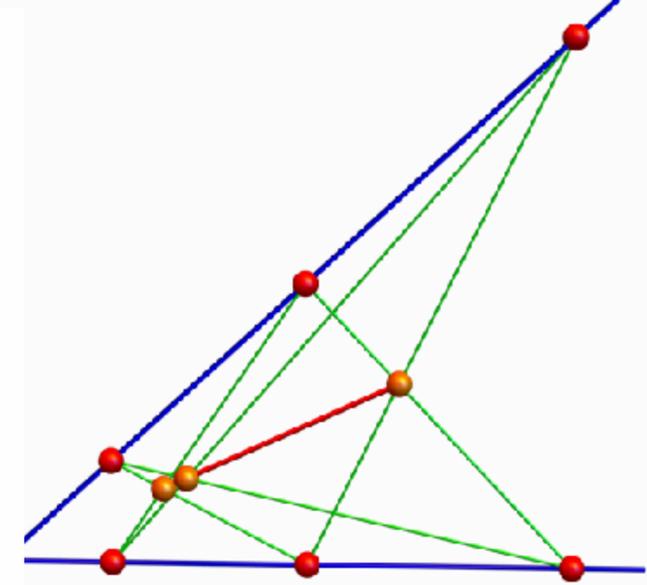
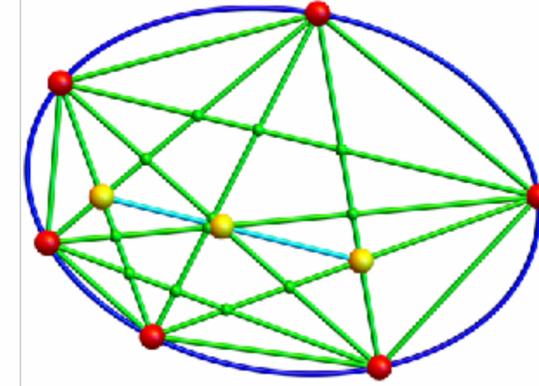
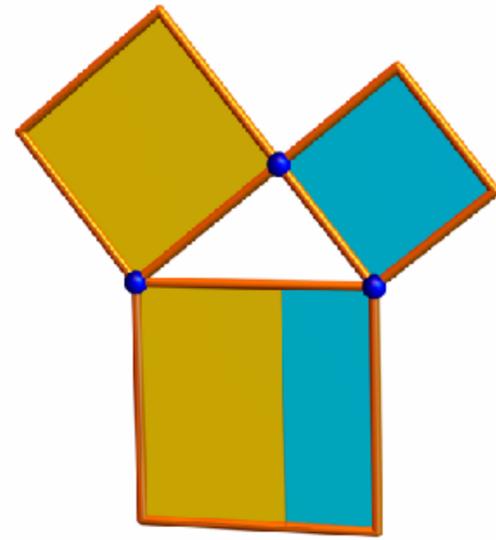
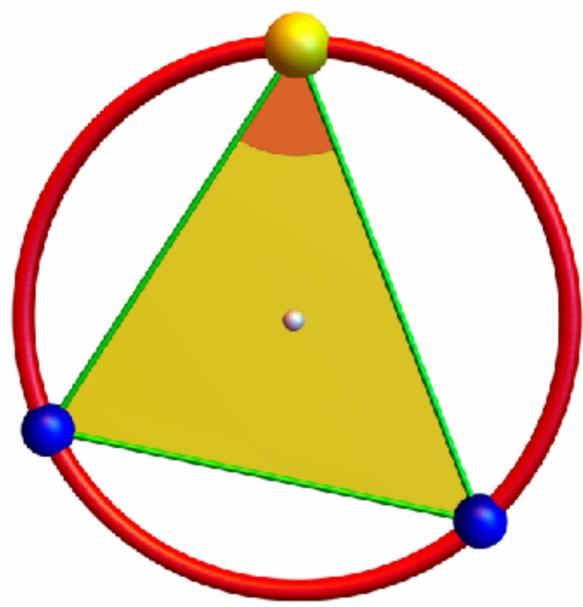
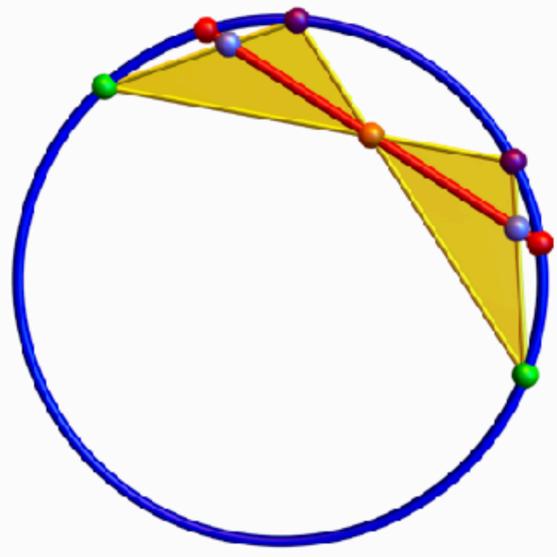
$\frac{1}{10}$

$\frac{1}{12}$

$\frac{1}{12}$

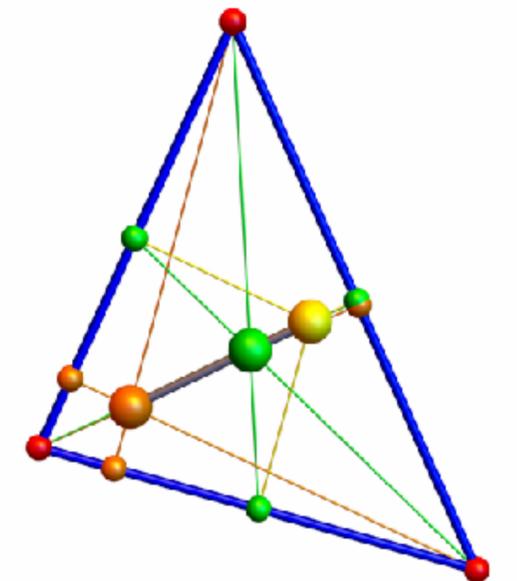
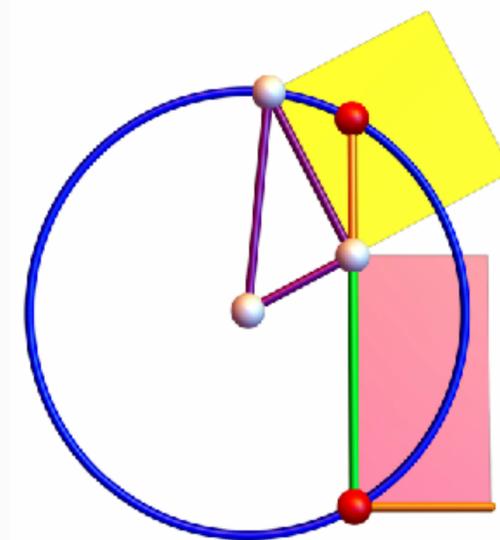
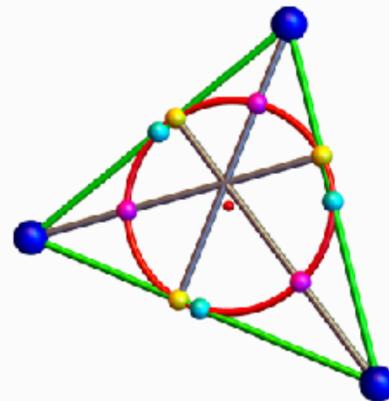
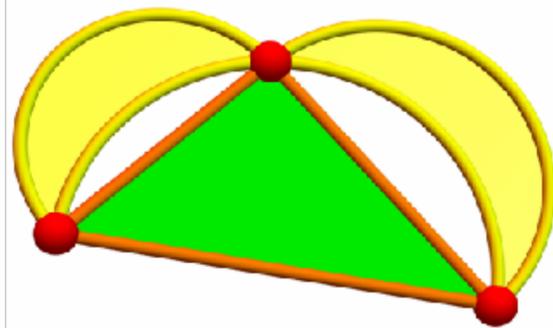
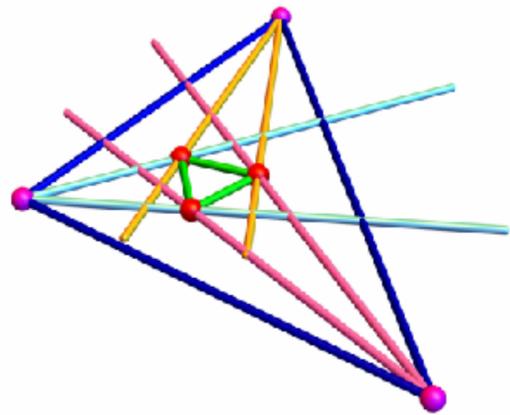
$$\begin{array}{r} 4 \\ + 1 \\ + 1 \\ + 1 \\ \hline 7 \end{array}$$
  
$$\begin{array}{r} 21 \\ + 1 \\ + 1 \\ \hline 23 \end{array}$$





# Ten Theorems in Planar Geometry

Oliver Knill  
Harvard  
Math E 320  
Spring 2022



$$\sum_{k=0}^n a_k x^k = 0$$

$$\sum_{k=0}^n \binom{n}{k} x^k$$

$$\sum_{k=1}^n k$$

$$\sum_{k=1}^n x^k$$

$$(a + b)(c + d)$$

# Top Ten Polynomial Identities

$$\frac{1}{(x - a)(x - b)}$$

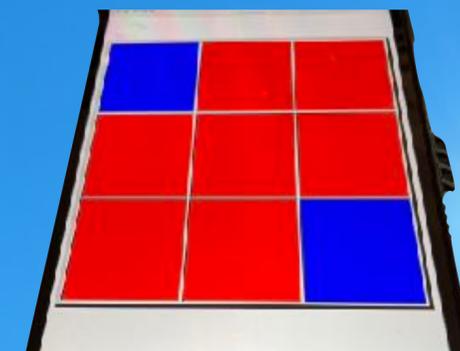
Oliver Knill Math E 320 Spring 2022

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2}$$

$$x^3 + cx + d = 0$$

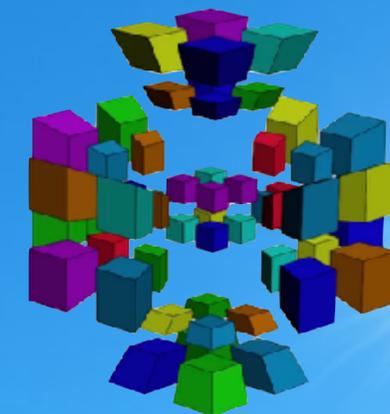
$$x^4 + cx^2 + dx + e = 0$$

$$x^5 - x + 1 = 0$$



# Ten Puzzles that are also Groups

Oliver Knill Math E 320 Spring 2022



$$n = \prod_{k=1}^m p_k$$

$$a^p = a \pmod{p}$$

$$(n-1)! + 1 = 0 \pmod{n}$$

$$\pi(n) \sim \frac{n}{\log(n)}$$

$$\binom{p}{q} \binom{q}{p} = (-1)^{\frac{p-1}{2} \frac{q-1}{2}}$$

# Ten Theorems in Number Theory

$$p = a^2 + b^2$$
$$p = 4k + 1$$

Oliver Knill Math E 320 Spring 2022

$$n = 2^{p-1} (2^p - 1)$$

$$n = a^2 + b^2 + c^2 + d^2$$

$$p = ak + b$$

prime

$$x = a \pmod{n}$$
$$x = b \pmod{m}$$

$$\frac{d}{dx} \int_0^x f(t) dt = f(x)$$

$$f(x+t) = f(x) + f'(x)t + \frac{f''(x)}{2}t^2 + \dots$$

$$\sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}$$

$$V = \frac{4\pi r^3}{3}$$

$$f(x)=0$$

# Ten Calculus Results

$$1 + \frac{1}{2} + \frac{1}{3} + \dots$$

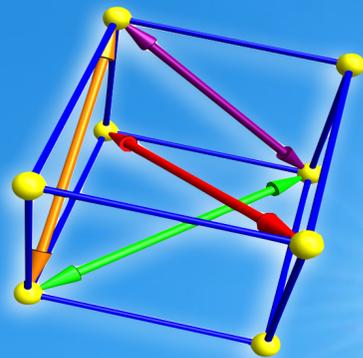
Oliver Knill Math E 320 Spring 2022

$$\pi = 3.1415926\dots$$

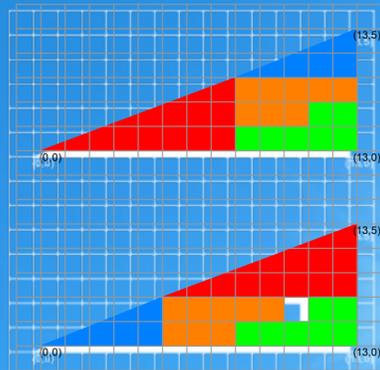
$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$f(x) = \sum_k b_k \sin(kx)$$

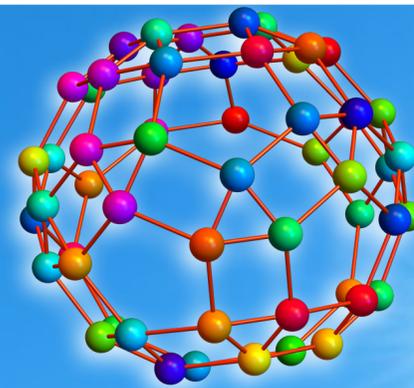
$$\sum_k a^k \cos(b^k x)$$



Experimental Math



Artificial Intelligence



Algorithms



Limits of Computing



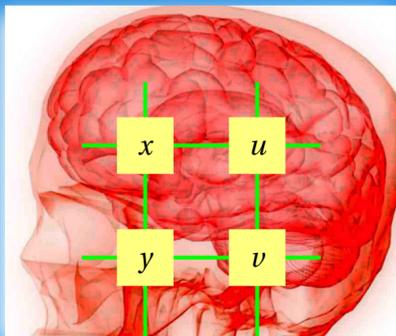
Complexity Theory

# Computer Science

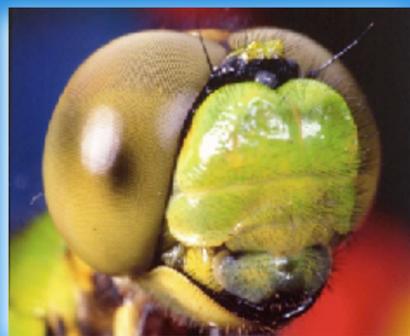
3	7	4	2
1	2	3	1
4	1	3	7

Data Structures

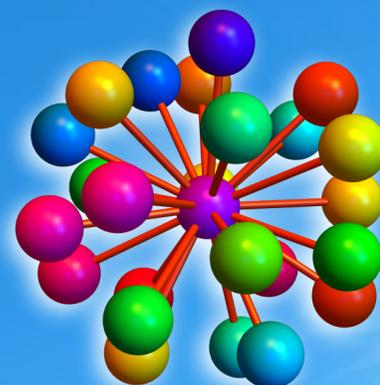
Oliver Knill Math E 320 Spring 2022



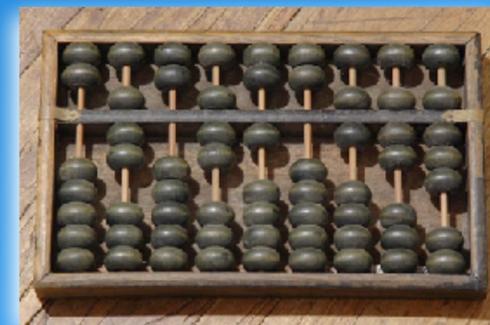
Inverse Problems



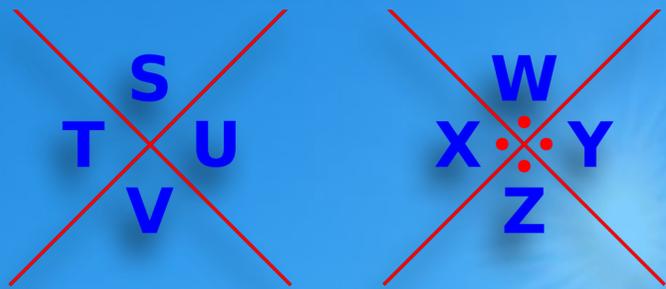
Computer Vision



Network Structures



Computer Museum



Polybius



Scytale



Caesar

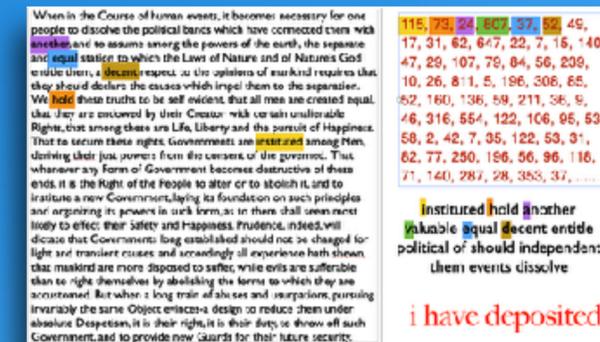


Trithemius

# Crypto Systems



Vigenaire



Ottendorf

Oliver Knill Math E 320 Spring 2022



Enigma



DES

$$X = a^x \pmod{p}$$

Diffie-Hellman

$$y = x^a \pmod{n}$$

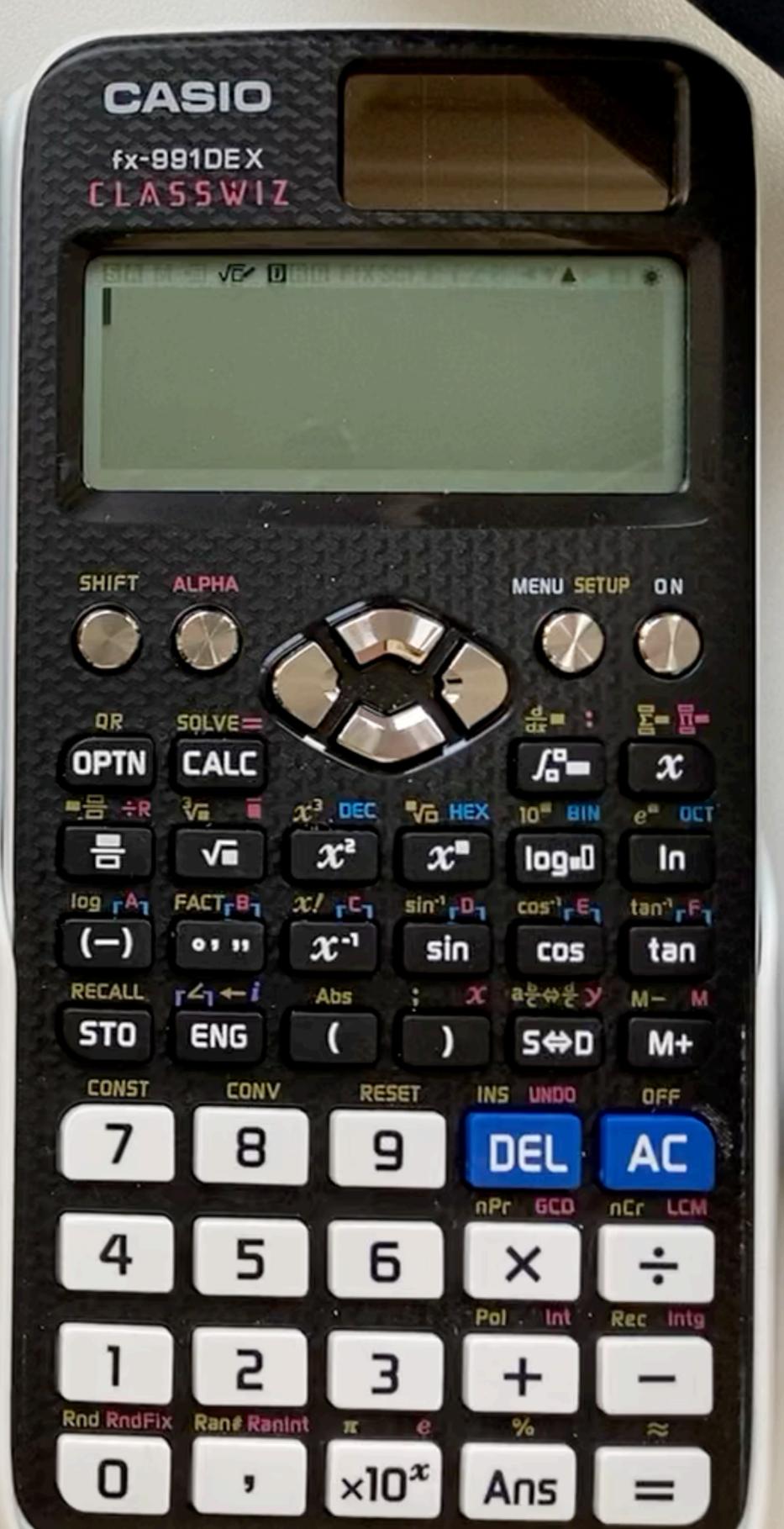
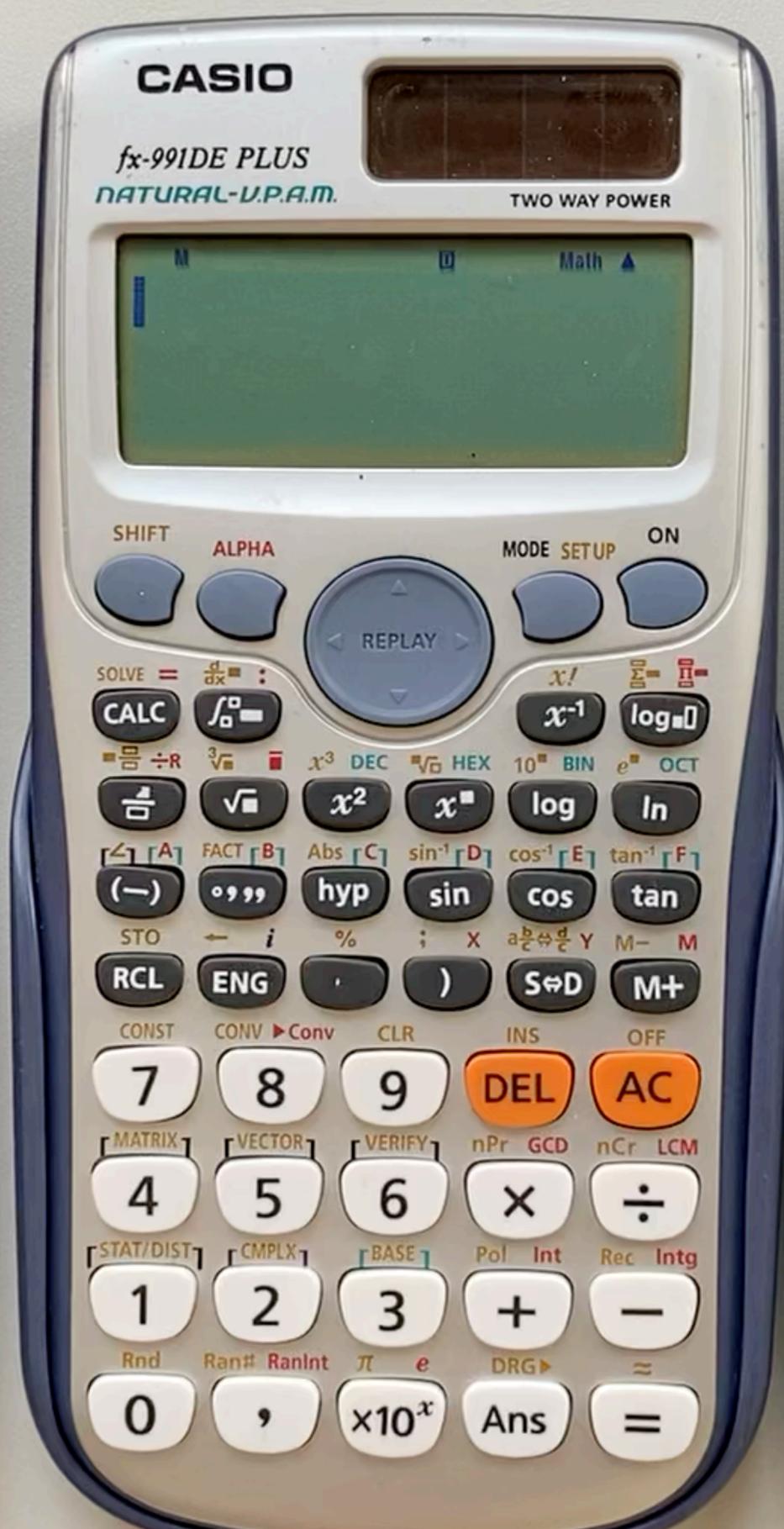
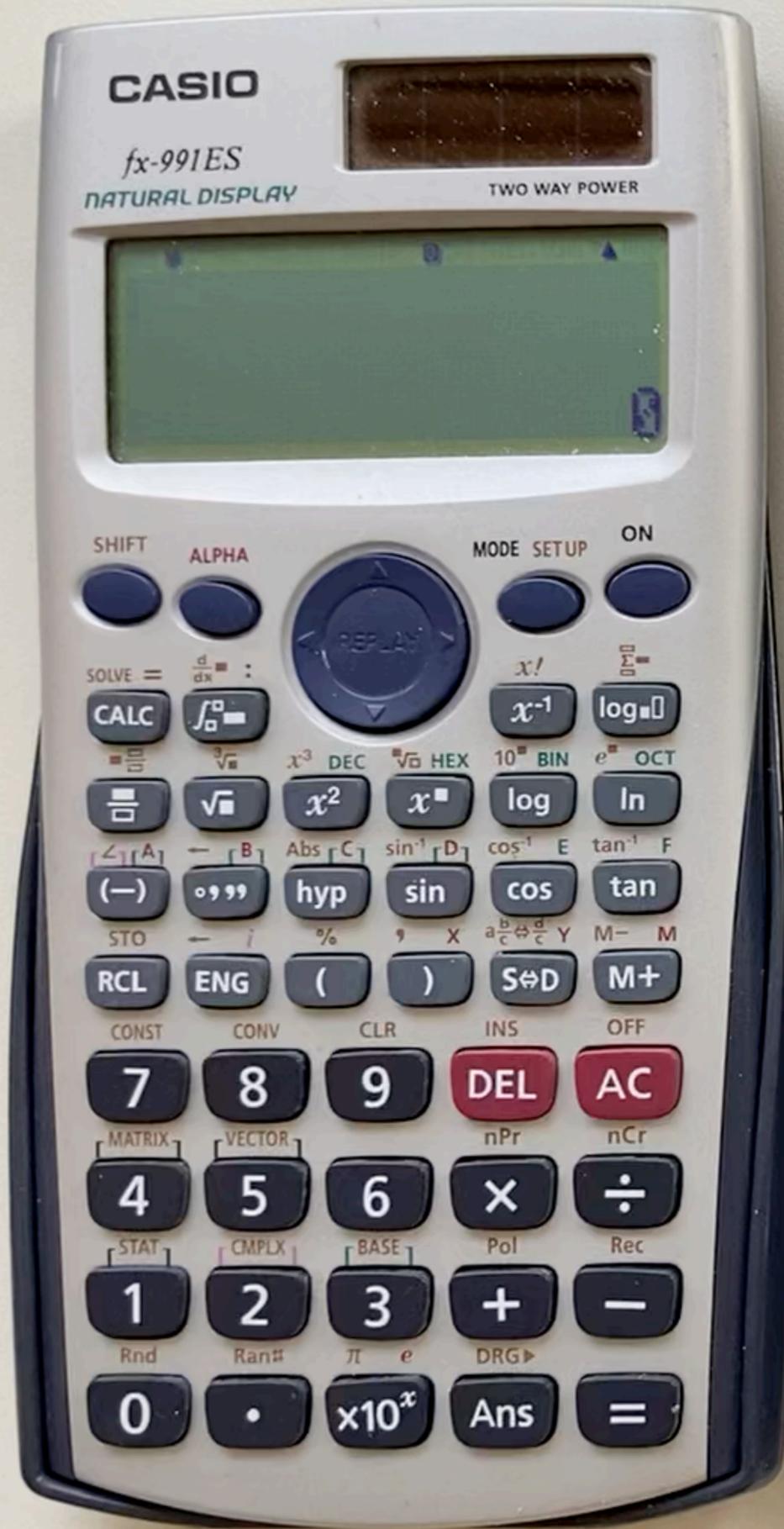
RSA

**PEMDAS WARS 2014-**

*What is  $6 \div 2(2 + 1)$ ?*

*Battles about  
mathematical Syntax*

*Oliver Knill, Stem club, April 28, 2018*



Why Warm-Ups?



# Why Warm-Ups?

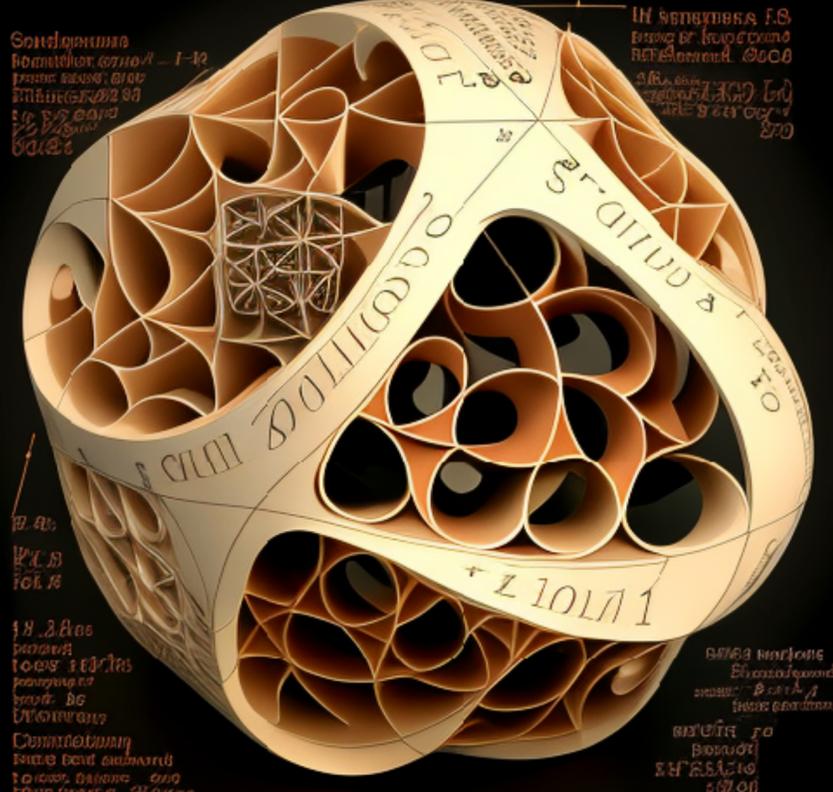
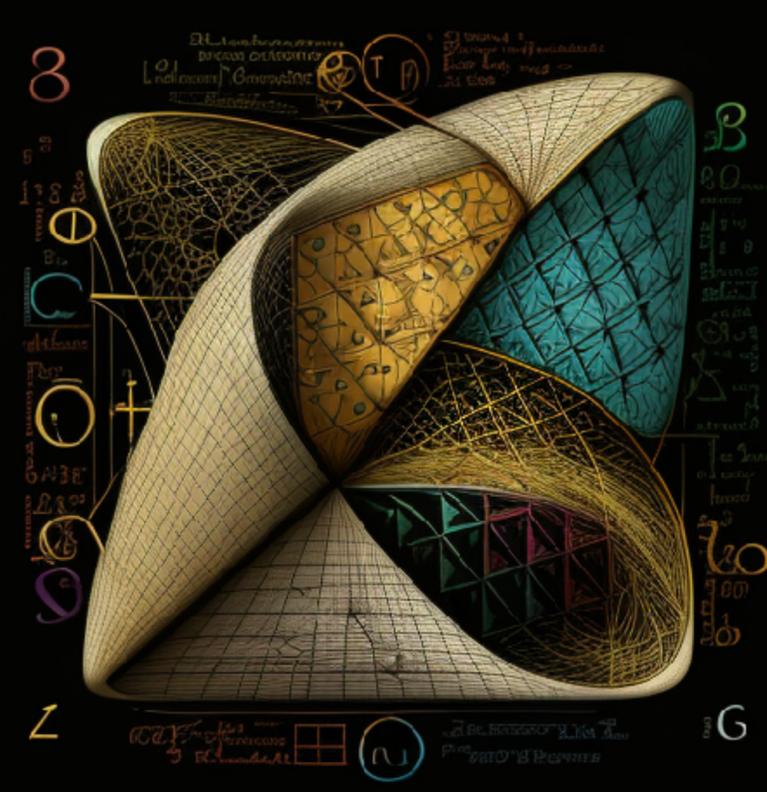


High school math teaching quality varies,  
even within schools.

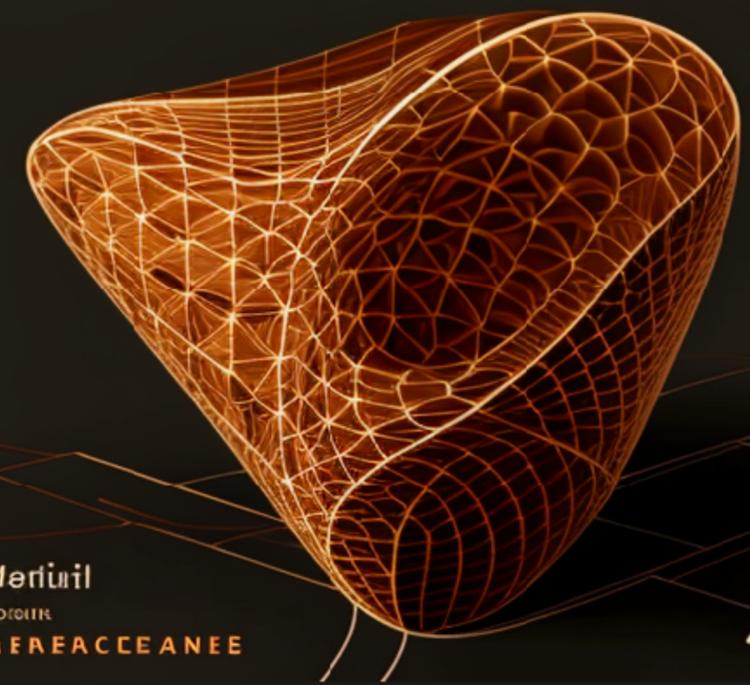
We have a more diverse student  
population today.

Students have a more diverse math  
background today

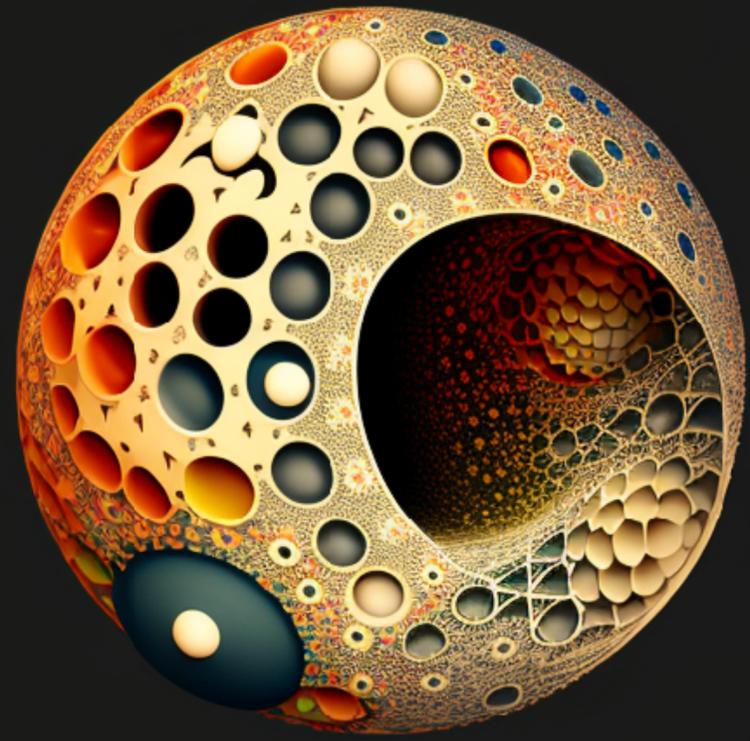
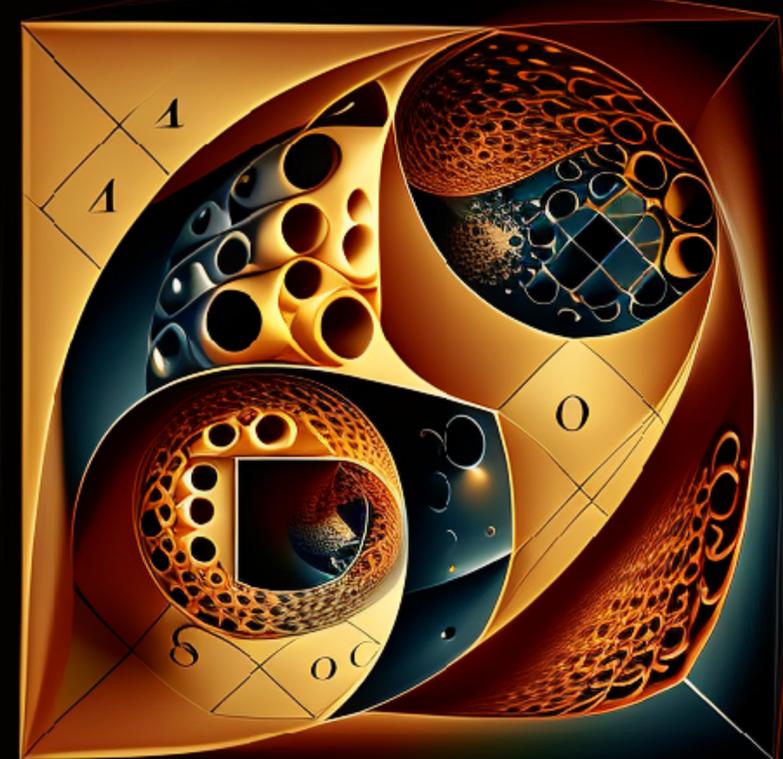
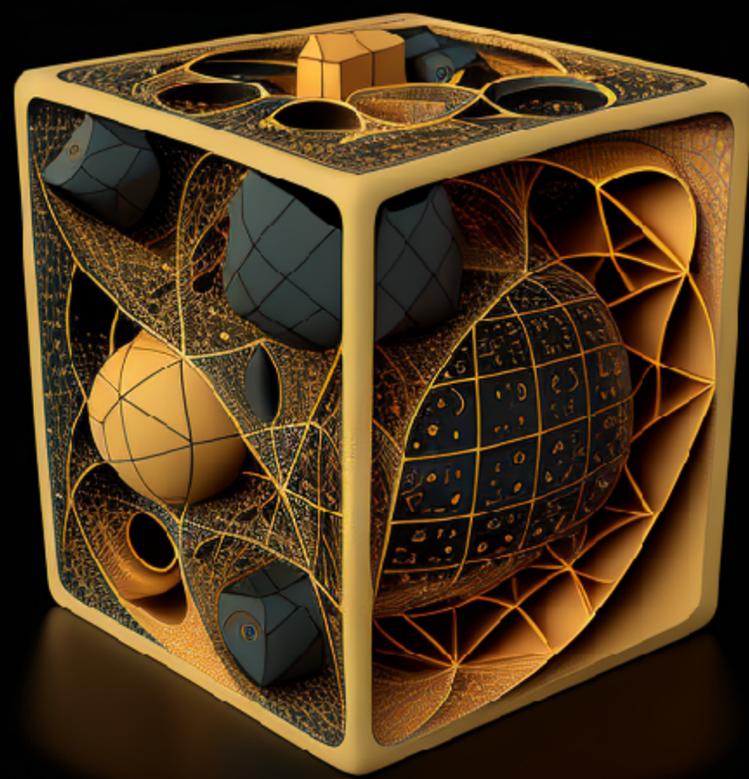




# Kce Kanntihe s. /Atkingcs Suallk F nll k -Satnes



Meriul  
S OOTIR  
AERACCEANEE



Education changed enormously in the last 30 years. Dramatically in the last 5.

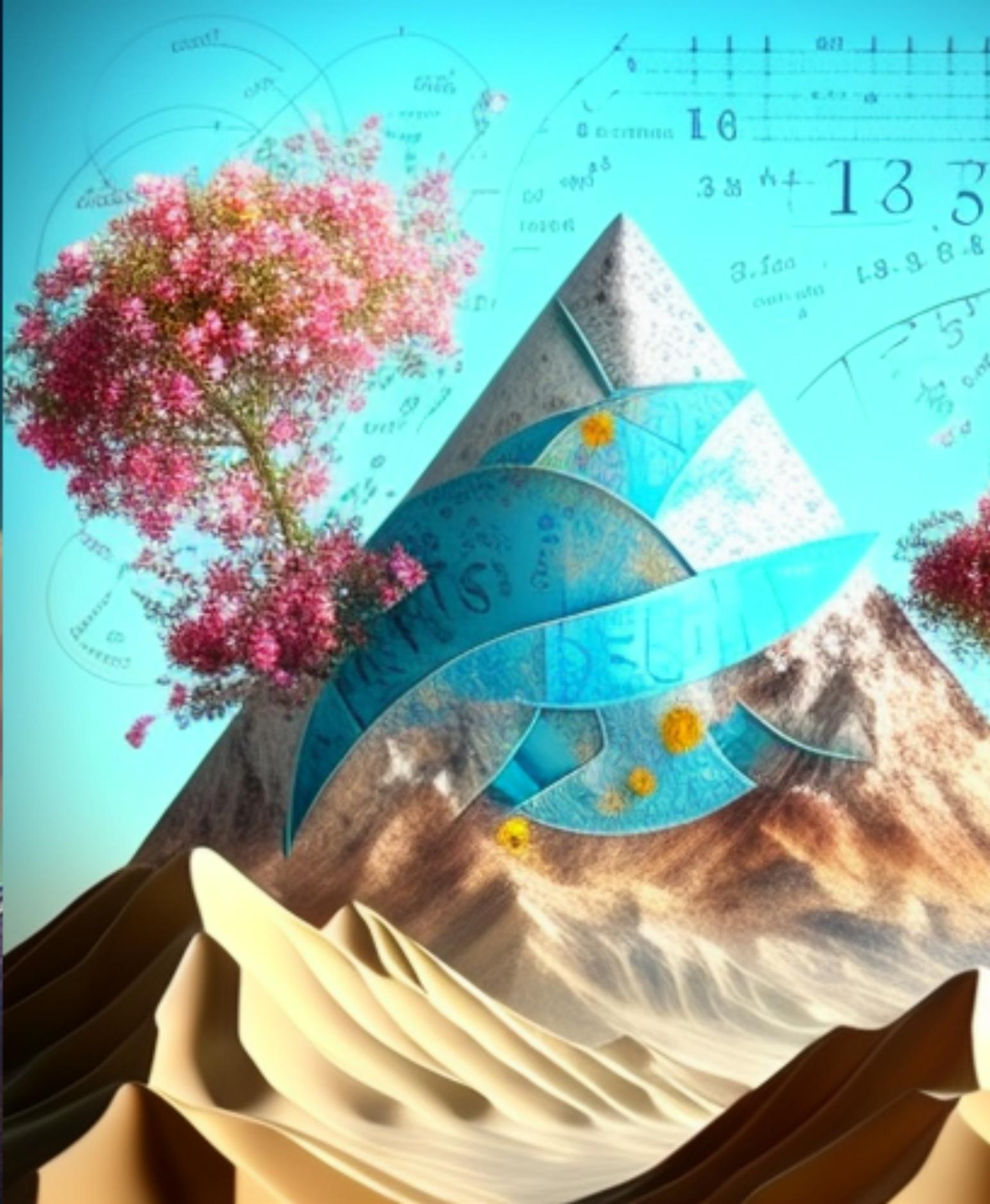
New structures, new companies, new ways to learn have emerged.

Information overflow has dramatically increased. Quality of information varies.

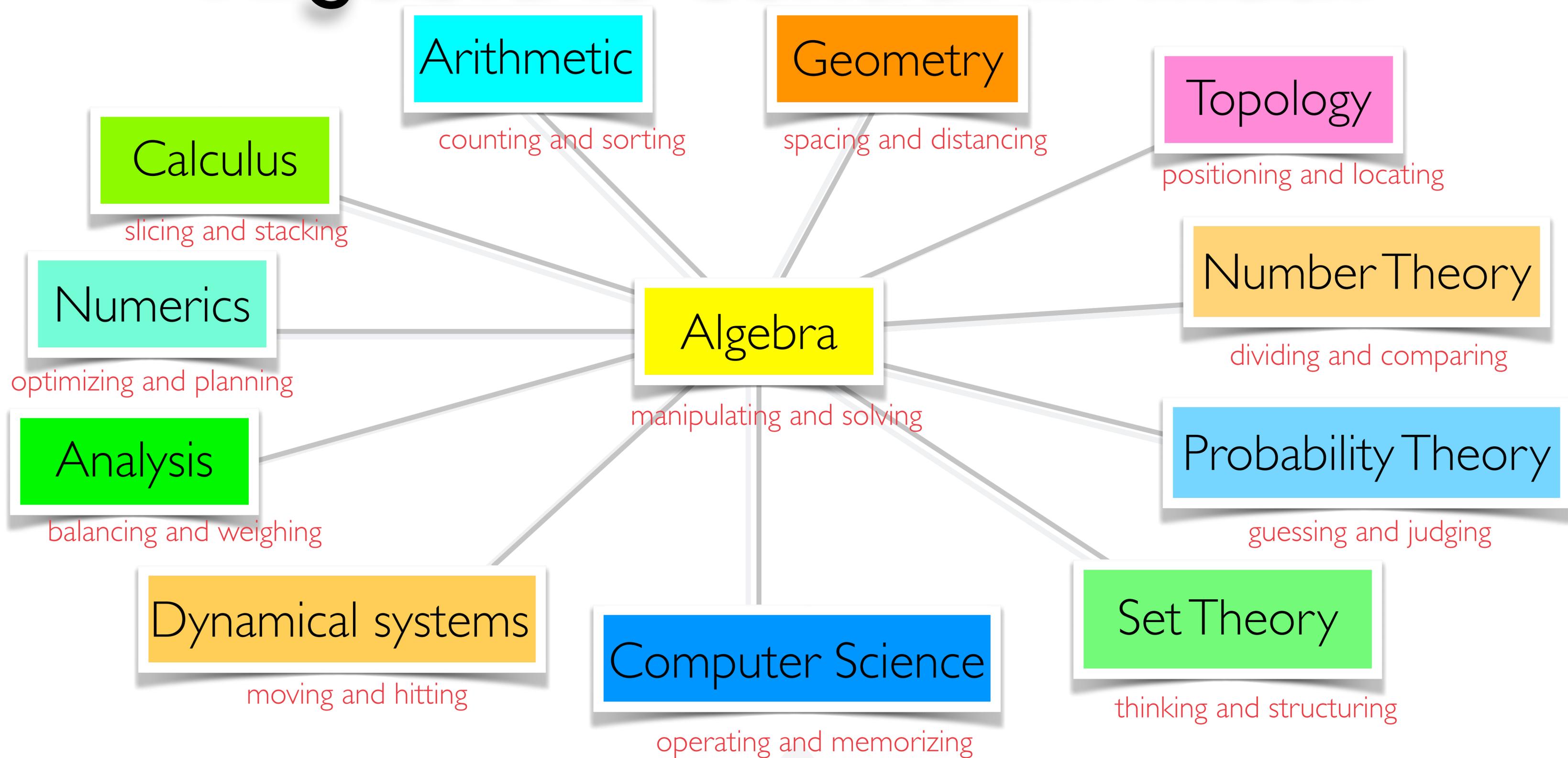


Why is Algebra important

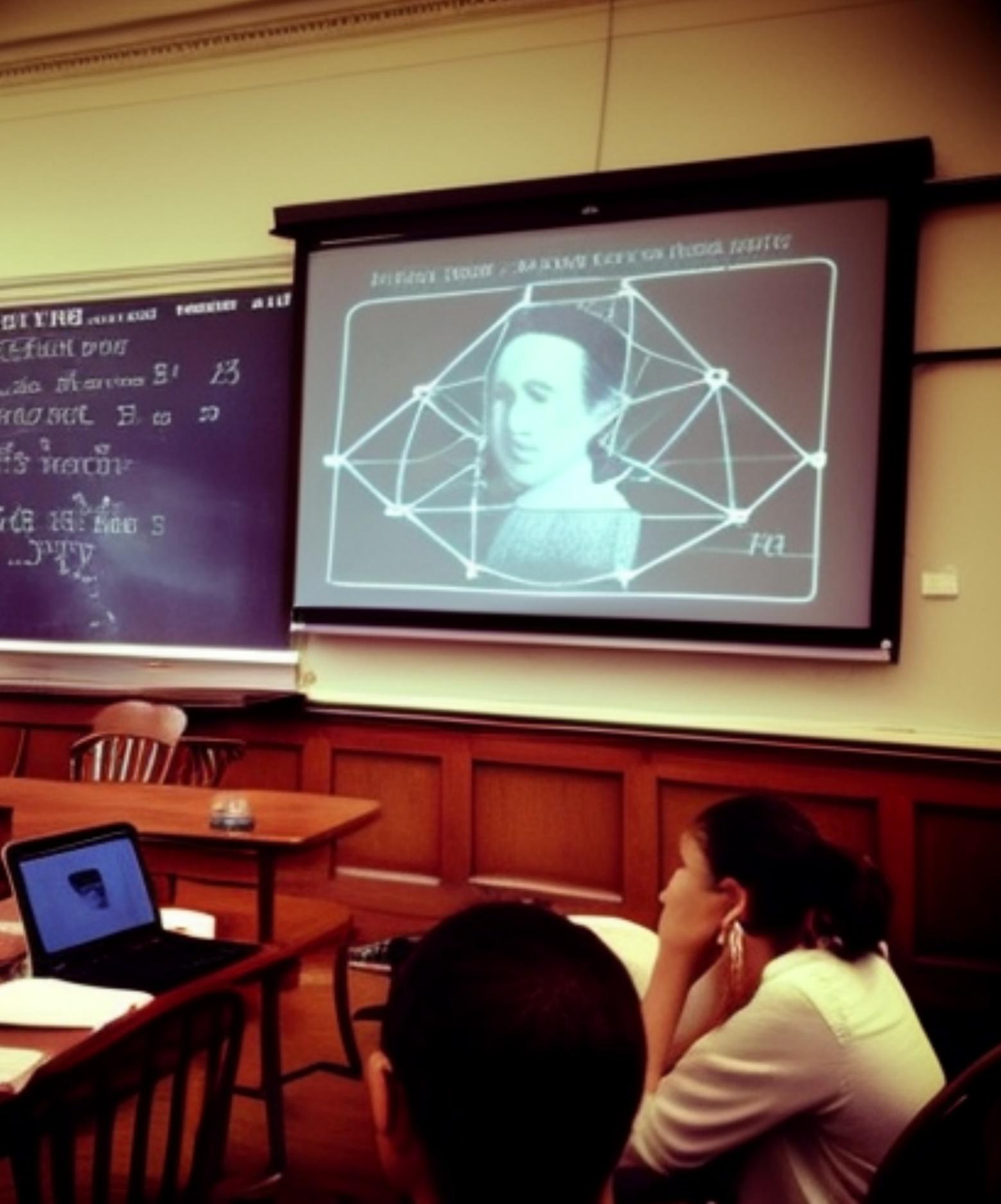




# Algebra is Central in Math



# Landscape of Students





# Which Topics?

Lets try to distill out the major 5 points



# Topics

# Algebra

Variables

Introduce  $x$  for unknown

Rules

Order of operations

Separation

Move unknowns to one side

Manipulation

Expand, factor and simplify

Powers

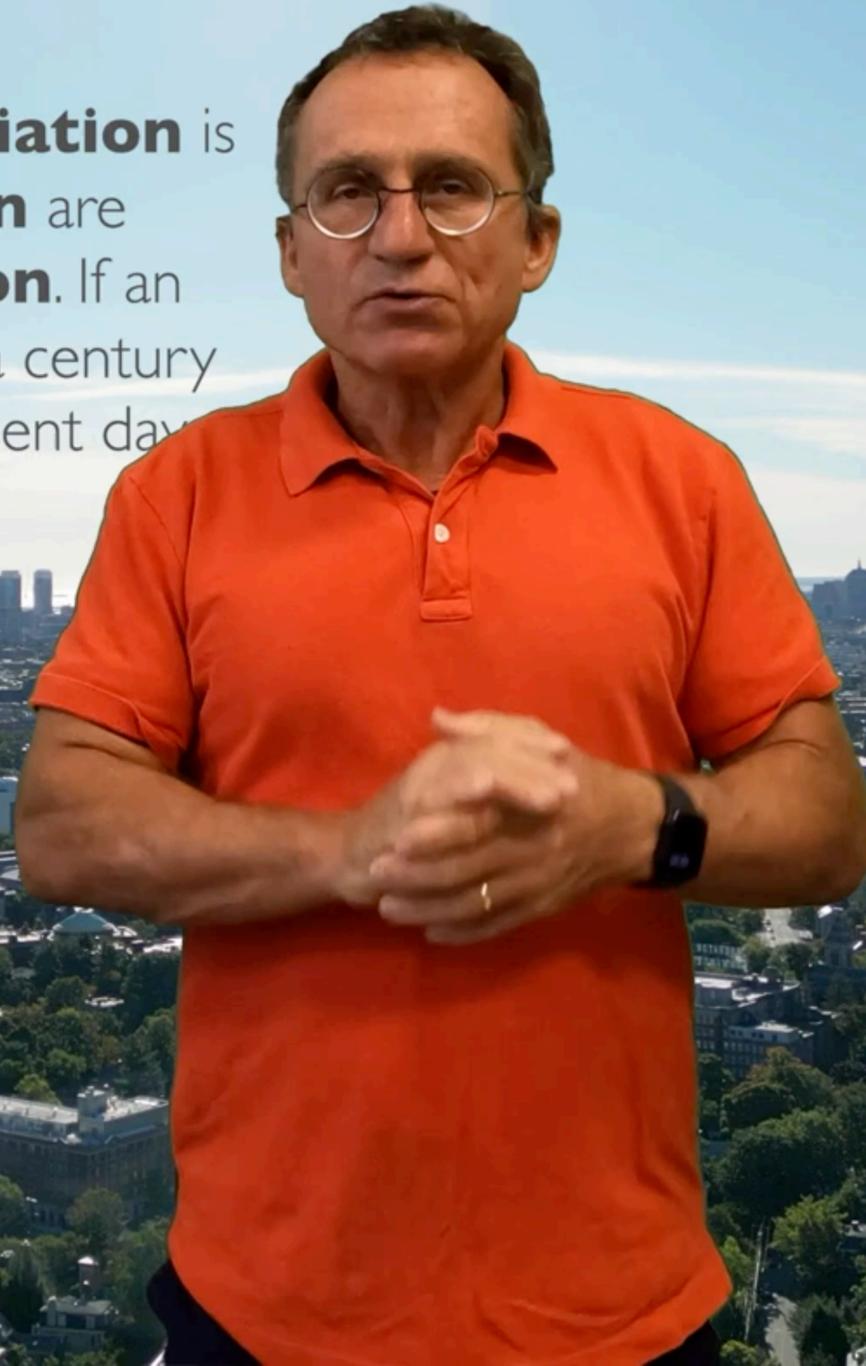
Power and multiplication rules

# 3) ORDER OF OPERATIONS

3

## PEMDAS

The algebra rule PEMDAS states that **exponentiation** is done before **multiplication and division** are performed before **addition and subtraction**. If an expression is ambiguous, **use brackets**. This is a century old recommendation which is valid until the present day.





Plot the graph of the exponential function  $e^x$

**Exponential growth** forward and and **exponential decay** backward.



# Two exploratory movies

## Elementary algebra

---

- 1) Variables
- 2) Separation
- 3) Order of operation
- 4) Multiplying out
- 5) Factor expressions
- 6) Adding fractions
- 7) Multiplying powers
- 8) Powers of powers
- 9) Linear equations
- 10) Quadratic equation

## Intermediate algebra

---

- 1) Functions
- 2) Drawing Graphs
- 3) Inverting functions
- 4) Finding roots
- 5) Exponentials
- 6) Logarithms
- 7) Systems of equations
- 8) Binomials
- 9) Rational functions
- 10) Geometric sums

Training with short formats

# Matrices

$$A = \begin{bmatrix} 8 & 8 & 8 & 1 & 1 & 1 & 1 \\ 8 & 1 & 8 & 1 & 1 & 1 & 1 \\ 8 & 8 & 8 & 1 & 1 & 1 & 1 \\ 8 & 1 & 8 & 1 & 1 & 1 & 1 \\ 8 & 1 & 8 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 8 & 8 & 1 \\ 1 & 1 & 1 & 1 & 8 & 1 & 8 \\ 1 & 1 & 1 & 1 & 8 & 8 & 1 \\ 1 & 1 & 1 & 1 & 8 & 1 & 8 \\ 1 & 1 & 1 & 1 & 8 & 8 & 1 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 9 & 9 & 9 & 2 & 9 & 9 & 2 \\ 9 & 2 & 9 & 2 & 9 & 2 & 9 \\ 9 & 9 & 9 & 2 & 9 & 9 & 2 \\ 9 & 2 & 9 & 2 & 9 & 1 & 9 \\ 9 & 2 & 9 & 2 & 9 & 9 & 2 \end{bmatrix}$$

Linear Spaces #shorts  
408 views

# Example 1 van der Pool

$$f'' + 2f' + f = 0$$

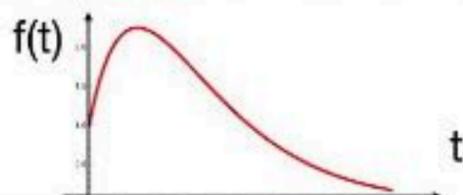
$$\lambda^2 + 2\lambda + 1$$

has the roots

$$\lambda_1 = -1, \lambda_2 = -1$$

The general solution is

$$f(t) = Ae^{-t} + Bte^{-t}$$



Homogenous Second Order  
Differential Equations...  
347 views

$$x' = y$$

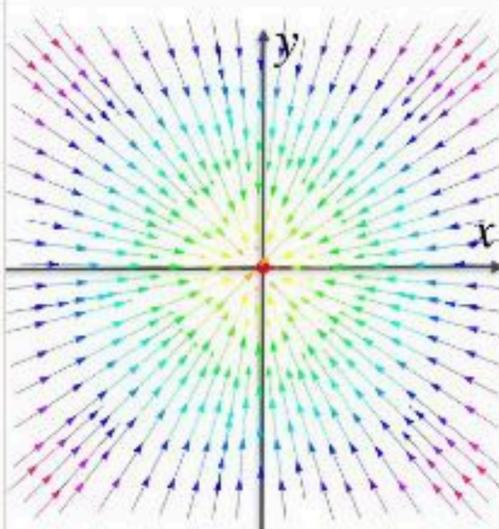
$$y' = -x - (x^2 - 1)y$$



Nonlinear Systems #shorts  
157 views

$$\vec{x}'(t) = A\vec{x}(t)$$

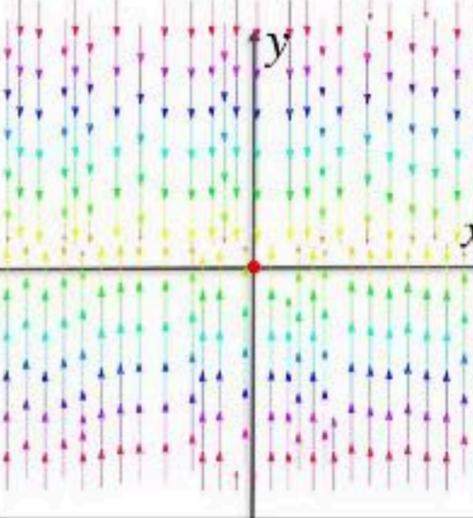
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Stability #shorts  
174 views

$$E \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

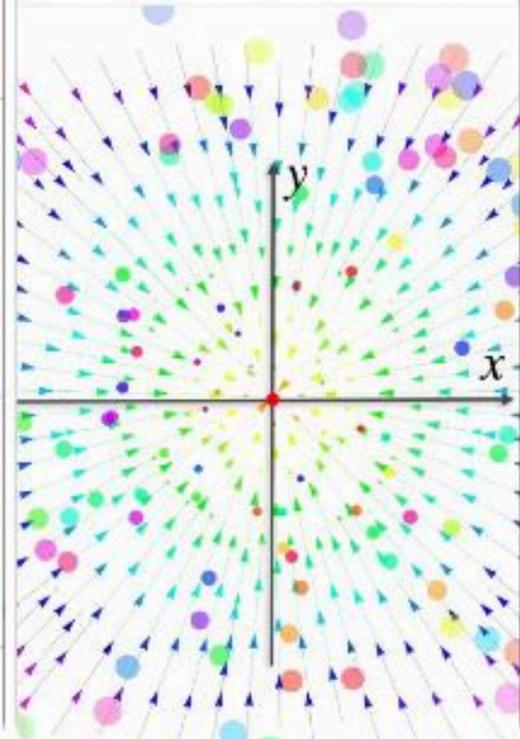
One zero, one negative eigenvalue



unstable  
(=not asymptotically stable)

Matrix Exponentiation  
#shorts  
268 views

$$2 \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Differential Equations  
#shorts  
284 views

# Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \text{ symmetric but not simple spectrum.}$$

$$\lambda_1 = 14, \lambda_2 = 0, \lambda_3 = 0$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$A(t) = \begin{bmatrix} 1+t & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \text{ has simple spectrum for small } t.$$

# Example

$$A = \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

An example of a **real matrix** with **complex eigenvalues** which is **not diagonalizable**.

$$\lambda_1 = i, \lambda_2 = -i, \lambda_3 = -1, \lambda_4 = -1$$

# Euler

$$1 + e^{i\pi} = 0$$

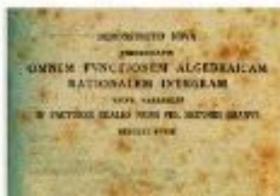


# Main Theorem

A polynomial P factors into linear polynomials.

$$a_0 + a_1\lambda + \dots + \lambda^n = (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n)$$

C.F. Gauss (1799): We can write P as



# Shear

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The **shear** is a prototype of a matrix which is not diagonalizable. An other example is following

**Jordan block:**

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Example

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 7 \end{bmatrix}$$

**Characteristic polynomial**

$$f_A(\lambda) = \det \begin{bmatrix} 4-\lambda & 5 \\ 2 & 7-\lambda \end{bmatrix}$$

$$= \lambda^2 - 11\lambda + 18$$

$$= (\lambda - 9)(\lambda - 2)$$

$$E_9 = \ker(A - 9I) = \ker \begin{bmatrix} -5 & 5 \\ 2 & -2 \end{bmatrix}$$

$$= \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

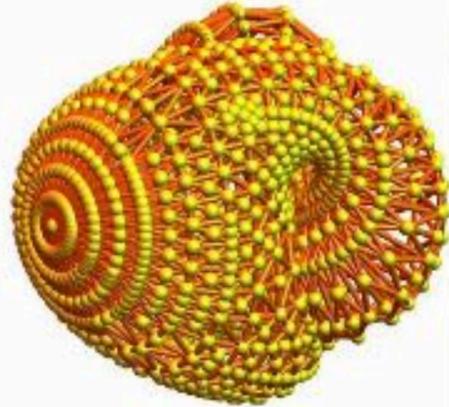
# Fibonacci



Leonardo of Pisa, 1170-1250

# Bryant-Kusner

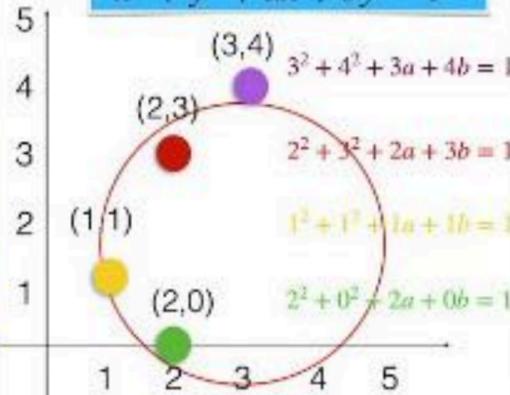
An animation of the Bryant-Kusner parametrization of the real projective plane. It is a rational parametrization of the Boy surface, minimizing the Willmore energy. The surface is famous because it is known as the "Oberwolfach surface", installed in 1991.



```
Mathematica implementation:
r[s_] := Module[{z = s*Exp[I t]},
  Z = z^6 + Sqrt[5] z^3 - 1;
  g = (-Im[z(1-z^4)/Z]^3/2,
  -Re[z(1+z^4)/Z]^3/2,
  Im[(1+z^6)/Z]-1/2]; g/(g.g);
```

# Circle Fit

Find the best circle  
 $x^2 + y^2 + ax + by = 1$



$$A = \begin{bmatrix} 3 & 4 \\ 2 & 0 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -24 \\ -3 \\ -1 \\ -12 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 151 \\ 437 \end{bmatrix} / 107$$



# Laplace

$$\det(A) = a_{11}\det(A_{1,1}) - a_{21}\det(A_{2,1}) + a_{31}\det(A_{3,1})$$

$$+ \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 7 & 8 & 0 \end{bmatrix} A_{1,1}$$

$$- \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 7 & 8 & 0 \end{bmatrix} A_{2,1}$$

$$+ \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 7 & 8 & 0 \end{bmatrix} A_{3,1}$$

# QR

$$A = QR$$

$$A = Q \cdot R$$



Columns are original basis    Columns are orthonormal basis    Upper triangular matrix

An experiment:

```
A = Table[GCD[i, j], {i, 50}, {j, 50}];
```



Discrete Dynamical Systems #shorts

#shorts

776 views

Bryant-Kusner #shorts

249 views

Data Fitting #shorts

673 views

How to Study #shorts

286 views

Determinants #shorts

355 views

Gram Schmidt Process #shorts

551 views

# 1D Projection

$$P\vec{x} = c\vec{u} \quad \text{with } c = \vec{u} \cdot \vec{x}$$

assuming that  $\vec{u}$  has length 1.



$$P\vec{x} = \vec{u}c = \vec{u}(\vec{u} \cdot \vec{x}) = \vec{u}(\vec{u}^T \vec{x}) = (\vec{u}\vec{u}^T)\vec{x} = QQ^T \vec{x}$$

$$P = \vec{u}\vec{u}^T = \begin{bmatrix} a^2 & ba \\ ab & b^2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}$$

# Steinitz Exchange

If  $\mathcal{U}$  spans  $V$  and  $\mathcal{W}$  is linearly independent, then  $|\mathcal{W}| \leq |\mathcal{U}|$ .

It is not possible for example to have 3 independent vectors in a plane.



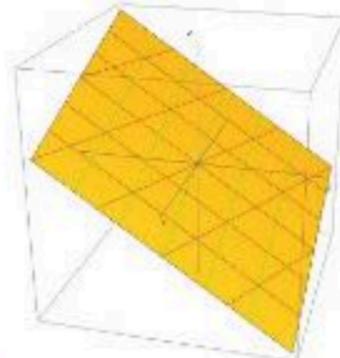
# Question F

Let  $X$  be the union of all planes going through all possible interger lattice points.

rational Lasagne



# Writing Planes



Contour picture

Parametric picture

As kernel

As image

# S matrix

$$S = [\vec{v}_1 \cdots \vec{v}_m]$$

This coordinate change matrix is invertible if and only if  $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$  is a basis

standard basis

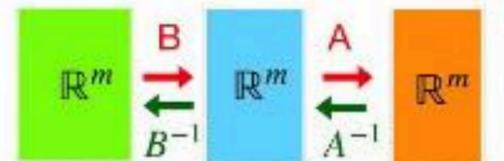
new basis  $\mathcal{B}$



# Composition

If  $A, B$  are  $n \times n$  matrices which are invertible then the product is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

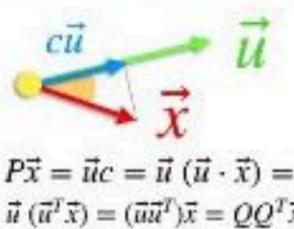


If you fly from London to Rome

# 1D Projection

$$P\vec{x} = c\vec{u} \quad \text{with } c = \vec{u} \cdot \vec{x}$$

assuming that  $\vec{u}$  has length 1.



$$P = \vec{u}\vec{u}^T = \begin{bmatrix} a^2 & ba \\ ab & b^2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}$$

$P = \begin{bmatrix} a^2 & ba \\ ab & b^2 \end{bmatrix}$  is the projection onto  $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$  if  $\vec{u}$  has length 1.

# Steinitz Exchange

If  $\mathcal{U}$  spans  $V$  and  $\mathcal{W}$  is linearly independent, then  $|\mathcal{W}| \leq |\mathcal{U}|$ .

It is not possible for example to have 3 independent vectors in a plane.



Ernst Steinitz 1871-1928

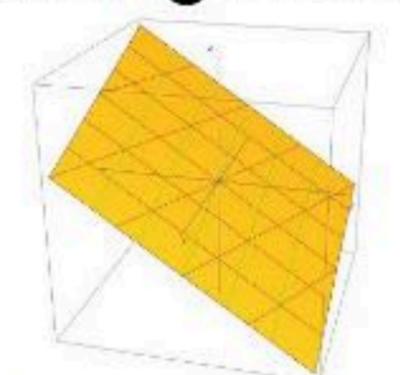
# Question F

Let  $X$  be the union of all planes going through all possible interger lattice points.



- $\vec{0} \in V$
- $\vec{x}, \vec{y}$  in  $V$  then  $\vec{x} + \vec{y}$  in  $V$
- $\vec{x}$  in  $V$  then  $k\vec{x}$  in  $V$

# Writing Planes



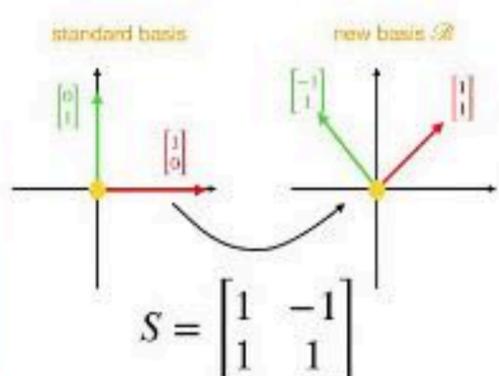
Contour picture:  $x + 2y + 3z = 0$   
 As kernel:  $\ker[1, 2, 3]$

Parametric picture:  $\vec{r} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} -3s - 2t \\ t \\ s \end{bmatrix}$   
 As image:  $\text{im} \begin{bmatrix} -3 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

# S matrix

$$S = [\vec{v}_1 \cdots \vec{v}_m]$$

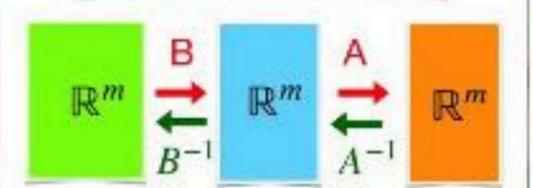
This coordinate change matrix is invertible if and only if  $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_m\}$  is a basis



# Composition

If  $A, B$  are  $n \times n$  matrices which are invertible then the product is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$



If you fly from London to Rome over Zurich then you return from Rome first to Zurich and then to London.

Orthogonal Projection #shorts  
413 views

Rank Nullity theorem #shorts  
365 views

Basis and Linear Spaces #shorts  
294 views

Image and Kernel #shorts  
307 views

Changing Coordinates #shorts  
344 views

Matrix inverse #shorts  
458 views

# Dot Product

$$\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$$

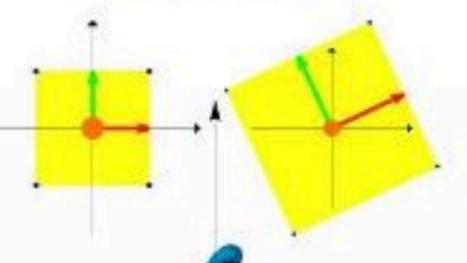
$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = 8$$

The dot product between two column

# Rotation Dilation

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

preserves angles



# Basis

A collection of vectors  $\vec{v}_1, \dots, \vec{v}_m$  spans  $\mathbb{R}^m$  if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m = \vec{b}$$

has a solution for all  $\vec{b}$ . (The system  $A\vec{c} = \vec{b}$ , must be consistent.) Spanning means that the matrix  $\text{ref}(A)$  will have a leading 1 in every row.

$$A = \begin{bmatrix} | & | & \dots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ | & | & \dots & | \end{bmatrix}$$

# Matrix



(Modified from "What does the Fox say" Pop song 2013)

$$T(\vec{e}_k) = \vec{v}_k \text{ is the } k\text{'th column}$$

The  $k$ 'th column of  $A$  is image of  $k$ 'th basis vector under  $T$

# Application



$$\begin{bmatrix} 2 & 1 & 60 \\ 1 & 2 & 75 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 75 \\ 2 & 1 & 60 \end{bmatrix} \xrightarrow{R_2(-2)} \begin{bmatrix} 1 & 2 & 75 \\ 0 & -3 & -90 \end{bmatrix} \xrightarrow{R_2(-1/3)} \begin{bmatrix} 1 & 2 & 75 \\ 0 & 1 & 30 \end{bmatrix} \xrightarrow{R_1(-2)} \begin{bmatrix} 1 & 0 & 15 \\ 0 & 1 & 30 \end{bmatrix}$$

# Solution

leading variables:  $x, y, z$       free variables:  $w$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x &= -3 + w \\ y &= 6 - w \\ z &= 7 - w \\ w &= w \end{aligned}$$

# Matrices

A=

8	8	8	1	1	1	1
8	1	8	1	1	1	1
8	8	8	1	1	1	1
8	1	8	1	1	1	1
8	1	8	1	1	1	1

B=

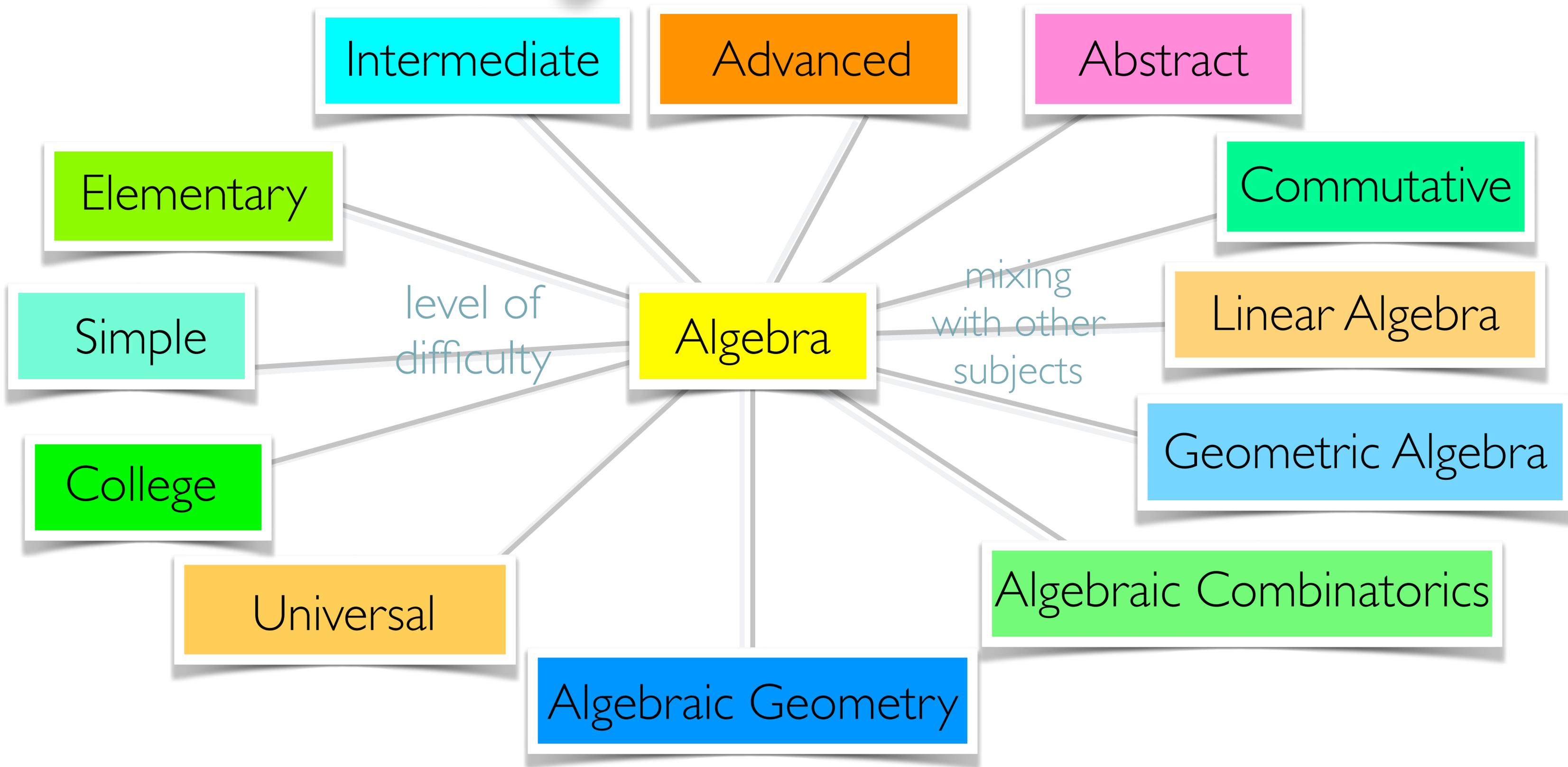
1	1	1	1	8	8	1
1	1	1	1	8	1	8
1	1	1	1	8	8	1
1	1	1	1	8	1	8
1	1	1	1	8	8	1

A+B =

9	9	9	2	9	9	2
9	2	9	2	9	2	9
9	9	9	2	9	9	2
9	2	9	2	9	1	9
9	2	9	2	9	9	2

for the last Math 21b class wednesday

# Algebra Flavors



# Geometry

Triangles

Special points, angles, similarity

Lines

Slope form, Ruler construction

Circles

Center Radius, Compass

Conics

Ellipse, hyperbola, parabola

Solids

Sphere, cylinder, pyramid, cone



# Trigonometry

Unit Circle

right 45-45, 30-60 triangles

Functions

cos, sin, tan, cot

Angles

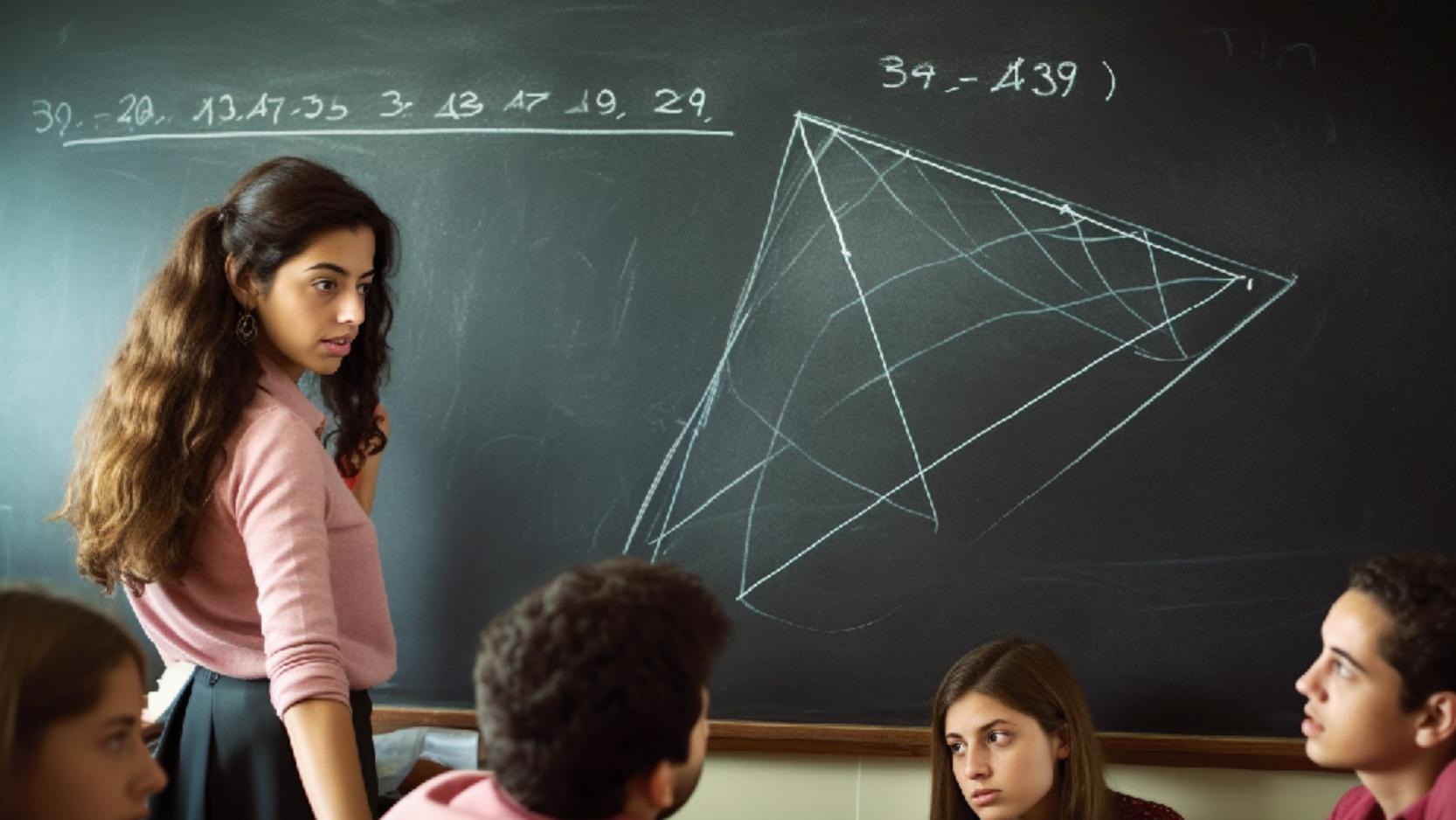
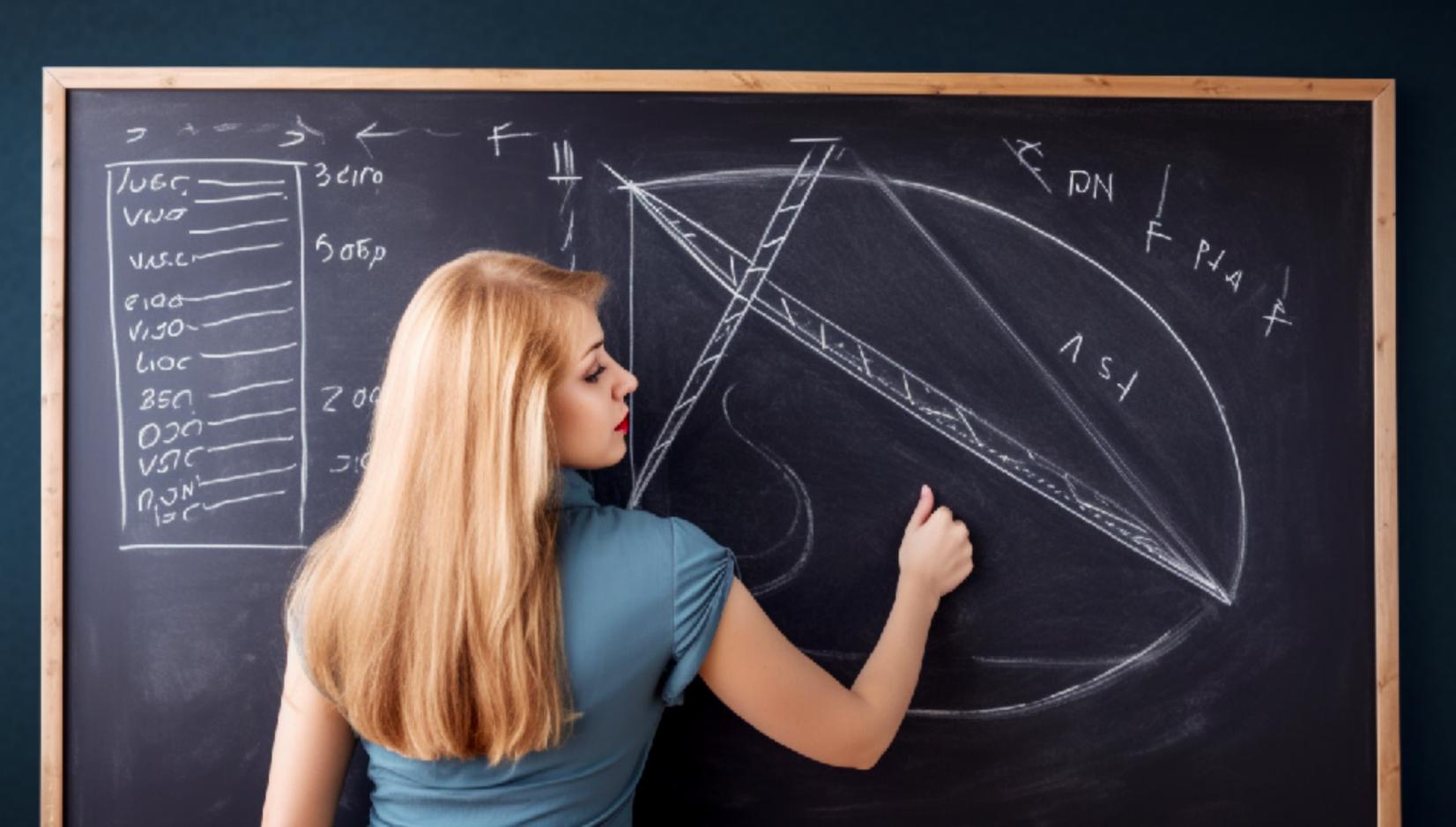
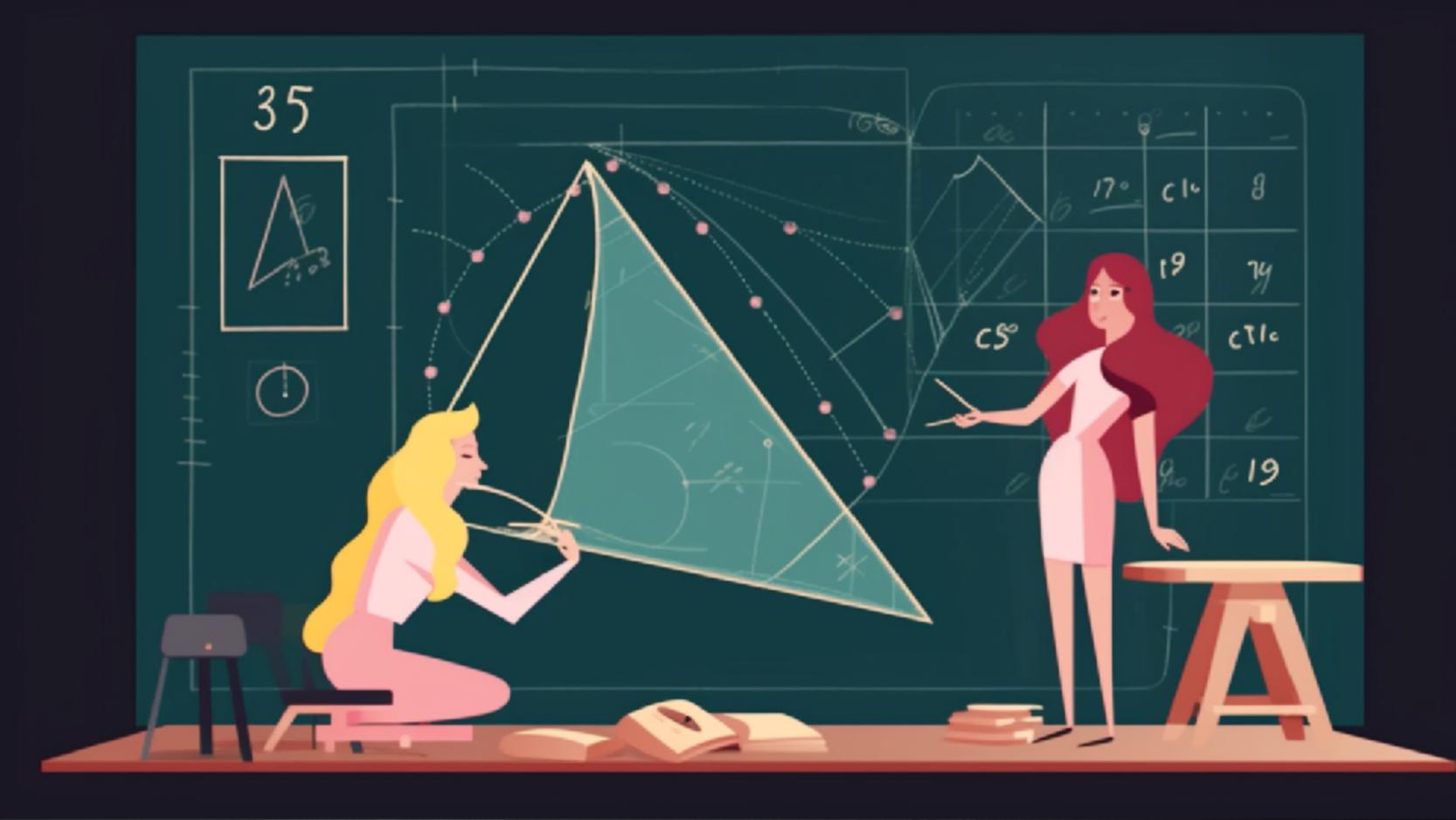
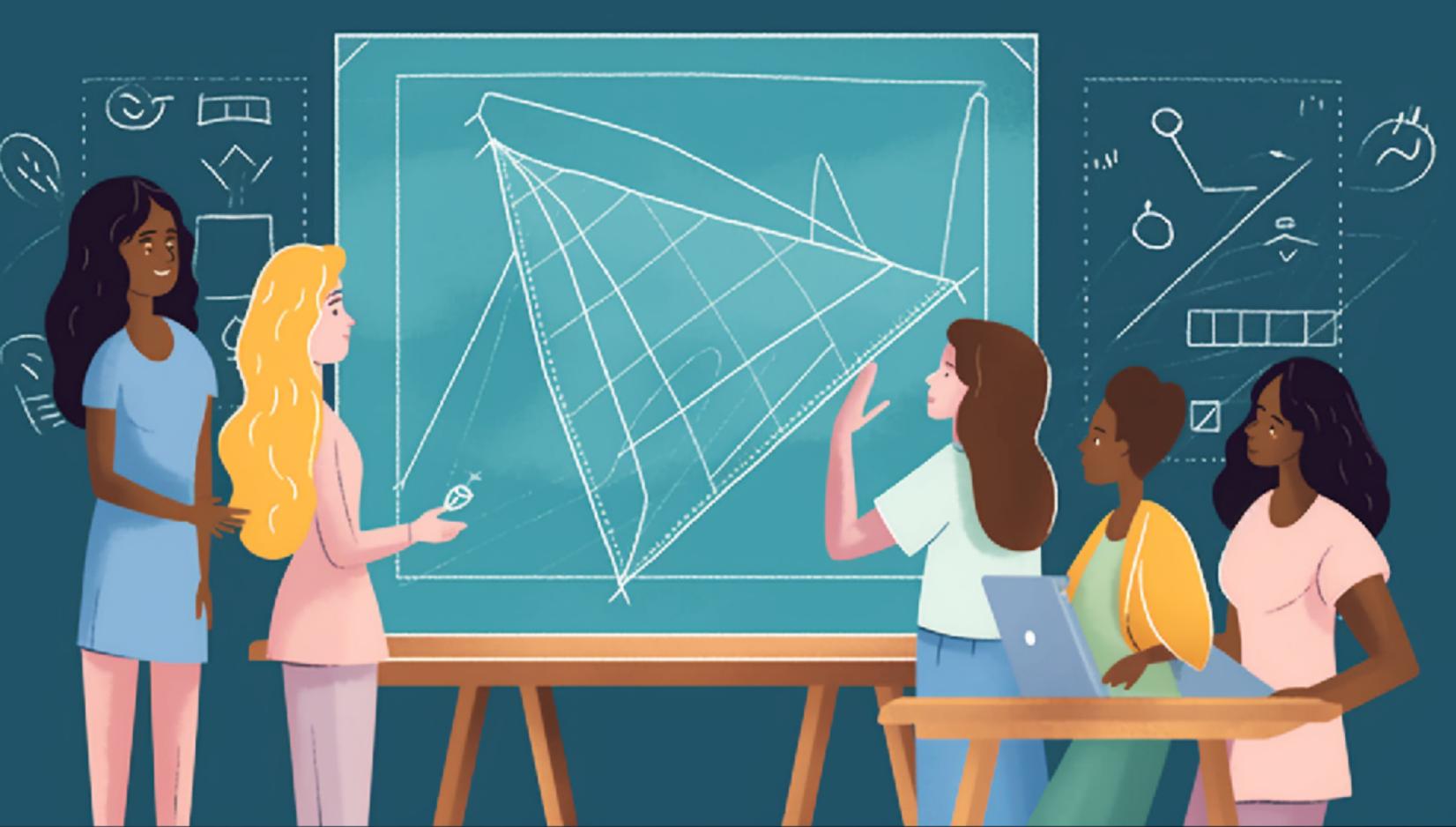
Degree-Radian 0-360, 0-2pi

Inverse trig

arcsin, arccos, arctan

Identities

$\cos^2 + \sin^2$ , double angles



# Equations

Linear

$$ax+b = 0$$

Quadratic

$$ax^2+bx+c=0$$

Trig

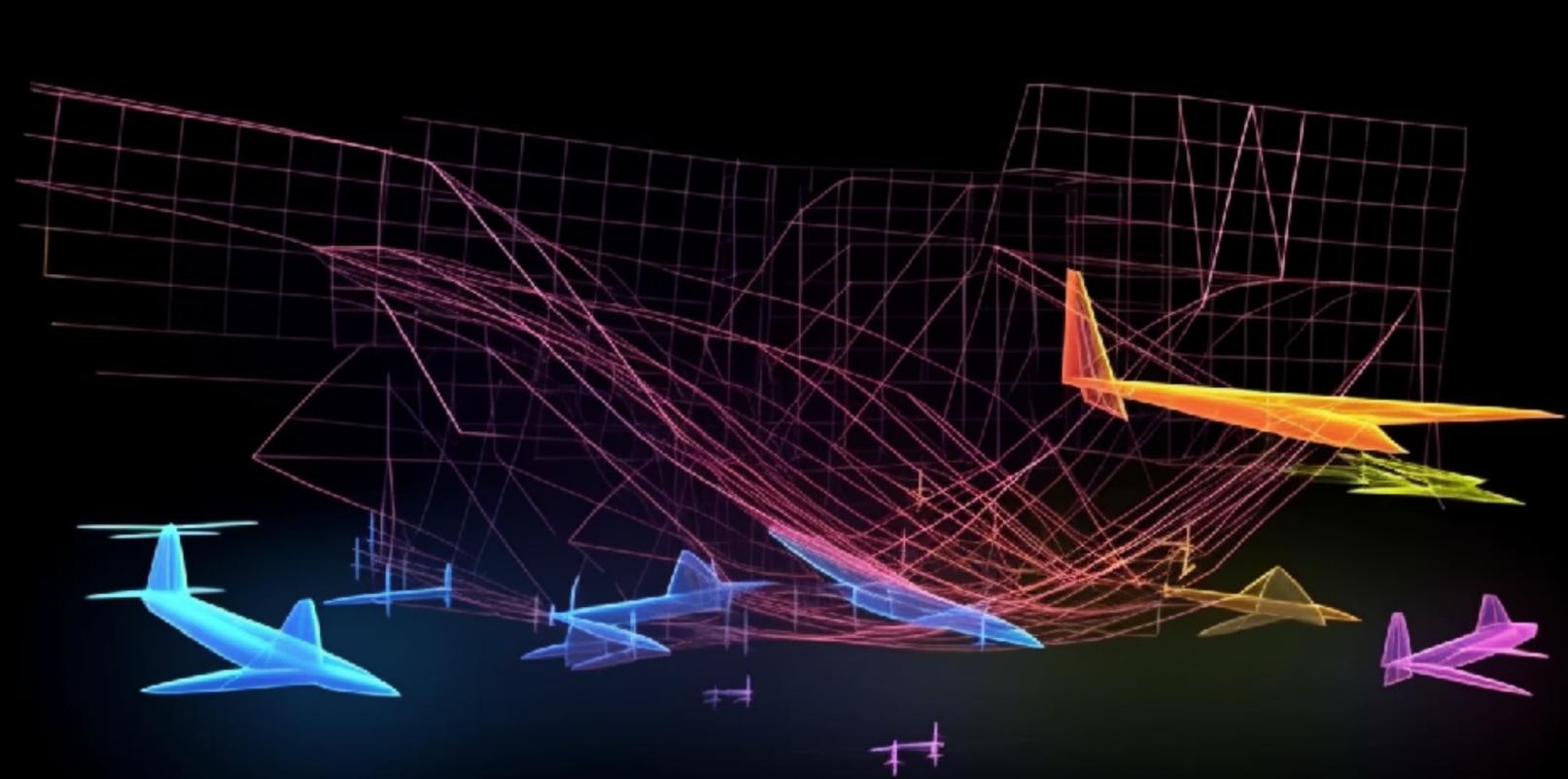
$$\cos(x)=0, \sin(x)=0, \tan(x)=0$$

Complex

$$e^{ix} = \cos(x) + i \sin(x)$$

Numerics

Numerical root finding



# Logarithm

Definition

start with examples base 10

Slide rule idea

multiplication becomes addition

Log rules

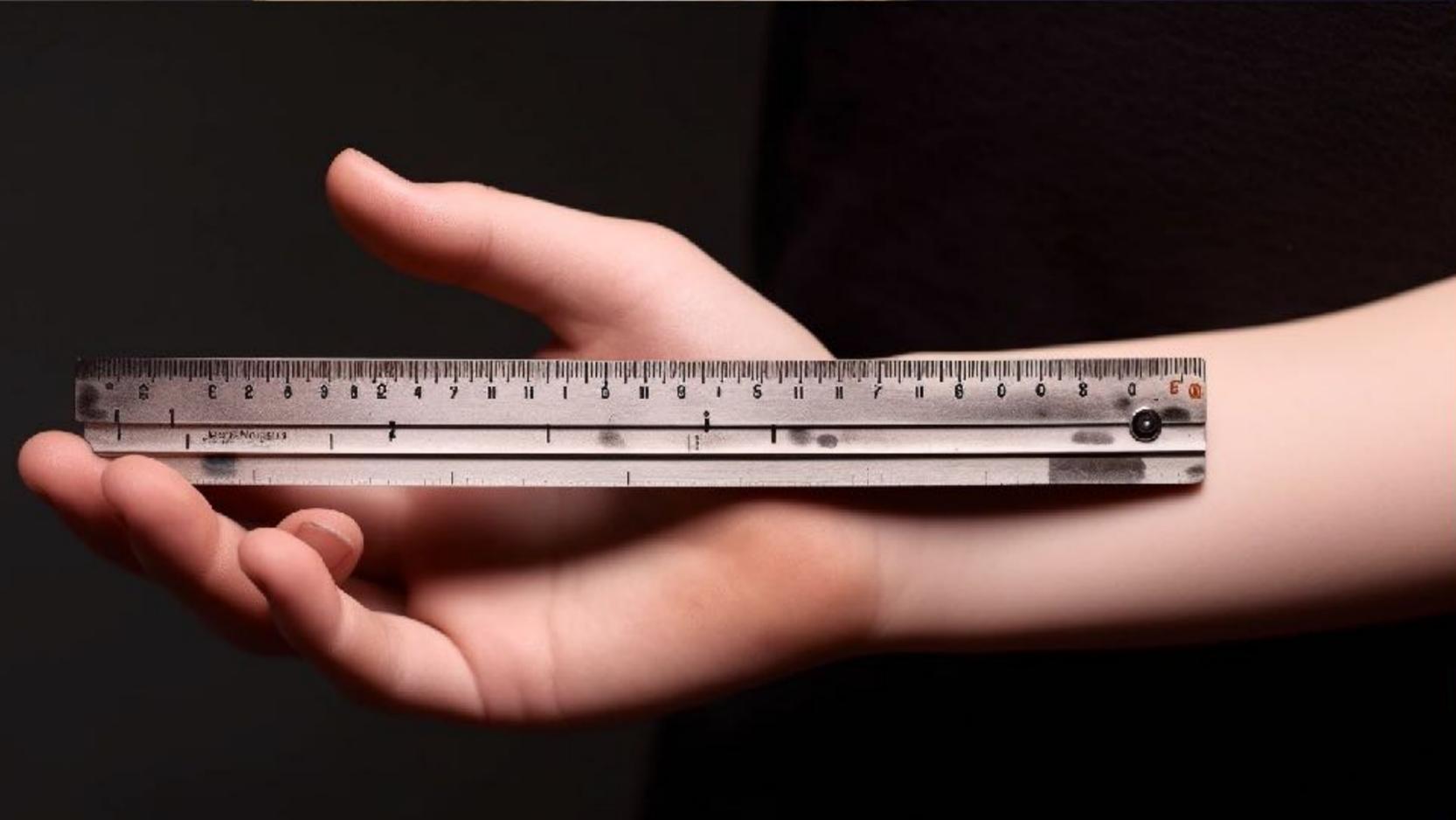
$\log(a \cdot b)$ ,  $\log(a^b)$

Change base

natural log, log 10, log base 2

Application

integration, natural coordinates



# Modeling

Applications

Capturing quantitative change

Prototypes

Hooke's law, Cooling law

Connections

Example: number theory

Relevance

Simplicity versus accuracy

Errors

How good is the model?



# Work Tips

Techniques

Prepare, Prelearn, Review

Library

Organize knowledge

Diaries

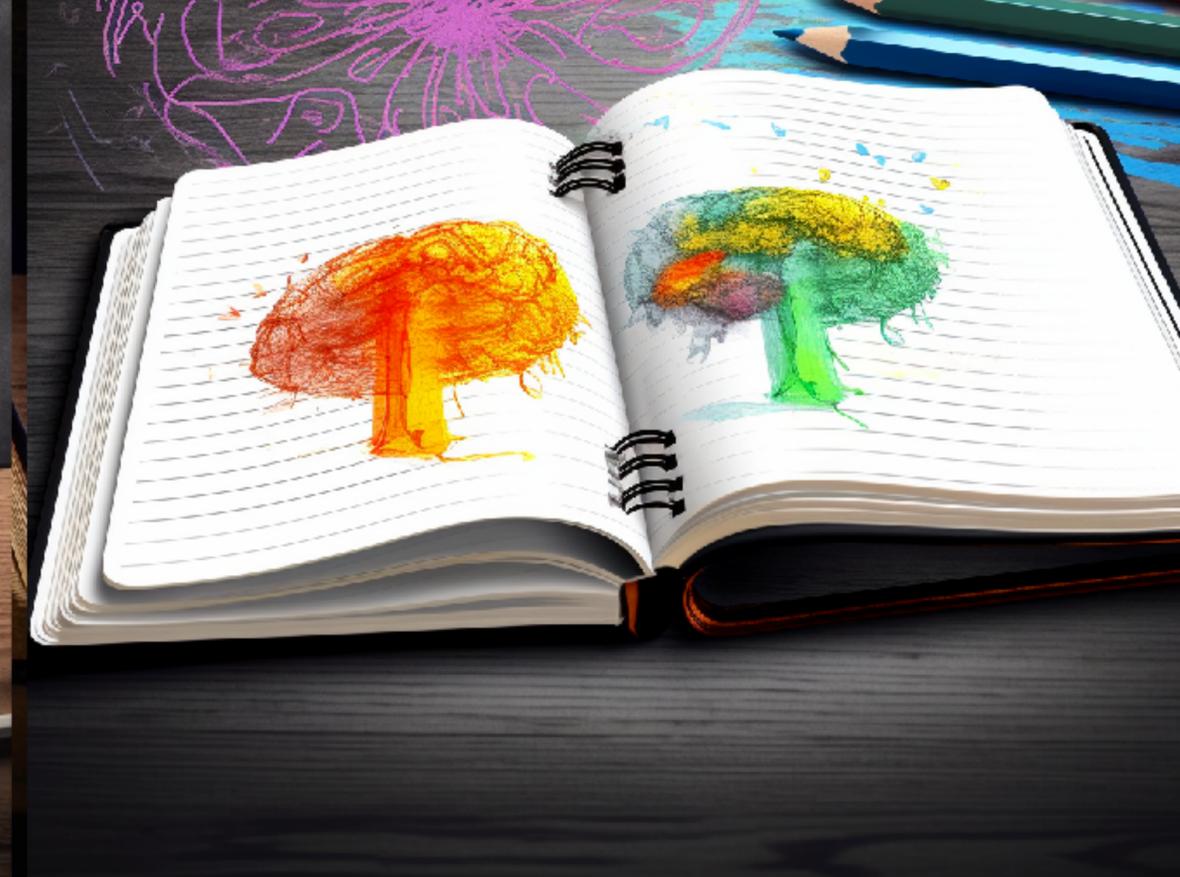
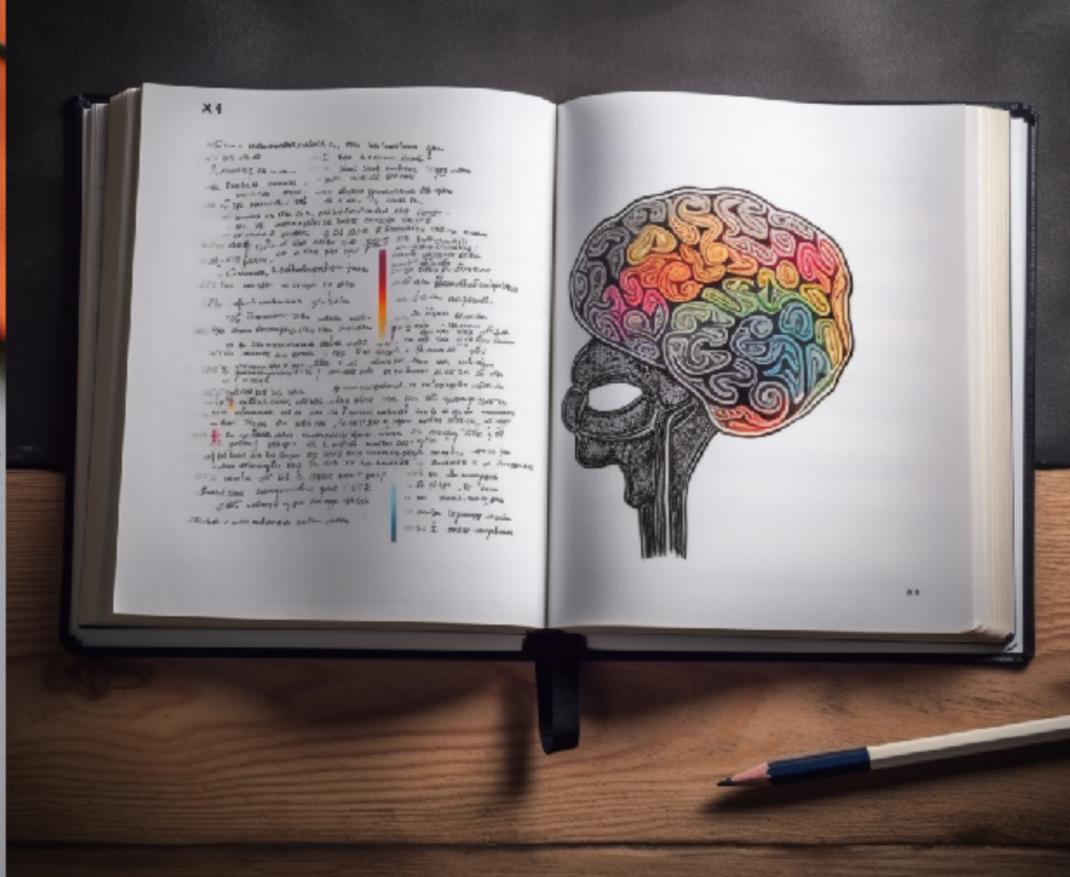
Notes, Ideas, Fragments

Questions

Stay curious

Resilience

Know about bias or psychology



Possible additional Topics

# Probability

Counting

counting with intersection

Rule of sum

when do probabilities add

Rule of Product

when do probabilities multiply

Choosing

choose 2 from 10

Averaging

add up and divide by number



# Numbers

Natural

0, Primes, even odd

Integers

Negative numbers

Fractions

Adding fractions

Reals

Pi, E, square roots

Complex

The imaginary  $i$





# Technology

Writing

From paper to tablets

CAS

Online or Programs

Games

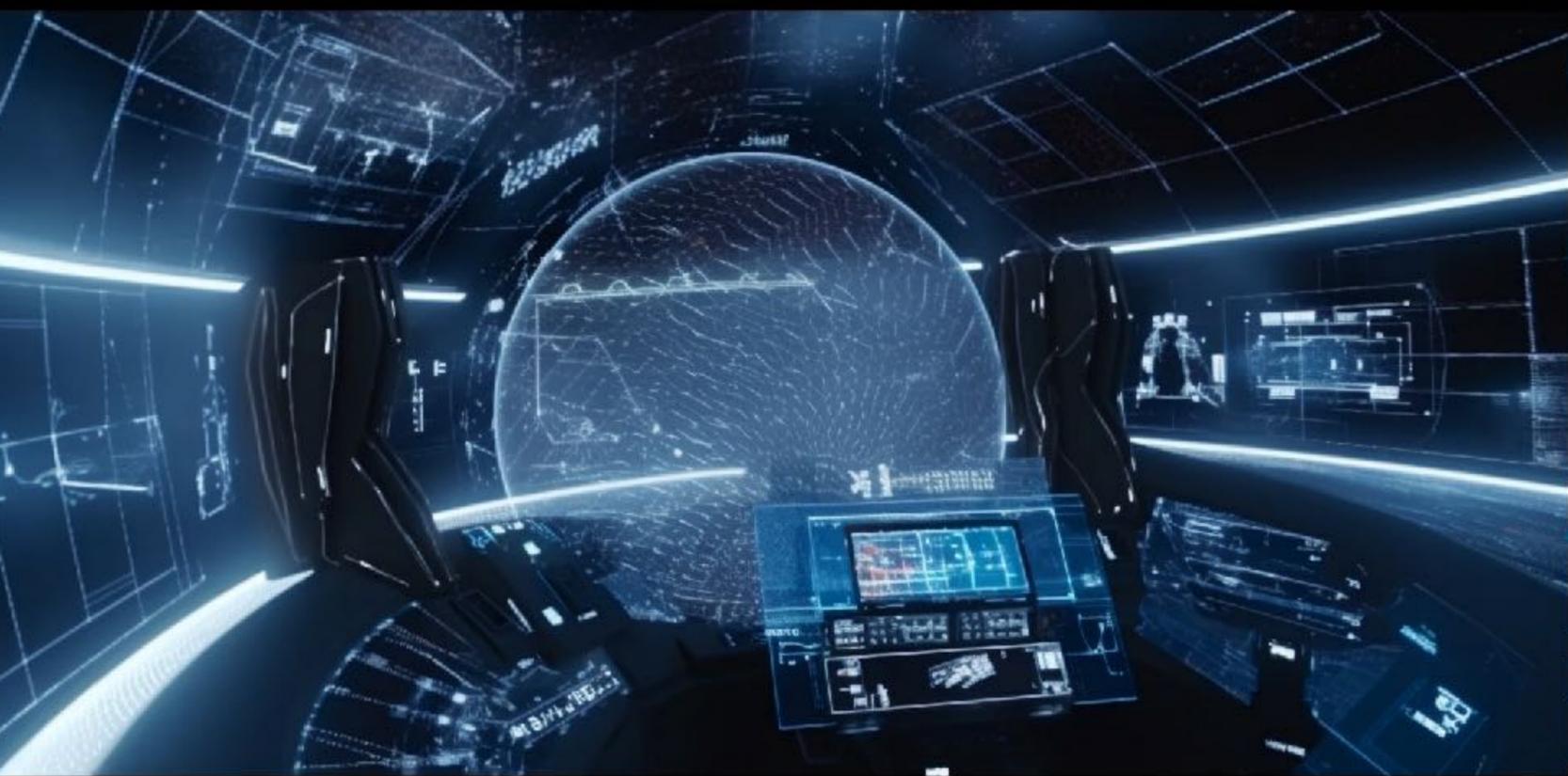
Developments

Media

Texts, Movies, Databases

AI

Bots and Generators



# History

Algebra

Roots in different places

Geometry

Planimetry

Calculus

Paradoxa

Diff equations

Predictions

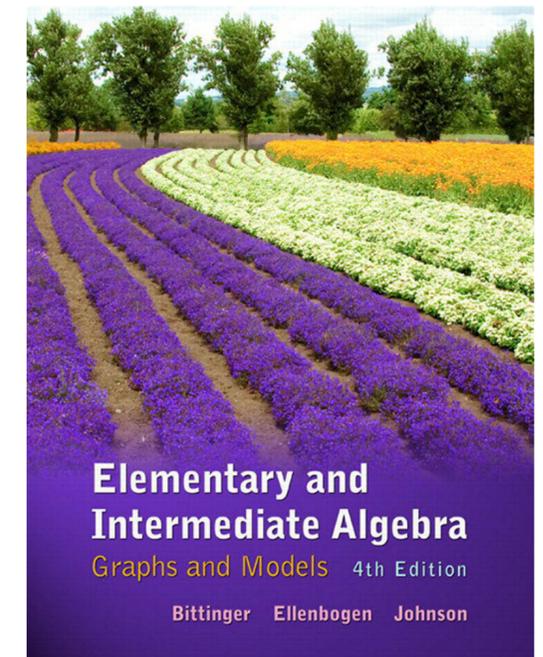
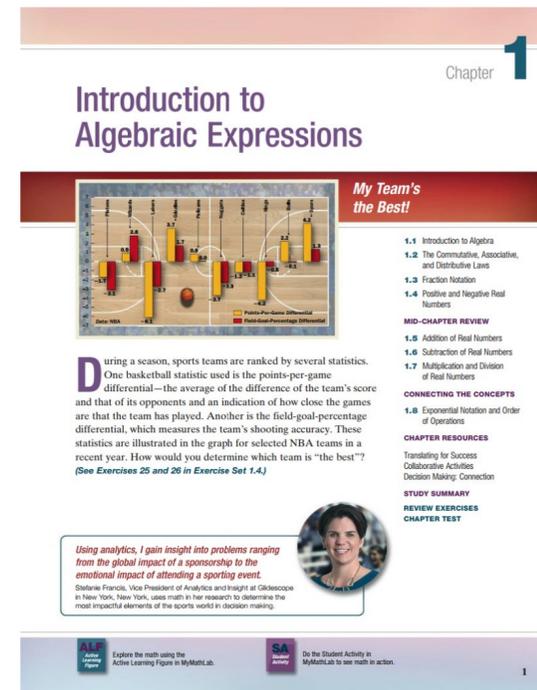
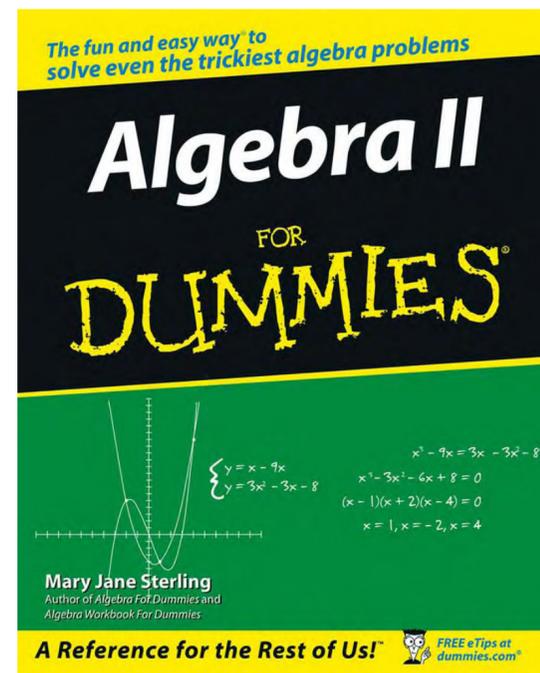
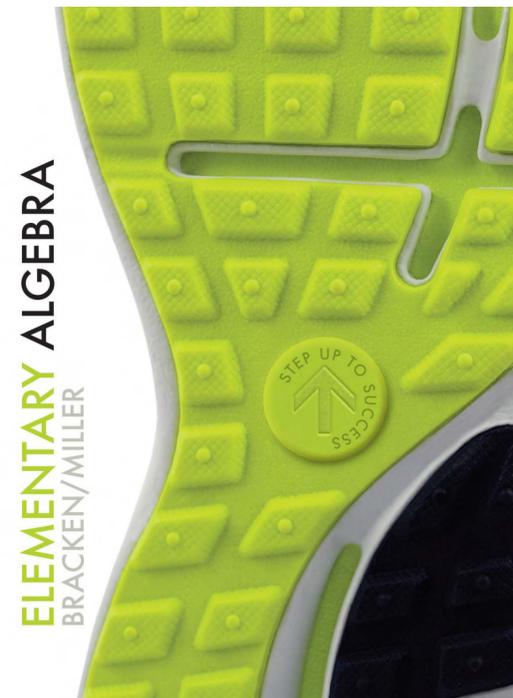
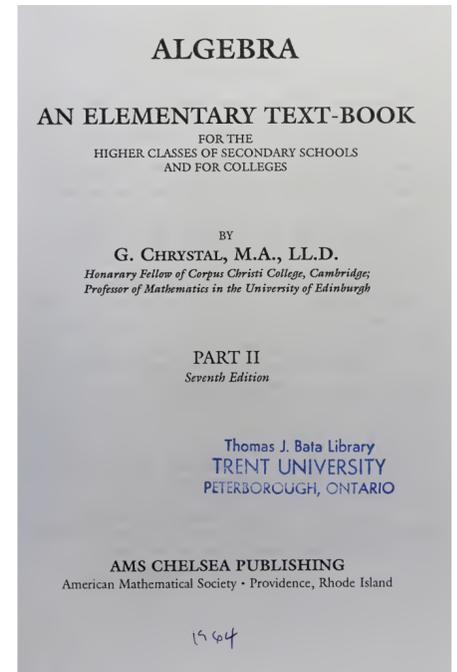
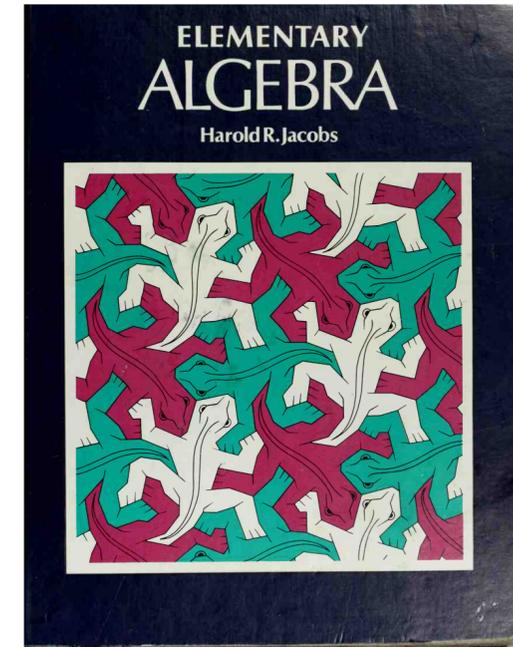
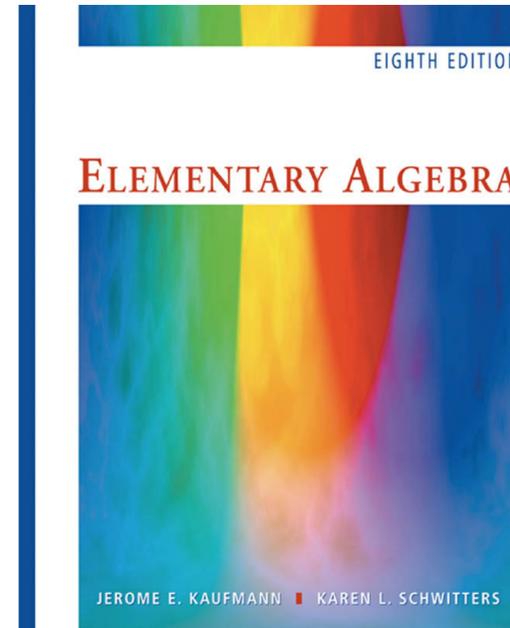
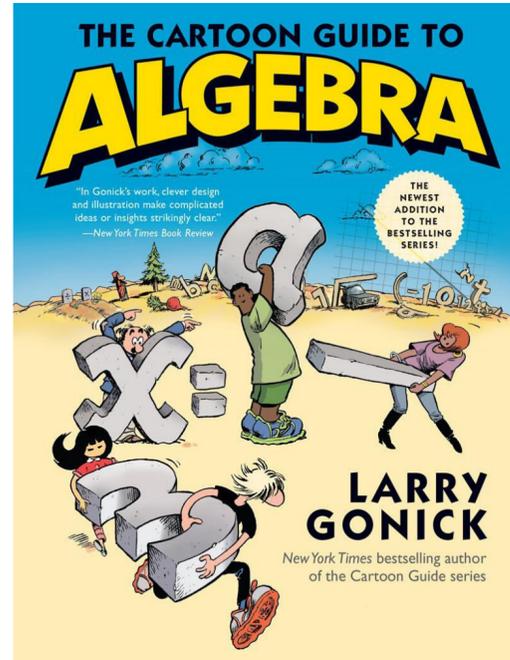
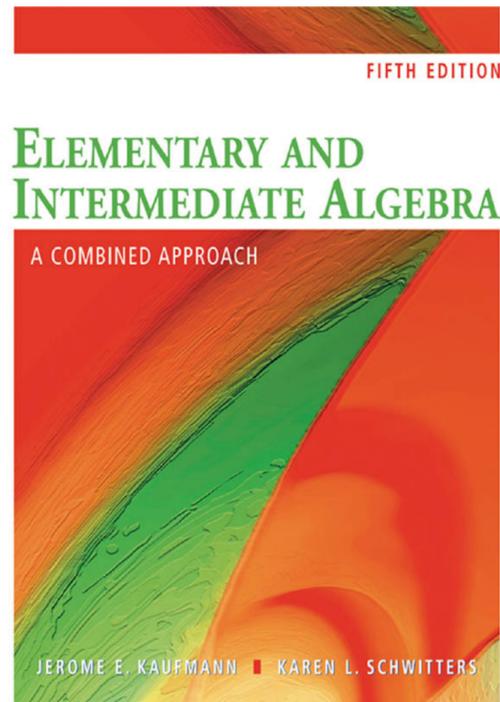
Probability

Gambling

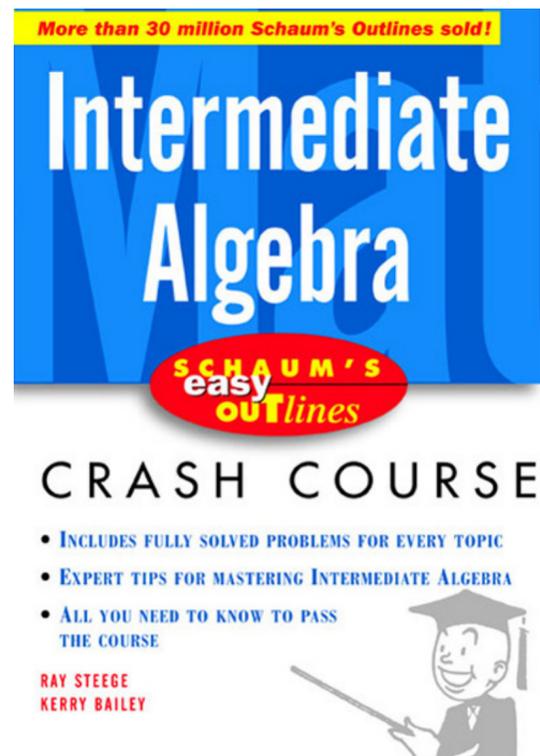
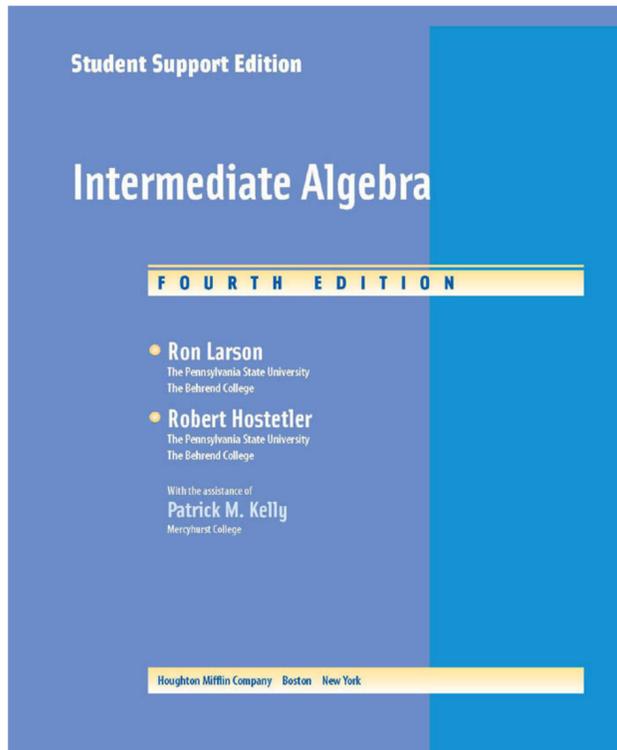
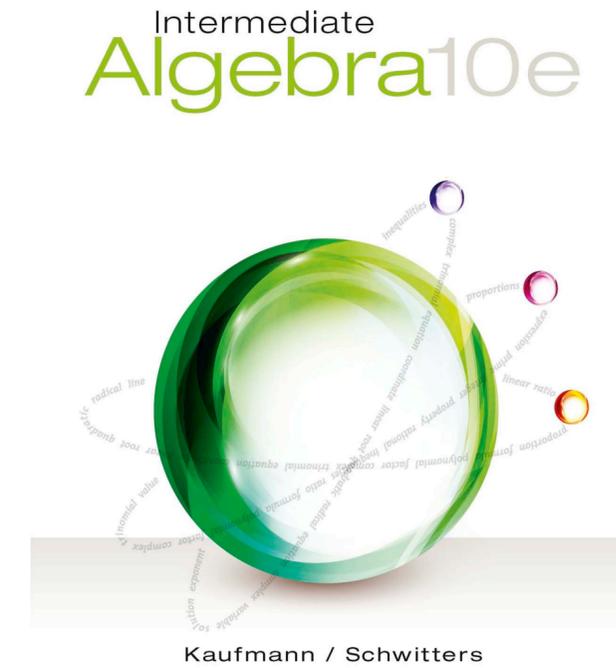
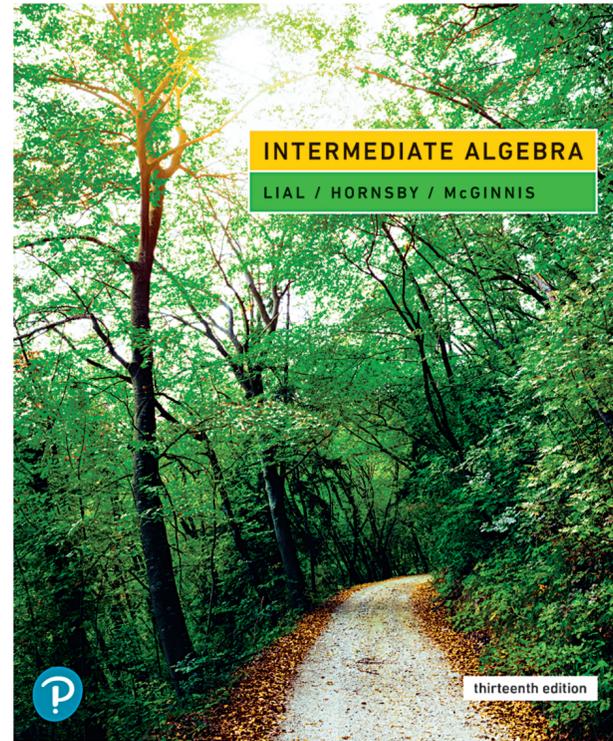
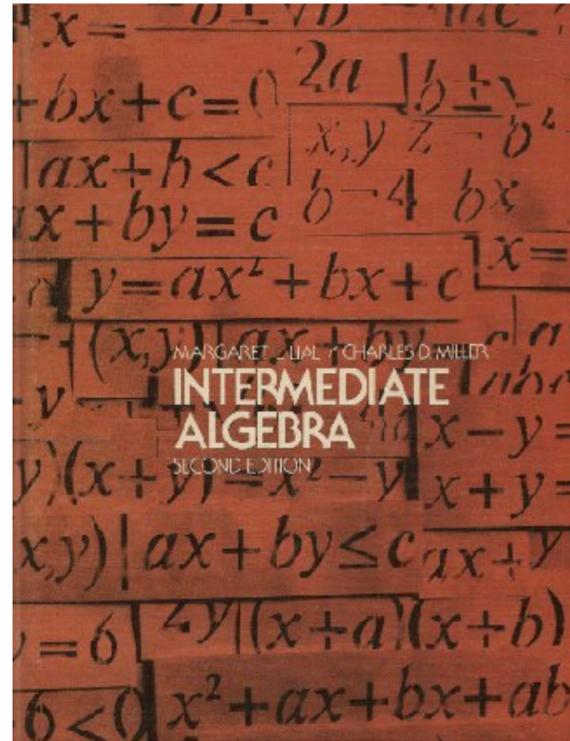


More About Algebra

# ELEMENTARY ALGEBRA BOOKS



# INTERMEDIATE ALGEBRA BOOKS



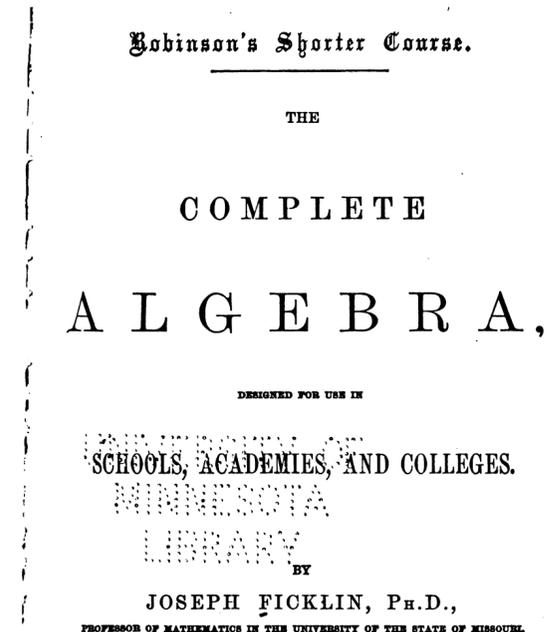
## INTERMEDIATE ALGEBRA

Ralph S. Underwood, *Professor of Mathematics, Texas Technological College*

Thomas R. Nelson, *Associate Professor of Mathematics, Agricultural and Mechanical College of Texas*

Samuel Selby, *Head of the Department of Mathematics, College of Engineering, University of Akron*

THE MACMILLAN COMPANY • NEW YORK • 1947



# MATH BASICS

SPARK CHART

## NUMBER SYSTEMS

### NATURAL NUMBERS

The **natural numbers** are the numbers we count with: 1, 2, 3, 4, ... Zero is not included.

### WHOLE NUMBERS

The **whole numbers** are the numbers we count with and zero: 0, 1, 2, 3, ...

### INTEGERS

The **integers** are the natural numbers, their **negatives**, and zero: ..., -3, -2, -1, 0, 1, 2, 3, 4, ...

- The **positive integers** are the natural numbers.
- The **negative integers** are the "minus" natural numbers: -1, -2, -3, -4, ...

### RATIONAL NUMBERS

The **rational numbers** are all the numbers that can be expressed as **fractions** (positive or negative, proper or improper).

Any rational number can be expressed as  $\frac{\text{integer}}{\text{non-zero integer}}$ .

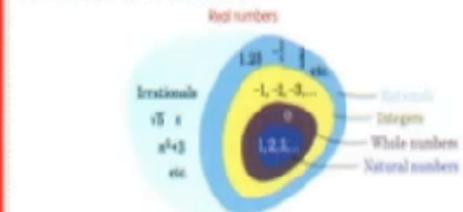
- All integers are rational. Ex:  $4 = \frac{4}{1}$
- All "terminating" decimals are rational. Ex:  $5.125 = \frac{5125}{1000}$

### REAL NUMBERS

The **real numbers** are all those that can be represented as points on a number line.

All rational numbers are real, but the **real number line** has many, many points that are "between" rational numbers.

Ex:  $\sqrt{2}$ ,  $\pi$ ,  $\sqrt{3} - 5$ , 0.1213211211321132...



### IMAGINARY NUMBERS

The **imaginary numbers** are square roots of negative numbers. They do not exist on the real number line.

- All of them are some real number multiplied by  $i = \sqrt{-1}$ .
- Ex:  $\sqrt{-49}$  is imaginary and is equal to  $7i$  or  $-7i$ .

### COMPLEX NUMBERS

The **complex numbers** are all possible sums of real and imaginary numbers. They are written as  $a + bi$ , where  $a$  and  $b$  are real and  $i = \sqrt{-1}$  is imaginary.

- All real numbers are complex (with  $b = 0$ ), all imaginary numbers are complex (with  $a = 0$ ).
- We represent the complex numbers on a 2-dimensional **complex plane**, with the horizontal axis representing the real and the vertical axis representing the imaginary numbers. The number  $a + bi$  is represented by the point  $(a, b)$ .

## NAMING WHOLE NUMBERS

### DIGITS VS. NUMBERS

- Digits** are symbols. Our number system (the Arabic system) uses 10 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
- Numbers** are actual values represented by some arrangement of digits. This is an abstract concept.

### PLACE VALUE

How much each digit is worth depends on its location within the number—in **place value**.

- Place values go up by powers of 10. The arrangement of digits in 234 can be represented as  $(2 \times 10^2) + (3 \times 10^1) + (4 \times 10^0)$ .
- Commas**. In numbers with more than 4 digits, commas separate off each group of three digits, starting from the right. These groups are read of together.

**Reading numbers**: The number 1,803,170,562 is read as "one billion, eight hundred and three million, one hundred and seventy thousand, five hundred and sixty-two."

The digit 7 in the ten thousands place is worth  $7 \times 10,000 = 70,000$ .

### ROUNDING

A **round number** ends with one or several zeros; how many zeros depends on the context. To **round** a number is to approximate it with the nearest round number. We specify what place—tens, hundreds, etc.—to round to.

- To "round a number to the nearest ... place," look at the place immediately to the right.
- If the digit there is 5 or larger, **round up**: increase the digit in the place being rounded to by 1 and drop all digits to the right.

## ARITHMETIC

### ADDITION

In the equation  $3 + 4 = 7$ , the numbers 3 and 4 are the **addends**, and 7 is their **sum**.

### SUBTRACTION

In the equation  $15 - 7 = 8$ , the number 15 is the **minuend**, 7 is the **subtrahend**, and 8 is their **difference**.

### MULTIPLICATION

In the equation  $4 \times 5 = 20$ , the number 4 is the **multiplicand**, 5 is the **multiplier**, and 20 is their **product**. Also 4 and 5 are both **factors** of the product 20.

- Multiplication by a natural number is "repeated addition":  $3 \times 4$  means "3 added to itself 4 times, or  $3 + 3 + 3 + 3$ ."
- Multiplication is **commutative**. Miraculously,  $4 \times 3$  (or 4 added to itself 3 times) gives the same answer:  $3 + 3 + 3 + 3 = 4 + 4 + 4 = 12$ . The fact that  $a \times b = b \times a$  is called the "commutative" property of multiplication (the two numbers can "move past" or commute with, each other).

### WAYS TO EXPRESS MULTIPLICATION

- Cross:  $4 \times 5 = 20$ .
- Dot:  $3 \cdot 9 = 27$ .
- Double pair of parentheses:  $(7)(9) = 56$ .
- Single pair of parentheses:  $9(6) = 54$ .

### DIVISION

In the equation  $36 \div 3 = 12$ , the number 36 is the **dividend**, 3 is the **divisor**, and 12 is the **quotient**.

- When working with whole numbers, we can **divide with remainder**:  $71 \div 5 = 9$ , remainder 3.
- In this example, the number 9 is still called the **quotient**, and 3 is the **remainder**.

- The quotient 9 above is sometimes called the **partial quotient** to distinguish from the **total quotient** (9) in the equation  $75 \div 3 = 25$ .

### DIVIDING BY ZERO IS NOT ALLOWED

- Ways to express division:
- Division sign:  $72 \div 9 = 8$ .
- Slash:  $45/3 = 15$ .
- Fraction:  $\frac{7}{4} = 1.75$ .
- Long division sign:  $6 \overline{)30} = 5$ .
- Colon (rare):  $30 : 6 = 5$ .

### ORDER OF OPERATIONS

Arithmetic operations (+, -,  $\times$ ,  $\div$ , and raising to powers) are always performed in a specific order. However, expressions enclosed in parentheses are evaluated first—also according to order of operation.

1. Parentheses
2. Exponents
3. Multiplication and Division (left to right)
4. Addition and Subtraction (left to right)

### AMERICAN REMINDS

This is sometimes expanded into the phrase "Please Excuse My Dear Aunt Sally." The phrase is somewhat misleading: multiplication and division have equal priority as do addition and subtraction.

Ex:  $3 + 2 \times 2^2 - (4 + 5) \times 2$

1. Parentheses: Evaluate  $(4 + 5) \times 2$ , performing multiplication before addition to get  $(4 + 5) = 14$ . We have  $3 + 2 \times 2^2 - 14$ .
2. Exponents: Evaluate  $2^2 = 4$ . We have  $3 + 2 \times 4 - 14$ .
3. Multiplication and division:  $2 \times 4 = 8$ . We have  $3 + 8 - 14$ .
4. Addition and subtraction:  $3 + 8 = 11$  and  $11 - 14 = -3$ , which is the answer.

### INEQUALITIES AND SIGNS

**Whichever property** if two numbers are not equal, then one of them is greater than the other one.

Sign	Meaning	Example
<	less than	$1 < 2$ and $4 < 56$ and $-29 < -3$
>	greater than	$2 > 0$ and $56 > 4$ and $-3 > -29$
≤	less than or equal to	$1 \leq 1$ and $1 \leq 2$
≥	greater than or equal to	$1 \geq 1$ and $3 \geq -29$
≠	not equal to	$0 \neq 5$ and $-4 \neq 1$

The **diver** end of an inequality sign points toward the smaller number; the open part towards the larger.

**AMERICAN**: You see the signs a mouth. The mouth wants to eat the larger number.

## CIRCLES (continued)

### POWER OF A POINT THEOREMS

These theorems are about segments formed by intersections of chords, secants, or tangents with a circle and with one another.

**ARC/CHORD**: Relative to a circle, every point  $P$  has a constant "power"—the product of the distances between  $P$  and the two intersection points with the circle along any line.

### INTERSECTION OF TWO CHORDS

The product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other chord.



$$(AP)(PC) = (BP)(PD)$$

### INTERSECTION OF TWO SECANTS

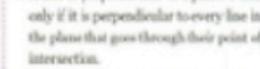
When two secant segments share an endpoint outside the circle, the products of the lengths of each secant segment with its external part are equal.



$$(AP)(BP) = (CP)(DP)$$

### INTERSECTION OF SECANT AND TANGENT

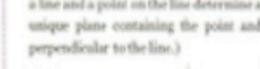
When a secant segment and a tangent segment share an endpoint not on the circle, the length of the tangent segment squared is equal to the product of the secant segment and its external part.



$$AP^2 = BP \cdot CP$$

### INTERSECTION OF TWO TANGENTS

When two tangent segments share an endpoint outside the circle, they are congruent.



$$AP = BP$$

### CIRCLES AND POLYGONS

A circle is said to be "**inscribed** in a polygon" if all of the sides of the polygon are tangent to the circle. (We can also say that the polygon is "circumscribed about the circle.")



A circle is said to be "**circumscribed** about a polygon" if all of the vertices of the polygon are on the circle. (We can also say that the polygon is "inscribed in the circle.")

The length of the tangent segment squared is equal to the product of the secant segment and its external part.

**Intersection of two tangents**: Tangent segments that share an endpoint outside the circle are congruent.



Circle  $O$  is circumscribed about heptagon  $ABCDEF$ .

## LINES AND PLANES IN SPACE

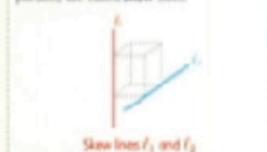
### TWO LINES

Lines either do or do not lie in the same plane.

Two lines that lie in the same plane either **intersect** or are **parallel**. Conversely...

- If two lines in space intersect, then they lie in the same plane.
- If two lines in space are parallel, then they lie in the same plane.

Two lines that do not lie in the same plane (equivalently, neither intersect nor are parallel) are called **skew lines**.



Skew lines  $f_1$  and  $f_2$

### LINE AND PLANE

A line and a plane in space always either **intersect** or are **parallel**.

- If they are parallel, they never intersect.
- A line is perpendicular to a plane if and only if it is perpendicular to every line in the plane that goes through their point of intersection.

For every point in the plane, there is a unique line perpendicular to the plane that goes through that point. (Similarly, a line and a point on the line determine a unique plane containing the point and perpendicular to the line.)

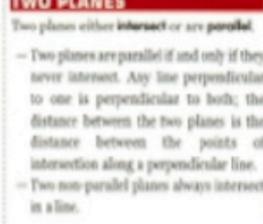


Line  $l$  is parallel to plane  $p$ .

### TWO PLANES

Two planes either **intersect** or are **parallel**.

- Two planes are parallel if and only if they never intersect. Any line perpendicular to one is perpendicular to both; the distance between the two planes is the distance between the points of intersection along a perpendicular line.
- Two non-parallel planes always intersect in a line.



Planes  $p$  and  $q$  are perpendicular. Line  $l$  is their intersection.

## SOLIDS IN SPACE

### SOLIDS IN SPACE

A **polyhedron** (pl. **polyhedra**) is a solid figure formed by the intersection of several (at least four) **planes**. The planes intersect in polygonal **faces** whose sides are called **edges** and whose corners are the **vertices** of the polyhedron.

**Surface Area**: Total area of all of the faces of the polyhedron.

**Volume**: A measure of how much space fits inside a solid figure, calculated in cubic units.

**Parallelepiped**: A polyhedron whose six faces are parallelograms lying in pairs of parallel planes. Rectangular prisms are parallelepipeds whose adjacent faces lie in perpendicular planes.

**Volume**:  $V = (\text{length}) \times (\text{width}) \times (\text{height})$

**Surface Area**:  $SA = 2(lw + lh + wh)$

**Rectangular Solid**: A polyhedron with six rectangular faces. Adjacent faces intersect at right angles. Has three measurements: length, width, height.

**Volume**:  $V = (\text{length}) \times (\text{width}) \times (\text{height})$

**Surface Area**:  $SA = 2(lw + lh + wh)$

**Parallelepiped**: A polyhedron whose six faces (the **bases**) are congruent polygons lying in parallel planes; the other faces (the **lateral faces**) are parallelograms that join corresponding sides on the congruent polygons. The sides that join the lateral parallelograms to each other are called **lateral edges**.

**Right triangular prism**: Two of three lateral faces are shaded blue.

**Lateral area** (for right prism):  $LA = (\text{perimeter of base}) \times (\text{height})$

**Special types of prisms**: Parallelepipeds are prisms with a parallelogram base. Rectangular solids and cubes are right

### SOLIDS IN SPACE

**Volume**:  $V = (\text{side length})^3$

**Surface Area**:  $SA = 6(\text{side length})^2$

**Height**: The (perpendicular) distance between the bases (or, rather, the planes containing them).

**Volume**:  $V = (\text{Base area}) \times (\text{height})$

**Lateral area**: The area of the lateral faces of a prism. (Lateral Area) + 2(Base Area) = Surface Area

**Right prism**: A prism whose lateral edges are perpendicular to the planes containing the bases of the prism.

**Volume**:  $V = (\text{Base area}) \times (\text{height})$

**Lateral area** (for right prism):  $LA = (\text{perimeter of base}) \times (\text{height})$

**Special types of prisms**: Parallelepipeds are prisms with a parallelogram base. Rectangular solids and cubes are right

### CYLINDER

A **cylinder** is analogous to a prism, except its bases are circular. A **cylinder is not a polyhedron** (just like a circle is not a polygon). The lateral area, the height—and all of the computational formulas—are defined analogously.

The **axis** of a cylinder is the line connecting the centers of the two bases.

**Volume**:  $V = (\text{Base area}) \times (\text{height})$

**Lateral area**: The area of the lateral faces of a prism. (Lateral Area) + 2(Base Area) = Surface Area

**Right prism**: A prism whose lateral edges are perpendicular to the planes containing the bases of the prism.

**Volume**:  $V = (\text{Base area}) \times (\text{height})$

**Lateral area** (for right prism):  $LA = (\text{perimeter of base}) \times (\text{height})$

**Special types of prisms**: Parallelepipeds are prisms with a parallelogram base. Rectangular solids and cubes are right

## EXPONENTS AND POWERS

**Exponential notation** is shorthand for repeated multiplication.

$2 \cdot 3 = 2^2$  and  $(-2)(-2)(-2)(-2) = (-2)^4$ . In the notation  $a^n$ ,  $a$  is the **base**, and  $n$  is the **exponent**. The whole expression is " $a$  to the  $n$ th power," or the " $n$ th power of  $a$ , or simply, " $a$  to the  $n$ ."

$a^2$  is " $a$  squared,"  $a^3$  is " $a$  cubed."

$(-a)^2$  is not necessarily the same as  $-a^2$ .

Ex:  $(-4)^2 = 16$ , whereas  $-4^2 = -16$ . Following the order of operation rules,  $-4^2 = -(4^2)$ .

### RULES OF EXPONENTS

**Product of powers**:  $a^m a^n = a^{m+n}$ . If the bases are the same, then to multiply, simply add their exponents. Ex:  $2^3 \cdot 2^2 = 2^5$ .

**Quotient of powers**:  $\frac{a^m}{a^n} = a^{m-n}$ . If the bases of two powers are the same, then to divide, subtract their exponents.

**Exponentiation powers**:  $(a^m)^n = a^{m \cdot n}$ . To raise a power to a power, multiply exponents.

### POWER OF A PRODUCT

$(ab)^n = a^n b^n$ . Exponentiation distributes over multiplication and division, but not over addition or subtraction. Ex:  $(3+2)^2 = 5^2 = 25$ , but  $3^2 + 2^2 = 9 + 4 = 13$ .

**Zeroth power**:  $a^0 = 1$ . To be consistent with all the other exponent rules, we set  $a^0 = 1$  unless  $a = 0$ . The expression  $0^0$  is undefined.

**Negative powers**:  $a^{-n} = \frac{1}{a^n}$ . We define negative powers as reciprocals of positive powers. This works well with all other rules. Ex:  $2^3 \cdot 2^{-4} = 2^{-1} = \frac{1}{2}$ . Also,  $2^3 \cdot 2^{-1} = 2^{3-1} = 2^2 = 4$ .

**Fractional powers**:  $a^{\frac{1}{n}} = \sqrt[n]{a}$ . This definition, too, works well with all other rules.

## ROOTS AND RADICALS

**Using root undoing raising to powers**:  $\sqrt{9} = 3$  because  $3^2 = 9$ . The expression  $\sqrt[n]{a}$  means "the  $n$ th root of  $a$ ."

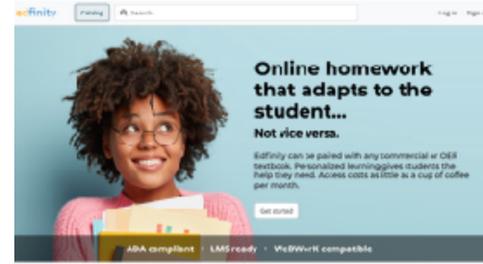
—The radical in the root sign  $\sqrt{\quad}$ .

Companies

# Brave new world



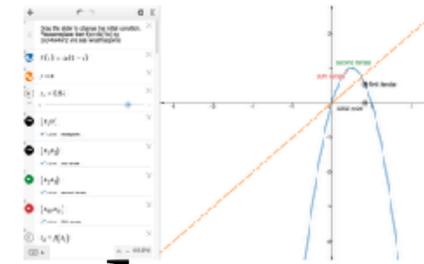
canvas



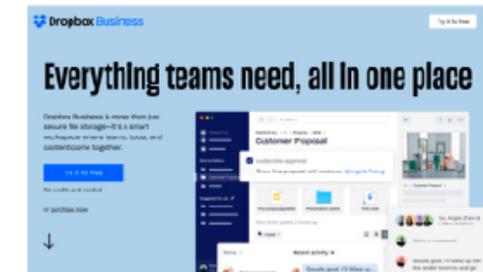
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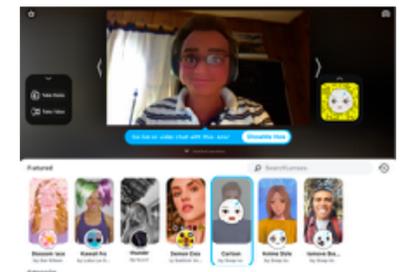
gradescope



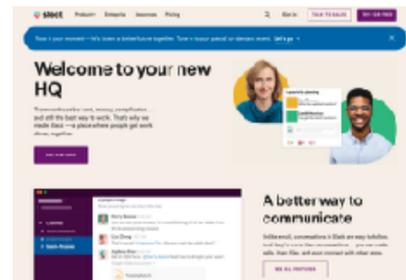
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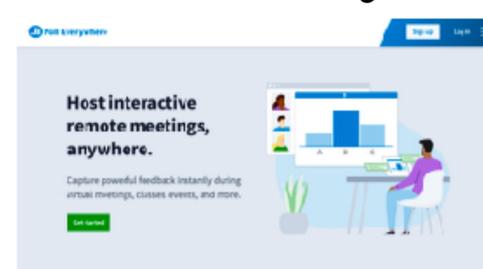
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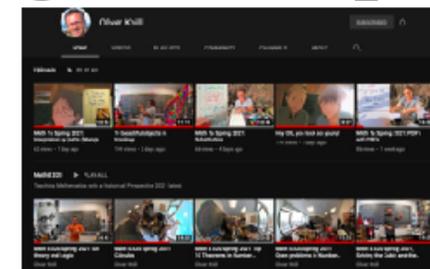
snapchat



slack



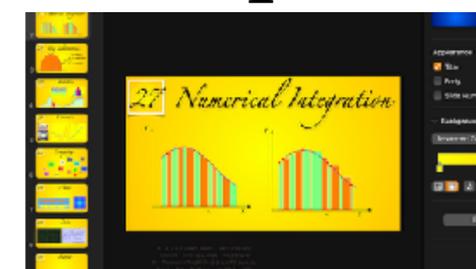
pollev



youtube



notability



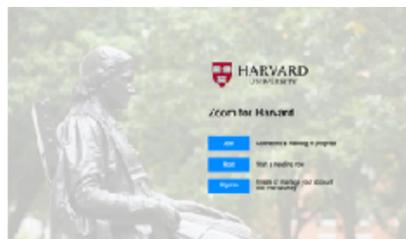
keynote



jamboard



overleaf



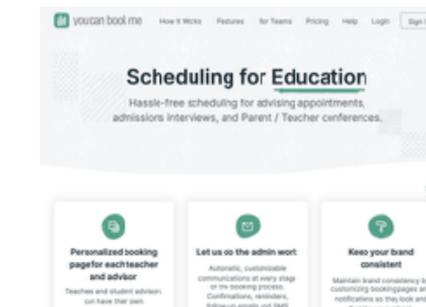
zoom



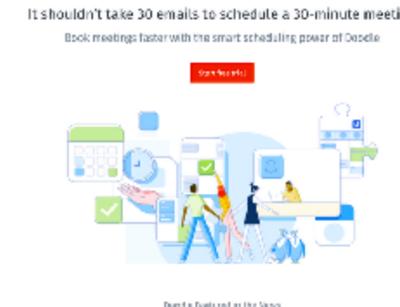
Wolfram



panopto



youcanbookme



doodle

# New Kids in Town

Chat GPT

Google Bard

Snapchat

Wolfram Alpha

Midjourney

# Google Bard



find the anti derivative of the function  $x^5 \sin(x)$



View other drafts 



The antiderivative of the function  $x^5 \sin(x)$  is  $-\frac{x^6 \cos(x)}{6} + C$ , where  $C$  is an arbitrary constant.

To find the antiderivative, we can use the following formula:

$$\int x^n \sin(x) dx = -\frac{x^{n+1} \cos(x)}{n+1} + C$$

In this case,  $n = 5$ , so we have:

$$\int x^5 \sin(x) dx = -\frac{x^6 \cos(x)}{6} + C$$

This is the antiderivative of the function  $x^5 \sin(x)$ .

# Chat GPT and AI ....

 SIGN IN / UP

The Register®



**OpenAI's ChatGPT is a morally corrupting influence**

No principles or virtues, people accepting everything it says – this

**colleagues, says watchdog boss**

Surely there are quicker and more effective ways to end your co-w

**New IT boss decided to 'audit everything you guys are doing wrong'. Which went wrong**

**ON CALL** Pet consultant took down the datacenter in attempt to find other people's errors

**Hey, online pharmacies: Quit spreading around everyone's data already**

# Chat GPT

$$c^2 = a^2 + b^2$$

Proof:

Draw a right triangle with legs of length  $a$  and  $b$  and hypotenuse of length  $c$ .

Draw a square of side length  $c$  around the triangle, as shown in the figure.

Draw two squares of side lengths  $a$  and  $b$  inside the square of side length  $c$ , as shown in the figure.

The area of the square of side length  $c$  is  $c^2$ , the area of the square of side length  $a$  is  $a^2$ , and the area of the square of side length  $b$  is  $b^2$ .

The area of the square of side length  $c$  is equal to the sum of the areas of the squares of side lengths  $a$  and  $b$ , since the smaller squares are completely contained inside the larger square.

So,  $c^2 = a^2 + b^2$ .

This completes the proof of the Pythagorean theorem. It is worth noting that the theorem is not limited to right triangles only, it holds for any triangle with the same sides and hypotenuse and legs, and it is called the law of cosines.

You use the  
Theorem to prove it!

In general,  
the result is different  
and reads then as

$$c^2 = a^2 + b^2 - 2ab \cos(\alpha)$$

# Automatic Problem Generation

Math 1A: Introduction to functions and calculus

Sofia Bot, 2014

```
(* Produce a random function g[x] of some complexity
(* and use it to generate differentiation problems
(* Oliver Knill, March 27, 2004 Sofia problem *)

$DisplayFunction=Identity;

RandomFraction:=Apply[Union,Array[Range[#-1]/#&,Random[Integer,10]+2]][[10]]
Ide[x_]:=x;
Pow[x_]:=x^Random[Integer,{2,5}];
Inv[x_]:=x^Random[Integer,{-3,-1}];
Sca[x_]:=Random[Integer,{-3,3}]*x;
Tra[x_]:=x+Random[Integer,{-4,4}];
FunctionList={Sin,Cos,Log,Exp,Tan,Sqrt,Pow,Inv,Sca,Tra};
RRandomFunction:=FunctionList[[Random[Integer,{1,Length[FunctionList]}]]];

ComposeF[f_,g_]:=Function[x,Evaluate[f[g[x]]];
ScaleF[f_]:=Function[x,Evaluate[Random[Integer,3]+1]*f[x]];
AddF[f_,g_]:=Function[x,Evaluate[f[x]+g[x]]];
SubtractF[f_,g_]:=Function[x,Evaluate[f[x]-g[x]]];
MultiplyF[f_,g_]:=Function[x,Evaluate[f[x]*g[x]]];
DivideF[f_,g_]:=Function[x,Evaluate[f[x]/g[x]]];
OpList={ComposeF,AddF,SubtractF,MultiplyF,DivideF};
RandomOperation[f_,g_]:=OpList[[Random[Integer,{1,Length[OpList]}]]][f,g];

BuildFunction[n_]:=Module[{f0=Ide,Do[f0=RandomOperation[f0,RRandomFunction],{n}];f0=ScaleF[f0];f0[x]}
BuildPolynomial[n_]:=Module[{f0=1,Do[f0=f0*(x-Random[Integer,6]-3),{k,2-Random[Integer,n]}];f0];

WS=WriteString;Fn="problem.tex";

(* random function *)

WriteMPProblem[L_]:=Module[{S1,S2,S3,S4,
g]=Function[y,Evaluate[BuildFunction[L]/.x->y]];
"worksheet.m" 153L, 8054B 1,1 Top
```

Mathematica code  
March 27, 2004

## Lecture 36: Worksheet

This worksheet as well as the solutions was generated by Sofia, a bot written in the academic year 2003/2004 using grant from the Harvard Provost together with Harvard students **Johnny Carlsson**, **Andrew Chi** and **Mark Lezama**. At that time, people have laughed at the chat bot idea. Now it is big business: Google, Siri, Cortana, Wolfram alpha: these are all AI bots which constantly become more and more sophisticated.

1 Differentiate the following functions:

- $f(x) = 2 \sin(x)$
- $f(x) = x - \cos(x)$
- $f(x) = 4$

Solution:

- $f'(x) = 2 \cos(x)$
- $f'(x) = \sin(x) + 1$
- $f'(x) = 0$

2 Integrate the following functions:

- $f(x) = -4(x \cot(x) - 1) \csc(x)$
- $f(x) = 0$
- $f(x) = 1$

Solution:

- $\int f(x) = 4x \csc(x) + C$
- $\int f(x) = 1 + C$
- $\int f(x) = x + 3 + C$

3 Differentiate the following functions:

- $f(x) = 3\sqrt{\log(x)}$
- $f(x) = 3(x-1)(x-e^x)$
- $f(x) = \frac{1}{x^3}$

Solution:

- $f'(x) = \frac{3}{2x\sqrt{\log(x)}}$
- $f'(x) = -3(e^x - 2)x - 3$
- $f'(x) = -\frac{3}{x^4}$

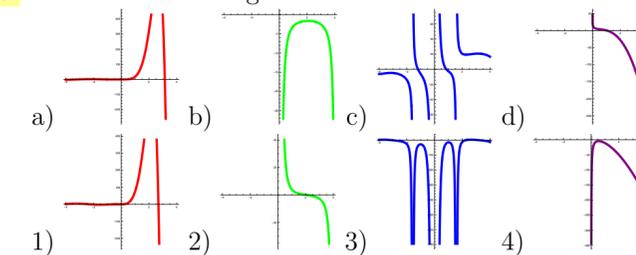
4 Integrate the following functions:

- $f(x) = \frac{2(5x^5+x-1)}{x}$
- $f(x) = -\frac{2^x}{\sqrt{x}}$
- $f(x) = \frac{3(x+e^x+(e^x+1)x \log(x))}{x}$

Solution:

- $\int f(x) = 2(x^5 + x - \log(x)) + C$
- $\int f(x) = 4(-\sqrt{x} - 1) + C$
- $\int f(x) = 3(x + e^x) \log(x) + C$

5 Match the following functions with derivatives:



Solution:

a → 1, b → 2, c → 3, d → 4

6 Find the critical points of the following functions:

- $f(x) = (x-9)^2(x-7)$
- $f(x) = (x-9)(x-7)^2$
- $f(x) = (x-6)(x-3)$

Solution:

- $f'(x) = \left\{ \left\{ x \rightarrow \frac{23}{3} \right\}, \left\{ x \rightarrow 9 \right\} \right\}$
- $f'(x) = \left\{ \left\{ x \rightarrow 7 \right\}, \left\{ x \rightarrow \frac{25}{3} \right\} \right\}$
- $f'(x) = \left\{ \left\{ x \rightarrow \frac{9}{2} \right\} \right\}$

# Filters

# FILTERS





ZOOM



# MUSIC



youngguru763 • [Follow](#)



youngguru763 I've been trying to tell everyone that this is where we are now with AI. For some reason this one got everyone's attention. So what do we do. On one hand I'm well aware that you can't stop technology. Once the genie is out of the box you can put him back in. On the other hand we have to protect the rights of the artist. Not only artist but everyone in society. People should not be able to take your Name, Image and Likeness without permission. We have to add the voice to this law. We have to learn from past mistakes. You would be a fool to chase every person that is going to do this. We learned that lesson with Napster. The only way I see to deal with it is to change the law. There



22,266 likes

4 DAYS AGO

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# Image Generators



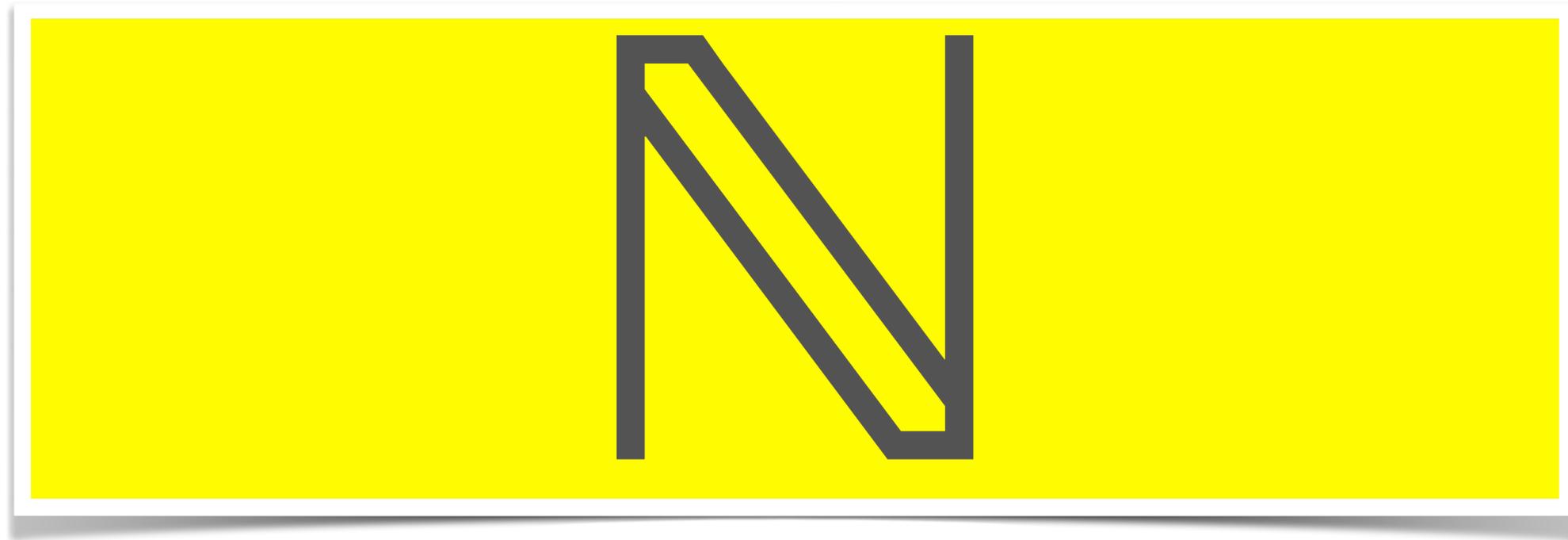
# Planning Stage

# TOP TEN ARITHMETIC SKILLS

A photograph of the Harvard University campus in Cambridge, Massachusetts. The central focus is the tall, white steeple of the Harvard Chapel, which rises above a cluster of red brick buildings. The foreground is filled with lush green trees, and the sky is a clear, bright blue. The text 'TOP TEN ARITHMETIC SKILLS' is overlaid in large, white, sans-serif capital letters across the top half of the image.

September 24, 2022  
Oliver Knill, Harvard University

# 1) NATURAL NUMBERS



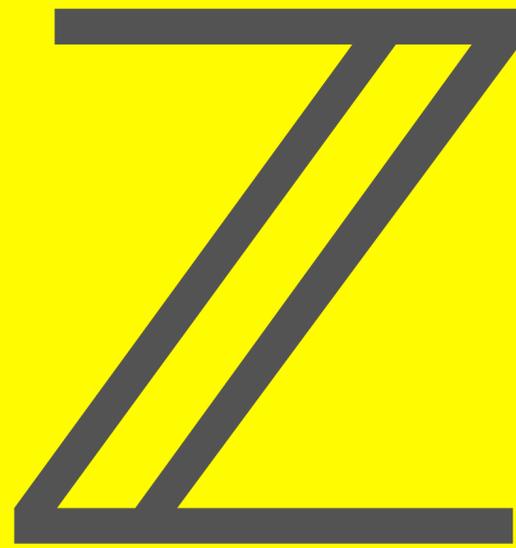
Natural numbers go back many thousands  
of years. Counting:

1, 2, 3, 4, .....

Primes are natural numbers  
that are only divisible by 1 or  
themselves

101 is a prime number

## 2) INTEGERS



Negative numbers

Zero

$$5x + 1 = 9 - 3x$$

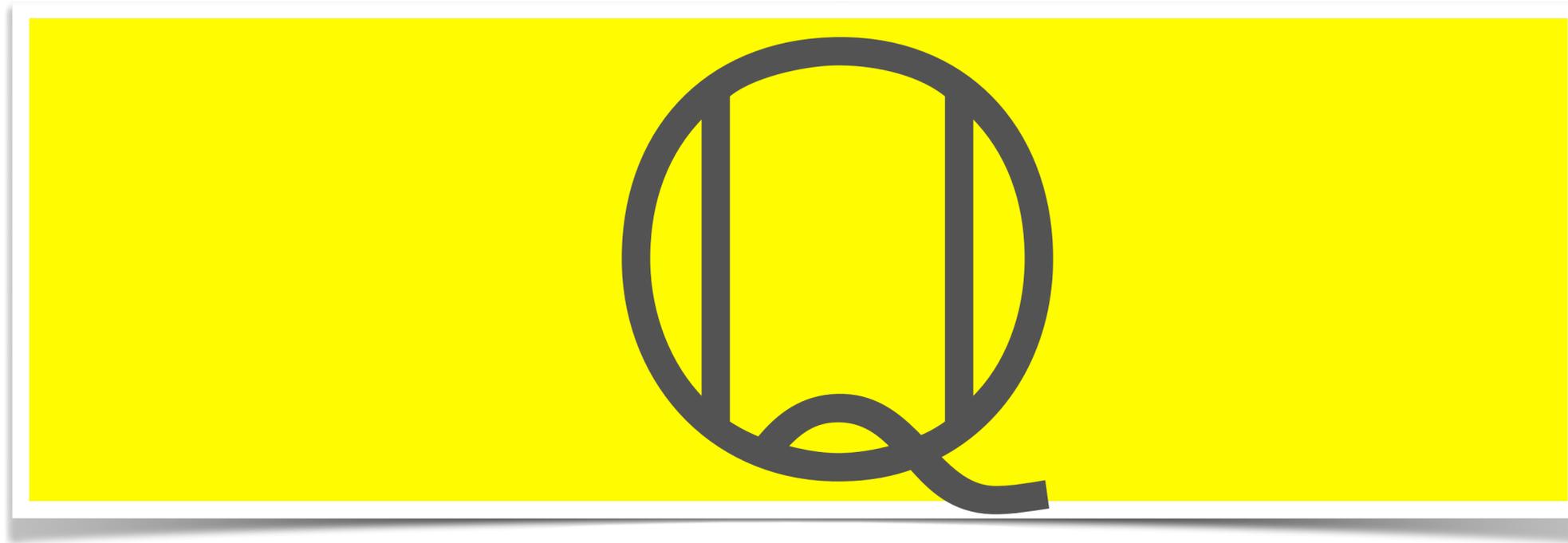
Bring all x variables to the left

$$5x + 3x = 9 - 1$$

Simplify

$$x = 1$$

# 3) FRACTIONS



When solving linear equations we are automatically  
led to fractions.  $p/q$

# Canvas Quizzes

[0 Section 1 \(25678\)](#) > [Quizzes](#) > [Lecture 03 Quiz](#)

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63 Student View

## Lecture 03 Quiz

Keep Editing This Quiz

This is a preview of the published version of the quiz

Started: Apr 6 at 9:30am

### Quiz Instructions

### Questions

[? Question 1](#)

[? Question 2](#)

[? Question 3](#)

[? Question 4](#)

[? Question 5](#)

Time Elapsed: [Hide Time](#)

Attempt due: Feb 8, 2022 at 11:59pm

0 Minutes, 7 Seconds



#### Question 1

1 pts

Who of the following mathematicians lived before the other three?

Hipocrates

Euclid

Archimedes

Thales

Pythagoras

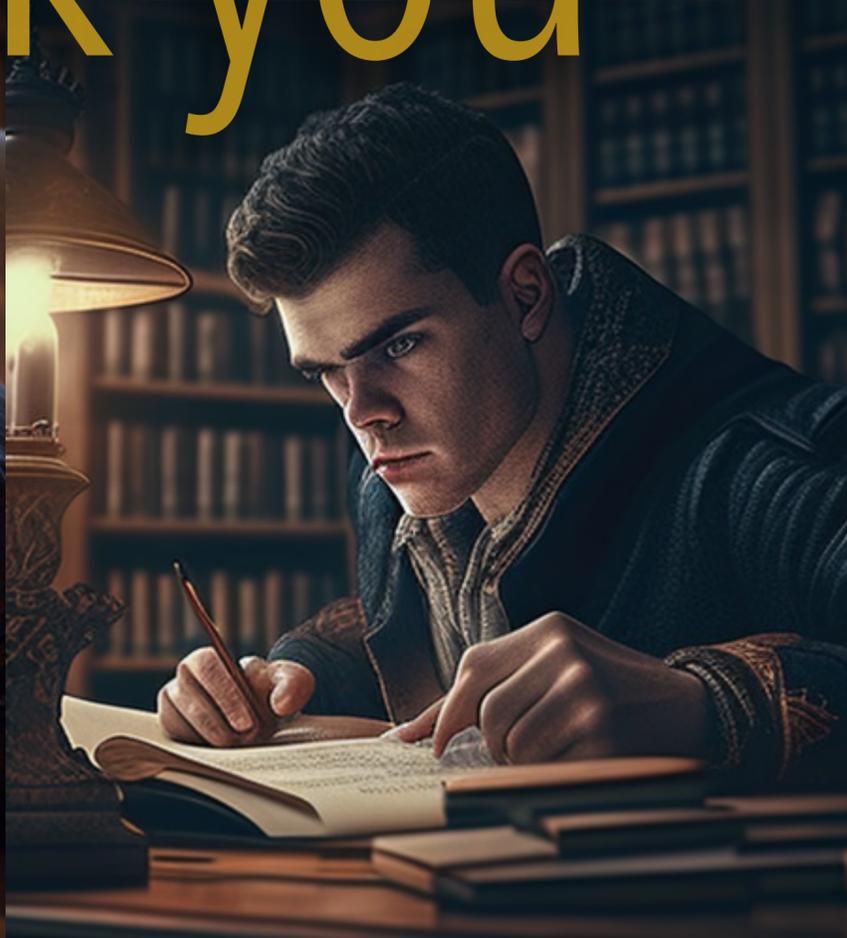


#### Question 2

1 pts



Thank you



# Additional Formats

# Media in the Classroom

Power point, keynote

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Laboratory setup

Computer Games

Social Media, tweet

Email, texting, slack

Desmos, Wolfram

Interactive whiteboards

White and chalk boards

Overhead projectors

Class video recordings

Document cameras

Calculators, CAS

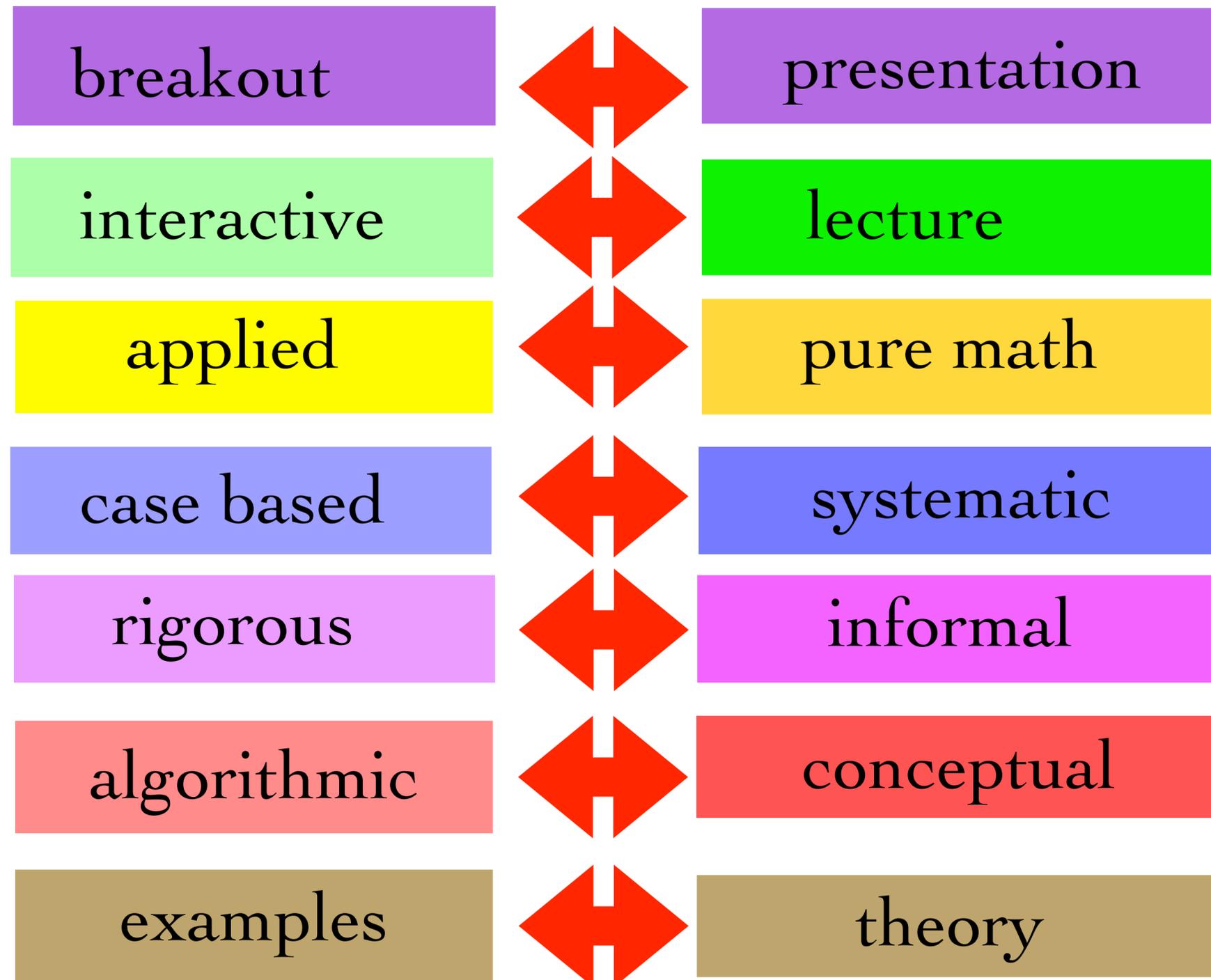
Video Games, virtual

Youtube like Kahn, TED,

AI agents, Siri, Alex

Geoalgebra, Desmos

# Parameters



# Objectives

Visual  
Geometric

Artistic  
Entertaining

Formal  
Algebraic

Numerical  
Algorithmic

Social  
Collaborative

Application  
Practical

Humor  
Fun

Historical  
Cultural

Challenge  
Discovery

Inquiry  
Exploratory